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On the Cauchy problem for a weakly dissipative μ-Degasperis-Procesi equation

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Abstract

In this paper, we study the Cauchy problem of a weakly dissipative μ-Degasperis-Procesi equation. We first present several blow-up results of strong solutions to the equation. Then, we give an improved global existence result to the equation. The obtained results for the equation improve considerably the earlier results. Finally, we discuss the global existence and uniqueness of weak solutions to the equation.

MSC: 35G25; 35L05

Keywords: a weakly dissipative μ-Degasperis-Procesi equation; blow-up; global existence; strong solutions; weak solutions

1 Introduction

The μ-Degasperis-Procesi equation

$$\mu(u)_t - u_{txx} + 3\mu(u)u_x - 3u_x u_{xx} - uu_{xxx} = 0 \tag{μDP}$$

can be formally described as evolution equations on the space of tensor densities over the Lie algebra of smooth vector fields on the circle [1], where $u(t,x)$ is a time-dependent function on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_{\mathbb{S}} u(t,x)\,dx$ denotes its mean. This equation is originally derived and studied in [2]. Recently, a new geometric explanation to the μDP equation has been given in [3]. It is notable that the physical significance of the μDP equation is a left open problem [1].

The μDP equation has close relation with the μ Burgers (μB) equation [2, 4]

$$-u_{txx} - 3u_x u_{xx} - uu_{xxx} = 0 \tag{μB}$$

and the Degasperis-Procesi (DP) equation [5]

$$u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0. \tag{DP}$$

In fact, with $y = Au$, $A = \mu - \partial_x^2$, one can rewrite the μDP equation as follows:

$$y_t + uy_x + 3u_x y = 0.$$

If $A = -\partial_x^2$, then the μDP equation becomes the μB equation and if $A = 1 - \partial_x^2$, then the μDP equation becomes the DP equation. Moreover, the μB equation is the high-frequency limit of the DP equation. In [2], the authors discussed the μB equation and its properties. The DP equation is a model for nonlinear shallow water dynamics. There are a lot of papers about the DP equation, *cf.* [6–10].

After the μDP equation appeared, it has been studied in several works [1, 3, 11]. In [1], the authors established the local well-posedness to the μDP equation, proved it has not only global strong solutions but also blow-up solutions. They also proved that the μDP equation is integrable, has bi-Hamiltonian structure and corresponding infinite hierarchy of conservation laws, admits shock-peakon solutions and multi-peakon solutions. Moreover, the shock-peakon solutions are similar to those of the DP equation formally [1]. In [11], the authors derived the precise blow-up scenario and the blow-up rate for strong solutions to the equation, presented several blow-up results of strong solutions and gave a geometric description to the equation.

In general, it is difficult to avoid energy dissipation mechanisms in the real world. So, it is reasonable to study the model with energy dissipation. In [12] and [13], the authors discussed the energy dissipative KdV equation from different aspects. The weakly dissipative Camassa-Holm (CH) equation and the weakly dissipative DP equation were studied in [14–16] and [17–20], respectively. In [21], the authors discussed the blow-up and blow-up rate of solutions to a weakly dissipative periodic rod equation. In [22], the authors investigated some properties of solutions to the weakly dissipative b-family equation. Recently, some results for a weakly dissipative μDP equation were proved in [23]. The author established local well-posedness for the weakly dissipative μDP equation by use of a geometric argument, derived the precise blow-up scenario, discussed the blow-up phenomena and global existence.

In this paper, we continue discussing the Cauchy problem of the following weakly dissipative μDP equation:

$$\begin{cases} y_t + uy_x + 3u_x y + \lambda y = 0, & t > 0, x \in \mathbb{R}, \\ y = \mu(u) - u_{xx}, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}, \end{cases} \tag{1.1}$$

or in the equivalent form:

$$\begin{cases} \mu(u)_t - u_{txx} + 3\mu(u)u_x - 3u_x u_{xx} - uu_{xxx} + \lambda(\mu(u) - u_{xx}) = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}. \end{cases} \tag{1.2}$$

Here the constant λ is assumed to be positive and the term $\lambda y = \lambda(\mu(u) - u_{xx})$ models energy dissipation. Firstly, based on the results in [23] and some new results, we present several new blow-up results of strong solutions and an improved global existence result to the equation. Then, we discuss the global existence and uniqueness of weak solutions.

The paper is organized as follows. In Section 2, we recall some useful lemmas and derive some new useful results to (1.1). In Section 3, we present some explosion criteria of strong solutions to equation (1.1) with general initial data and give the blow-up rate of strong solutions to the equation when blow-up occurs. In Section 4, we give an improved global existence result of strong solutions to equation (1.1). In Section 5, we establish global existence and uniqueness of weak solutions to equation (1.1) by use of smooth approximate to initial data and Helly's theorem.

Notation Throughout the paper, we denote by $*$ the convolution. Let $\|\cdot\|_Z$ denote the norm of Banach space Z, and let $\langle\cdot,\cdot\rangle$ denote the $H^1(\mathbb{S})$, $H^{-1}(\mathbb{S})$ duality bracket. Let $M(\mathbb{S})$ be the space of Radon measures on $\mathbb{S}$ with bounded total variation, and let $M^+(\mathbb{S})$ ($M^-(\mathbb{S})$) be the subset of $M(\mathbb{S})$ with positive (negative) measures. Finally, we write $BV(\mathbb{S})$ for the space of functions with bounded variation, $\mathbb{V}(f)$ being the total variation of $f \in BV(\mathbb{S})$. Since all spaces of functions are over $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, for simplicity, we drop $\mathbb{S}$ in our notations if there is no ambiguity.

2 Preliminaries

In this section, we recall some useful lemmas and derive some new useful results to (1.1). Firstly, one can reformulate equation (1.1) as follows:

$$\begin{cases} u_t + uu_x = -\partial_x A^{-1}(3\mu(u)u) - \lambda u, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $A = \mu - \partial_x^2$ is an isomorphism between H^s and H^{s-2} with the inverse $v = A^{-1}w$ given explicitly by [1, 3]

$$\begin{aligned} v(x) = {}& \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12}\right)\mu(w) + \left(x - \frac{1}{2}\right)\int_0^1\int_0^y w(s)\,ds\,dy \\ & - \int_0^x\int_0^y w(s)\,ds\,dy + \int_0^1\int_0^y\int_0^s w(r)\,dr\,ds\,dy. \end{aligned} \tag{2.2}$$

Since A^{-1} and ∂_x commute, the following identities

$$A^{-1}\partial_x w(x) = \left(x - \frac{1}{2}\right)\int_0^1 w(x)\,dx - \int_0^x w(y)\,dy + \int_0^1\int_0^x w(y)\,dy\,dx \tag{2.3}$$

and

$$A^{-1}\partial_x^2 w(x) = -w(x) + \int_0^1 w(x)\,dx \tag{2.4}$$

hold. If we rewrite the inverse of the operator $A = \mu - \partial_x^2$ in terms of Green's function, we find $(A^{-1}m)(x) = \int_0^1 g(x-x')m(x')\,dx' = (g * m)(x)$ for all $m \in L^2$. So, we get another equivalent form:

$$\begin{cases} u_t + uu_x = -\partial_x g * (3\mu(u)u) - \lambda u, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}, \end{cases} \tag{2.5}$$

where the Green's function $g(x)$ is given [2] by

$$g(x) = \frac{1}{2}x(x-1) + \frac{13}{12} \quad \text{for } x \in \mathbb{S}, \tag{2.6}$$

and is extended periodically to the real line. In other words,

$$g(x-x') = \frac{(x-x')^2}{2} - \frac{|x-x'|}{2} + \frac{13}{12}, \quad x, x' \in \mathbb{S}.$$

In particular, $\mu(g) = 1$.

Lemma 2.1 ([23]) *Given $u_0 \in H^s$, $s > \frac{3}{2}$, then there exists a maximal $T = T(\lambda, u_0) > 0$ and a unique solution u to* (2.1) (*or* (1.1)) *such that*

$$u = u(\cdot, u_0) \in C([0,T); H^s) \cap C^1([0,T); H^{s-1}).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$u_0 \to u(\cdot, u_0) : H^s \to C([0,T); H^s) \cap C^1([0,T); H^{s-1})$$

is continuous.

Remark 2.1 Similar to the proof of Theorem 2.3 in [24], we have that the maximal time of existence $T > 0$ in Lemma 2.1 is independent of the Sobolev index $s > \frac{3}{2}$.

Combining Remark 2.1 with Lemma 9 in [23], we have the following results.

Lemma 2.2 ([23]) *Let $u_0 \in H^s$, $s > \frac{3}{2}$ be given, and $u(t,x)$ is the solution of equation* (1.1) *with the initial data u_0. Then we have*

$$\mu(u) = \mu_0 e^{-\lambda t}$$

for $t \geq 0$ in the existence interval of u, where $\mu_0 = \mu(u_0) = \int_{\mathbb{S}} u_0(x)\,dx$.

Combining Lemma 2.2 and (2.5), we have another equivalent form of (2.1):

$$\begin{cases} u_t + uu_x = -\partial_x g * (3\mu_0 e^{-\lambda t} u) - \lambda u, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t, x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}. \end{cases} \tag{2.7}$$

Lemma 2.3 ([23]) *Let $u_0 \in H^s$, $s > \frac{3}{2}$ be given, and let T be the maximal existence time of the corresponding solution u to* (2.1) *with the initial data u_0. Then the corresponding solution blows up in finite time if and only if*

$$\liminf_{t \to T} \left\{ \min_{x \in \mathbb{S}} u_x(t,x) \right\} = -\infty.$$

Given $u_0 \in H^s$ with $s > \frac{3}{2}$, Lemma 2.1 ensures the existence of a maximal $T > 0$ and a solution u to (2.1) such that

$$u = u(\cdot, u_0) \in C([0,T); H^s) \cap C^1([0,T); H^{s-1}).$$

Consider now the following initial value problem:

$$\begin{cases} q_t = u(t,q), & t \in [0,T), \\ q(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{2.8}$$

Lemma 2.4 ([23]) *Let $u_0 \in H^s$ with $s > \frac{3}{2}$, $T > 0$ be the maximal existence time. Then equation* (2.8) *has a unique solution $q \in C^1([0,T) \times \mathbb{R};\mathbb{R})$ and the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = \exp\left(\int_0^t u_x\big(s,q(s,x)\big)\,ds\right) > 0, \quad (t,x) \in [0,T) \times \mathbb{R}.$$

Moreover, with $y = \mu(u) - u_{xx}$, we have

$$y\big(t,q(t,x)\big)q_x^3(t,x) = y_0(x)e^{-\lambda t}.$$

Lemma 2.5 *Let $u_0 \in H^s$, $s > \frac{3}{2}$ be given, and $u(t,x)$ is the solution of equation* (1.1) *with the initial data u_0. Then we have*

$$\int_{\mathbb{S}} u^2\,dx = \int_{\mathbb{S}} u_0^2\,dx \cdot e^{-2\lambda t} := \mu_1^2 e^{-2\lambda t},$$

where $\mu_1 = (\int_{\mathbb{S}} u_0^2\,dx)^{\frac{1}{2}}$.

Proof Let $v = (\mu - \partial_x^2)^{-1}u$. By the first equation in (2.1), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} u^2\,dx &= \int_{\mathbb{S}} u\big(-uu_x - 3\mu(u)\partial_x\big(\mu - \partial_x^2\big)^{-1}u - \lambda u\big)\,dx \\ &= -\int_{\mathbb{S}} u^2 u_x\,dx - 3\mu(u)\int_{\mathbb{S}} u\partial_x\big(\mu - \partial_x^2\big)^{-1}u\,dx - \lambda\int_{\mathbb{S}} u^2\,dx \\ &= -3\mu(u)\int_{\mathbb{S}} \big(\mu - \partial_x^2\big)v \cdot \partial_x v\,dx - \lambda\int_{\mathbb{S}} u^2\,dx \\ &= -3\mu(u)\mu(v)\int_{\mathbb{S}} v_x\,dx + 3\mu(u)\int_{\mathbb{S}} v_{xx}v_x\,dx - \lambda\int_{\mathbb{S}} u^2\,dx \\ &= -\lambda\int_{\mathbb{S}} u^2\,dx. \end{aligned}$$

This completes the proof of Lemma 2.5. □

Using Lemma 2.5, we have the following useful result.

Lemma 2.6 *Let $u_0 \in H^s$, $s > \frac{3}{2}$, be given and assume that T is the maximal existence time of the corresponding solution u to* (2.1) *with the initial data u_0. Then we have*

$$\|u(t,x)\|_{L^\infty} \le e^{-\lambda t}\left(\left(\frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_1\right)t + \|u_0\|_{L^\infty}\right).$$

Proof Applying a simple density argument, Remark 2.1 implies that we only need to consider the case $s = 3$. Combining the first equation in (2.1) with Lemma 2.2, we have

$$u_t + uu_x + \lambda u = -3\mu_0 e^{-\lambda t}\partial_x\left(\mu - \partial_x^2\right)^{-1}u.$$

From (2.8) it follows that

$$\begin{aligned}\frac{du(t,q(t,x))}{dt} + \lambda u\big(t,q(t,x)\big) &= (u_t + uu_x + \lambda u)\big(t,q(t,x)\big)\\ &= -3\mu_0 e^{-\lambda t}\left(\partial_x\left(\mu - \partial_x^2\right)^{-1}u\right)\big(t,q(t,x)\big),\end{aligned}$$

that is,

$$e^{\lambda t}\frac{du(t,q(t,x))}{dt} + \lambda e^{\lambda t}u\big(t,q(t,x)\big) = -3\mu_0\left(\partial_x\left(\mu - \partial_x^2\right)^{-1}u\right)\big(t,q(t,x)\big).$$

Combining (2.3) with Lemma 2.5, we have

$$\left|-3\mu_0\partial_x\left(\mu - \partial_x^2\right)^{-1}u\right| \le \frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_1 := C.$$

So we have

$$-C \le \frac{d}{dt}\left(e^{\lambda t}u\big(t,q(t,x)\big)\right) \le C.$$

Integrating all sides of this inequality from 0 to t, we obtain

$$\left|u\big(t,q(t,x)\big)\right| \le e^{-\lambda t}\left(Ct + \|u_0\|_{L^\infty}\right).$$

Noting that the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$, we have

$$\left\|u(t,x)\right\|_{L^\infty} = \left\|u(t,x)\right\|_{L^\infty(\mathbb{R})} = \left\|u\big(t,q(t,x)\big)\right\|_{L^\infty(\mathbb{R})} \le e^{-\lambda t}\left(Ct + \|u_0\|_{L^\infty}\right).$$

This completes the proof of Lemma 2.6. □

Lemma 2.7 *Let $u_0 \in H^s$, $s > \frac{3}{2}$, be given and assume that T is the maximal existence time of the corresponding solution u to* (2.1) *with the initial data u_0. Let*

$$E(t) = \int_{\mathbb{S}}\left(\frac{3}{2}\mu(u)\left(A^{-1}\partial_x u\right)^2 + \frac{1}{6}u^3\right)dx,$$

then for $\forall t_0 \in [0,T)$, we have $E(t) = E(t_0)e^{-3\lambda(t-t_0)}$. In particular, we have $E(t) = E(0)e^{-3\lambda t} := \mu_2 e^{-3\lambda t}$, where $\mu_2 = \int_{\mathbb{S}}(\frac{3}{2}\mu_0(A^{-1}\partial_x u_0)^2 + \frac{1}{6}u_0^3)\,dx$.

Proof Differentiating the first equation of (2.1) with respect to x, we have

$$u_{tx} + u_x^2 + uu_{xx} = 3\mu(u)u - 3\mu(u)^2 - \lambda u_x. \tag{2.9}$$

It follows that

$$\begin{aligned}
\frac{d}{dt}E(t) &= \int_{\mathbb{S}} \left(\frac{3}{2}\mu(u)_t \bigl(A^{-1}\partial_x u\bigr)^2 + 3\mu(u)\bigl(A^{-1}\partial_x u\bigr)\bigl(A^{-1}u_{tx}\bigr) + \frac{1}{2}u^2 u_t \right) dx \\
&= \frac{3}{2}\mu(u)_t \int_{\mathbb{S}} \bigl(A^{-1}\partial_x u\bigr)^2\, dx + 3\mu(u)\int_{\mathbb{S}} \bigl(A^{-1}\partial_x u\bigr)\cdot A^{-1}\bigl(-u_x^2 - uu_{xx} \\
&\quad + 3\mu(u)u - 3\mu(u)^2 - \lambda u_x\bigr)\, dx + \frac{1}{2}\int_{\mathbb{S}} u^2\bigl(-uu_x - 3\mu(u)\partial_x A^{-1}u - \lambda u\bigr)\, dx \\
&= \frac{3}{2}\bigl(-\lambda\mu(u)\bigr)\int_{\mathbb{S}} \bigl(A^{-1}\partial_x u\bigr)^2\, dx - \frac{3}{2}\mu(u)\int_{\mathbb{S}} \bigl(A^{-1}\partial_x u\bigr)\bigl(-u^2 + \mu\bigl(u^2\bigr)\bigr)\, dx \\
&\quad - 3\lambda\mu(u)\int_{\mathbb{S}} \bigl(A^{-1}\partial_x u\bigr)^2\, dx - \frac{3}{2}\mu(u)\int_{\mathbb{S}} u^2\partial_x A^{-1}u\, dx - \frac{\lambda}{2}\int_{\mathbb{S}} u^3\, dx \\
&= -\frac{9}{2}\lambda\int_{\mathbb{S}} \mu(u)\bigl(A^{-1}\partial_x u\bigr)^2\, dx - \frac{\lambda}{2}\int_{\mathbb{S}} u^3\, dx = -3\lambda E(t),
\end{aligned}$$

here we used the relation $uu_{xx} + u_x^2 = \frac{1}{2}\partial_x^2(u^2)$ and (2.4). Integrating this equality from t_0 to t for $\forall t_0, t \in [0,T)$, we know that the conclusions in Lemma 2.7 hold. □

3 Blow-up and blow-up rate

In this section, we discuss the blow-up phenomena of equation (1.1) and prove that there exist strong solutions to (1.1) which do not exist globally in time. At first, we give the following useful lemma.

Lemma 3.1 ([25]) *Let $t_0 > 0$ and $v \in C^1([0,t_0); H^2(\mathbb{R}))$. Then, for every $t \in [0,t_0)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$m(t) := \inf_{x\in\mathbb{R}}\bigl\{v_x(t,x)\bigr\} = v_x\bigl(t,\xi(t)\bigr),$$

and the function m is almost everywhere differentiable on $(0,t_0)$ with

$$\frac{d}{dt}m(t) = v_{tx}\bigl(t,\xi(t)\bigr) \quad \text{a.e. on } (0,t_0).$$

Theorem 3.1 *Let $u_0 \in H^s$, $s > \frac{3}{2}$, and T be the maximal time of the solution u to* (1.1) *with the initial data u_0. If $\mu_0\mu_2 \le 0$ and there is a point $\xi_0 \in \mathbb{S}$ such that $\mu_0 u_0(\xi_0) \le 0$ and $u_0'(\xi_0) < -\lambda$, then the corresponding solution to* (1.1) *blows up in finite time.*

Proof As mentioned earlier, here we only need to show that the above theorem holds for $s = 3$.

Firstly, we claim that for any fixed $t \in [0,T)$, there is $\xi(t) \in \mathbb{S}$ such that $\mu_0 u(t,\xi(t)) \le 0$. By the assumption of the theorem $\mu_0 u_0(\xi_0) \le 0$, we have that $\xi(0)$ exists when $t = 0$ and $\xi(0) = \xi_0$. Next, we claim that for any fixed $t \in (0,T)$, there is $\xi(t) \in \mathbb{S}$ such that $\mu_0 u(t, \xi(t)) \le 0$. If not, there exists $t_0 \in (0,T)$ such that $\mu_0 u(t_0,x) > 0$ for any $x \in \mathbb{S}$. By Lemma 2.7, we have

$$\begin{aligned}
\mu_0 E(t) &= e^{-3\lambda(t-t_0)}\mu_0 E(t_0) \\
&= e^{-3\lambda(t-t_0)}\int_{\mathbb{S}} \left(\frac{3}{2}\mu_0^2 e^{-\lambda t_0}\bigl(A^{-1}\partial_x u(t_0,x)\bigr)^2 + \frac{1}{6}\mu_0 u(t_0,x)\cdot u^2(t_0,x) \right) dx > 0.
\end{aligned}$$

From $\mu_0 E(t) = \mu_0\mu_2 e^{-3\lambda t}$ it follows that $\mu_0\mu_2 > 0$, which contradicts the assumption $\mu_0\mu_2 \leq 0$.

Since $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ and $\xi(t) \in \mathbb{S}$ for any fixed $t \in [0,T)$, there is $y(t) \in \mathbb{R}$ such that

$$q(t, y(t)) = \xi(t), \quad t \in [0,T).$$

Moreover, $y(0) = \xi(0) = \xi_0$. Define now $f(t) = u_x(t, q(t, y(t)))$. Evaluating (2.9) at $(t, q(t, y(t)))$, we obtain

$$\begin{aligned}\frac{df(t)}{dt} &= -f^2(t) + 3e^{-\lambda t}\mu_0 u(t, q(t, y(t))) - 3(\mu(u))^2 - \lambda f(t)\\ &= -f^2(t) + 3e^{-\lambda t}\mu_0 u(t, \xi(t)) - 3(\mu(u))^2 - \lambda f(t)\\ &\leq -f^2(t) - \lambda f(t)\\ &= -f(t)(f(t) + \lambda).\end{aligned}$$

Note that if $f(0) = u_0'(\xi_0) < -\lambda$, then $f(t) < -\lambda$ for all $t \in [0,T)$. From the above inequality we obtain

$$\left(1 + \frac{\lambda}{f(0)}\right)e^{\lambda t} - 1 \leq \frac{\lambda}{f(t)} \leq 0.$$

Since $\frac{f(0)}{f(0)+\lambda} > 1$, then there exists

$$0 < T \leq \frac{1}{\lambda}\ln\frac{f(0)}{f(0)+\lambda}$$

such that $\lim_{t\to T} f(t) = -\infty$. Lemma 2.3 implies that the solution u blows up in finite time. □

Theorem 3.2 *Let* $\epsilon > 0$ *and* $u_0 \in H^s$, $s > \frac{3}{2}$, *and* T *be the maximal time of the solution* u *to* (1.1) *with the initial data* u_0. *If*

$$\min_{x\in\mathbb{S}} u_0'(x) \leq -\frac{\lambda}{2} - (1+\epsilon)\cdot\frac{(\lambda^2 + (\sqrt{3\|u_0\|_{L^\infty}^2 + 2\sqrt{3}C\ln(1+\frac{2}{\epsilon})} + \sqrt{3}\|u_0\|_{L^\infty})^2)^{\frac{1}{2}}}{2}$$

with $C = \frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_1$, *then the corresponding solution to* (1.1) *blows up in finite time*.

Proof As mentioned earlier, here we only need to show that the above theorem holds for $s = 3$. Define now

$$m(t) := \min_{x\in\mathbb{S}}\{u_x(t,x)\}, \quad t \in [0,T),$$

and let $\xi(t) \in \mathbb{S}$ be a point where this minimum is attained by using Lemma 3.1. It follows that

$$m(t) = u_x(t, \xi(t)).$$

Clearly, $u_{xx}(t,\xi(t)) = 0$ since $u(t,\cdot) \in H^3(\mathbb{S}) \subset C^2(\mathbb{S})$. Evaluating (2.9) at $(t,\xi(t))$, by Lemma 2.6 we obtain

$$\begin{aligned}\frac{dm(t)}{dt} &= -m^2(t) + 3\mu(u)u - 3\mu(u)^2 - \lambda m(t) \\ &\leq -m^2(t) - \lambda m(t) + 3\|u\|_{L^\infty}^2 \\ &\leq -m^2(t) - \lambda m(t) + 3\big(Ct + \|u_0\|_{L^\infty}\big)^2. \end{aligned} \tag{3.1}$$

For fixed $\epsilon > 0$, taking

$$T_1 = \frac{\sqrt{3\|u_0\|_{L^\infty}^2 + 2\sqrt{3}C\ln(1+\frac{2}{\epsilon})} - \sqrt{3}\|u_0\|_{L^\infty}}{2\sqrt{3}C}$$

and

$$K(T_1) = \sqrt{3}\big(CT_1 + \|u_0\|_{L^\infty}\big),$$

we find that

$$2K(T_1)T_1 = \ln\left(1 + \frac{2}{\epsilon}\right).$$

From (3.1) it follows that

$$\frac{dm(t)}{dt} \leq -\big(m(t) + A + B\big)\big(m(t) + A - B\big), \quad \forall t \in [0, T_1],$$

where $A = \frac{\lambda}{2}$, $B = \frac{\sqrt{\lambda^2 + 4(K(T_1))^2}}{2}$. Note that if $m(0) \leq -A - (1+\epsilon)B < -A - B$, then $m(t) < -A - B$ for all $t \in [0, T_1] \cap [0, T)$. From the above inequality we obtain

$$\frac{m(0) + A + B}{m(0) + A - B}e^{2Bt} - 1 \leq \frac{2B}{m(t) + A - B} \leq 0, \quad \forall t \in [0, T_1] \cap [0, T).$$

Since $0 < \frac{m(0)+A+B}{m(0)+A-B} < 1$, $m(0) \leq -A - (1+\epsilon)B$ and $2K(T_1)T_1 = \ln(1 + \frac{2}{\epsilon})$, then there exists

$$0 < T \leq \frac{1}{2B}\ln\frac{m(0) + A - B}{m(0) + A + B} \leq T_1$$

such that $\lim_{t\to T} m(t) = -\infty$. Lemma 2.3 implies that the solution u blows up in finite time. □

Theorem 3.3 *Let $u_0 \in H^s$, $s > \frac{3}{2}$, and T be the maximal time of the solution u to* (1.1) *with the initial data u_0. If $u_0(x)$ is odd satisfies $u_0'(0) < -\frac{\lambda}{2} - \frac{\sqrt{\lambda^2 + 12\mu_0^2}}{2}$, then the corresponding solution to* (1.1) *blows up in finite time.*

Proof As mentioned earlier, here we only need to show that the above theorem holds for $s = 3$. By $\mu(-u(t,-x)) = -\mu(u(t,x))$, we have (1.2) is invariant under the transformation

$(u,x) \to (-u,-x)$. Thus we deduce that if $u_0(x)$ is odd, then $u(t,x)$ is odd with respect to x for any $t \in [0,T)$. By continuity with respect to x of u and u_{xx}, we have

$$u(t,0) = u_{xx}(t,0) = 0, \quad \forall t \in [0,T).$$

Evaluating (2.9) at $(t,0)$ and letting $h(t) = u_x(t,0)$, we obtain

$$\begin{aligned} \frac{dh(t)}{dt} &= -h^2(t) - \lambda h(t) - 3\mu(u)^2 \\ &\le -h^2(t) - \lambda h(t) + 3\mu_0^2 \\ &= -\bigl(h(t) + A + B\bigr)\bigl(h(t) + A - B\bigr), \end{aligned}$$

where $A = \frac{\lambda}{2}$, $B = \frac{\sqrt{\lambda^2 + 12\mu_0^2}}{2}$. Note that if $h(0) < -A - B$, then $h(t) < -A - B$ for all $t \in [0,T)$. From the above inequality we obtain

$$\frac{h(0) + A + B}{h(0) + A - B} e^{2Bt} - 1 \le \frac{2B}{h(t) + A - B} \le 0.$$

Since $\frac{h(0)+A-B}{h(0)+A+B} > 1$, then there exists

$$0 < T \le \frac{1}{2B} \ln \frac{h(0) + A - B}{h(0) + A + B}$$

such that $\liminf_{t\to T}\{\min_{x\in\mathbb{S}} u_x(t,x)\} \le \lim_{t\to T} h(t) = -\infty$. Lemma 2.3 implies that the solution u blows up in finite time. □

Similar to the proof of Theorem 3.1 in [21], we have the following blow-up rate result. This result shows that the blow-up rate of strong solutions to the weakly dissipative μDP equation is not affected by the weakly dissipative term even though the occurrence of blow-up of strong solutions to equation (1.1) is affected by the dissipative parameter, see Theorems 3.1-3.3.

Theorem 3.4 *Let $u_0 \in H^s$, $s > \frac{3}{2}$, and T be the maximal time of the solution u to* (1.1) *with the initial data u_0. If T is finite, we obtain*

$$\lim_{t\to T} (T - t) \min_{x\in\mathbb{S}} u_x(t,x) = -1.$$

4 Global existence

In this section, we present some global existence results. Firstly, we give a useful lemma.

Lemma 4.1 ([11, 26]) *If $f \in H^1(\mathbb{S})$ is such that $\int_{\mathbb{S}} f(x)\,dx = 0$, then we have*

$$\max_{x\in\mathbb{S}} f^2(x) \le \frac{1}{12} \int_{\mathbb{S}} f_x^2(x)\,dx.$$

Theorem 4.1 *If $y_0(x) = \mu_0 - u_{0,xx}(x) \in H^1$ does not change sign, then the corresponding solution u of the initial value u_0 exists globally in time.*

Proof Note that given $t \in [0, T)$, there is $\xi(t) \in \mathbb{S}$ such that $u_x(t, \xi(t)) = 0$ by the periodicity of u to x-variable. If $y_0(x) \geq 0$, then Lemma 2.4 implies that $y(t,x) \geq 0$. For $x \in [\xi(t), \xi(t) + 1]$, we have

$$\begin{aligned} -u_x(t,x) &= -\int_{\xi(t)}^x \partial_x^2 u(t,x)\,dx = \int_{\xi(t)}^x \big(y - \mu(u)\big)\,dx = \int_{\xi(t)}^x y\,dx - \mu(u)\big(x - \xi(t)\big) \\ &\leq \int_{\mathbb{S}} y\,dx - \mu(u)\big(x - \xi(t)\big) = \mu(u)\big(1 - x + \xi(t)\big) \leq |\mu_0|. \end{aligned}$$

It follows that $u_x(t,x) \geq -|\mu_0|$. On the other hand, if $y_0(x) \leq 0$, then Lemma 2.4 implies that $y(t,x) \leq 0$. Therefore, for $x \in [\xi(t), \xi(t) + 1]$, we have

$$\begin{aligned} -u_x(t,x) &= -\int_{\xi(t)}^x \partial_x^2 u(t,x)\,dx = \int_{\xi(t)}^x \big(y - \mu(u)\big)\,dx = \int_{\xi(t)}^x y\,dx - \mu(u)\big(x - \xi(t)\big) \\ &\leq -\mu(u)\big(x - \xi(t)\big) \leq |\mu_0|. \end{aligned}$$

It follows that $u_x(t,x) \geq -|\mu_0|$. This completes the proof by using Theorem 3.3. □

Corollary 4.1 *If the initial value* $u_0 \in H^3$ *such that*

$$\|\partial_x^3 u_0\|_{L^2} \leq 2\sqrt{3}|\mu_0|,$$

then the corresponding solution u *of* u_0 *exists globally in time.*

Proof Note that $\int_{\mathbb{S}} \partial_x^2 u_0\,dx = 0$, Lemma 4.1 implies that

$$\|\partial_x^2 u_0\|_{L^\infty} \leq \frac{\sqrt{3}}{6}\|\partial_x^3 u_0\|_{L^2}.$$

If $\mu_0 \geq 0$, then

$$y_0(x) = \mu_0 - \partial_x^2 u_0(x) \geq \mu_0 - \frac{\sqrt{3}}{6}\|\partial_x^3 u_0\|_{L^2} \geq \mu_0 - |\mu_0| = 0.$$

If $\mu_0 \leq 0$, then

$$y_0(x) = \mu_0 - \partial_x^2 u_0(x) \leq \mu_0 + \|\partial_x^2 u_0\|_{L^\infty} \leq \mu_0 + \frac{\sqrt{3}}{6}\|\partial_x^3 u_0\|_{L^2} \leq \mu_0 + |\mu_0| = 0.$$

Since $\mu_0 = \int_{\mathbb{S}} u_0(x)\,dx$ is a determined constant for given $u_0 \in H^3$, $y_0(x) \in H^1$ does not change sign. This completes the proof by using Theorem 4.1. □

5 Weak solutions

This section is concerned with global existence of weak solutions for (1.2) by use of smooth approximate to initial data and Helly's theorem. Before giving the precise statement of the main result, we first introduce the definition of a weak solution to problem (1.2).

Definition 5.1 A function $u(t,x) \in C(\mathbb{R}^+ \times \mathbb{S}) \cap L^\infty(\mathbb{R}^+; H^1)$ is said to be an admissible global weak solution to (1.2) if u satisfies the equations in (1.2) and $u(t,\cdot) \to u_0$ as $t \to 0^+$ in the sense of distributions on $\mathbb{R}_+ \times \mathbb{R}$. Moreover, $\mu(u) = \mu(u_0)e^{-\lambda t}$.

The main result of this paper can be stated as follows.

Theorem 5.1 *Let $u_0 \in H^1$. Assume that $y_0 = (\mu(u_0) - u_{0,xx}) \in M^+$, then equation* (1.2) *has a unique admissible global weak solution in the sense of Definition* 5.1. *Moreover,*

$$u \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}_+; W^{1,\infty}\big) \cap H^1_{\mathrm{loc}}\big(\mathbb{R}^+ \times \mathbb{S}\big).$$

Furthermore, $y = (\mu(u) - u_{xx}(t,\cdot)) \in M^+$ for a.e. $t \in \mathbb{R}^+$ is uniformly bounded on $\mathbb{S}$.

Remark 5.1 If $y_0 = (\mu(u_0) - u_{0,xx}) \in M^-$, then the conclusions in Theorem 5.1 also hold with $y = (\mu(u) - u_{xx}(t,\cdot)) \in M^-$.

Firstly, we will give some useful lemmas.

Lemma 5.1 *If $y_0 = \mu_0 - u_{0,xx} \in H^1$ does not change sign, then the corresponding solution u to* (2.7) *of the initial value u_0 exists globally in time, that is, $u \in C(\mathbb{R}^+, H^3) \cap C^1(\mathbb{R}^+, H^2)$. Moreover, the following properties hold*:

(1) *$y(t,x)$, $u(t,x)$ have the same sign with $y_0(x)$, and $\|u_x\|_{L^\infty(\mathbb{R}^+\times\mathbb{S})} \le |\mu_0|$,*
(2) $|\mu_0| e^{-\lambda t} = \|y_0\|_{L^1} e^{-\lambda t} = \|y(t,\cdot)\|_{L^1} = \|u(t,\cdot)\|_{L^1}$.

Proof Firstly, Lemma 2.4 and $u = g * y$, $g \ge 0$ imply that $y(t,x)$, $u(t,x)$ have the same sign with $y_0(x)$. Moreover, from the proof of Theorem 4.1, we have $u_x(t,x) \ge -|\mu_0|$. Now note that given $t \in [0,T)$, there is $\xi(t) \in \mathbb{S}$ such that $u_x(t,\xi(t)) = 0$ by the periodicity of u to x-variable. If $y_0 \ge 0$, then $y \ge 0$. For $x \in [\xi(t), \xi(t)+1]$, we have

$$\begin{aligned} u_x(t,x) &= \int_{\xi(t)}^x \partial_x^2 u(t,x)\,dx = \int_{\xi(t)}^x \big(\mu(u) - y\big)\,dx = \mu(u)\big(x - \xi(t)\big) - \int_{\xi(t)}^x y\,dx \\ &\le \mu(u)\big(x - \xi(t)\big) \le |\mu_0|. \end{aligned}$$

It follows that $u_x(t,x) \le |\mu_0|$. On the other hand, if $y_0 \le 0$, then $y \le 0$. Therefore, for $x \in [\xi(t), \xi(t)+1]$, we have

$$\begin{aligned} u_x(t,x) &= \int_{\xi(t)}^x \partial_x^2 u(t,x)\,dx = \int_{\xi(t)}^x \big(\mu(u) - y\big)\,dx \le \mu(u)\big(x - \xi(t)\big) - \int_{\mathbb{S}} y\,dx \\ &= \mu(u)\big(x - \xi(t)\big) - \int_{\mathbb{S}} \big(\mu(u) - u_{xx}\big)\,dx = \mu(u)\big(x - \xi(t) - 1\big) \le |\mu_0|. \end{aligned}$$

It follows that $u_x(t,x) \le |\mu_0|$. So we have $\|u_x\|_{L^\infty(\mathbb{R}^+\times\mathbb{S})} \le |\mu_0|$, this completes the proof of (1). By the first equation of (1.1), we have

$$\int_{\mathbb{S}} y(t,x)\,dx = \left(\int_{\mathbb{S}} y_0(x)\,dx\right) e^{-\lambda t} = \mu_0 e^{-\lambda t}.$$

If $y_0 \ge 0$, then $y \ge 0$ and $\mu_0 \ge 0$, we have

$$\|y\|_{L^1(\mathbb{S})} = \int_{\mathbb{S}} y(t,x)\,dx = \left(\int_{\mathbb{S}} y_0(x)\,dx\right) e^{-\lambda t} = \|y_0\|_{L^1(\mathbb{S})} e^{-\lambda t} = \mu_0 e^{-\lambda t}.$$

If $y_0 \le 0$, then $y \le 0$ and $\mu_0 \le 0$, we have

$$\|y\|_{L^1(\mathbb{S})} = -\int_{\mathbb{S}} y(t,x)\,dx = \left(\int_{\mathbb{S}} \bigl(-y_0(x)\bigr)\,dx\right)e^{-\lambda t} = \|y_0\|_{L^1(\mathbb{S})}e^{-\lambda t} = -\mu_0 e^{-\lambda t}.$$

Combining these two equalities above, we have $|\mu_0|e^{-\lambda t} = \|y_0\|_{L^1}e^{-\lambda t} = \|y(t,\cdot)\|_{L^1}$. A similar discussion implies $\|y(t,\cdot)\|_{L^1} = \|u(t,\cdot)\|_{L^1}$. This completes the proof of (2). □

Lemma 5.2 ([27]) *Assume that $X \subset B \subset Y$ with compact imbedding $X \to B$ (X, B and Y are Banach spaces), $1 \le p \le \infty$ and (1) F is bounded in $L^p(0,T;X)$, (2) $\|\tau_h f - f\|_{L^p(0,T-h;Y)} \to 0$ as $h \to 0$ uniformly for $f \in F$. Then F is relatively compact in $L^p(0,T;B)$ (and in $C(0,T;B)$ if $p = \infty$), where $(\tau_h f)(t) = f(t+h)$ for $h > 0$, if f is defined on $[0,T]$, then the translated function $\tau_h f$ is defined on $[-h, T-h]$.*

Lemma 5.3 (Helly's theorem [28]) *Let an infinite family of functions $F = f(x)$ be defined on the segment $[a,b]$. If all functions of the family and the total variation of all functions of the family are bounded by a single number $|f(x)| \le K$, $\bigvee_a^b(f) \le K$, then there exists a sequence $f_n(x)$ in the family F which converges at very point of $[a,b]$ to some function $\varphi(x)$ of finite variation.*

Lemma 5.4 ([29]) *Assume that $u(t,\cdot) \in W^{1,1}(\mathbb{R})$ is uniformly bounded in $W^{1,1}(\mathbb{R})$ for all $t \in \mathbb{R}^+$. Then, for a.e. $t \in \mathbb{R}^+$,*

$$\frac{d}{dt}\int_{\mathbb{R}} |\phi_n * u|\,dx = \int_{\mathbb{R}} (\phi_n * u_t)\operatorname{sgn}(\phi_n * u)\,dx$$

and

$$\frac{d}{dt}\int_{\mathbb{R}} |\phi_n * u_x|\,dx = \int_{\mathbb{R}} (\phi_n * u_{xt})\operatorname{sgn}(\phi_n * u_x)\,dx.$$

Lemma 5.5 ([29]) *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous and bounded. If $\mu \in M(\mathbb{R})$, then*

$$\bigl[\rho_n * (f\mu) - (\rho_n * f)(\rho_n * \mu)\bigr] \to 0 \quad \text{in } L^1(\mathbb{R}) \text{ as } n \to \infty.$$

Lemma 5.6 ([29]) *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous and bounded. If $g \in L^\infty(\mathbb{R})$, then*

$$\bigl[\rho_n * (fg) - (\rho_n * f)(\rho_n * g)\bigr] \to 0 \quad \text{in } L^\infty(\mathbb{R}) \text{ as } n \to \infty.$$

Now we consider the approximate equation of (2.7) as follows:

$$\begin{cases} u_t^n + u^n u_x^n = -\partial_x g * (3\mu_0^n e^{-\lambda t}u^n) - \lambda u^n, & t > 0, x \in \mathbb{R}, \\ u^n(0,x) = u_0^n(x), & x \in \mathbb{R}, \\ u^n(t,x+1) = u^n(t,x), & t \ge 0, x \in \mathbb{R}, \end{cases} \tag{5.1}$$

where $u_0^n(x) = \phi_n * u_0 \in H^\infty$ for $n \ge 1$ and $\mu_0^n = \int_{\mathbb{S}} u_0^n(x)\,dx$. Here $\{\phi_n\}_{n\ge1}$ are the mollifiers

$$\phi_n(x) := \left(\int_{\mathbb{R}} \phi(\xi)\,d\xi\right)^{-1} n\phi(nx), \quad x \in \mathbb{R}, n \ge 1,$$

where $\phi \in C_c^\infty(\mathbb{R})$ is defined by

$$\phi(x) = \begin{cases} e^{1/(x^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Obviously, $\|\phi_n\|_{L^1(\mathbb{R})} = 1$. Clearly, we have

$$u_0^n \to u_0 \quad \text{in } H^1 \text{ as } n \to \infty \tag{5.2}$$

and

$$\begin{aligned} &\|u_0^n\|_{L^2} \leq \|u_0\|_{L^2}, \qquad \|u_{0,x}^n\|_{L^2} \leq \|u_{0,x}\|_{L^2}, \\ &\|u_0^n\|_{H^1} \leq \|u_0\|_{H^1}, \qquad \|u_0^n\|_{L^1} \leq \|u_0\|_{L^1} \end{aligned} \tag{5.3}$$

in view of Young's inequality. Note that

$$\begin{aligned} \mu_0^n = \mu(u_0^n) &= \int_{\mathbb{S}} u_0^n(x)\,dx = \int_{\mathbb{S}}\int_{\mathbb{R}} \phi_n(y)u_0(x-y)\,dy\,dx = \int_{\mathbb{R}}\int_{\mathbb{S}} \phi_n(y)u_0(x-y)\,dx\,dy \\ &= \int_{\mathbb{R}} \phi_n(y)\cdot\left(\int_{\mathbb{S}} u_0(x-y)\,dx\right)dy \\ &= \int_{\mathbb{R}} \phi_n(y)\cdot\left(\int_{\mathbb{S}} u_0(z)\,dz\right)dy \\ &= \int_{\mathbb{R}} \phi_n(y)\mu(u_0)(x-y)\,dy \\ &= \phi_n * \mu(u_0) = \mu(u_0) = \mu_0. \end{aligned}$$

Using this identity, we can rewrite (5.1) as follows:

$$\begin{cases} u_t^n + u^n u_x^n = -\partial_x g * (3\mu_0 e^{-\lambda t} u^n) - \lambda u^n, & t > 0, x \in \mathbb{R}, \\ u^n(0,x) = u_0^n(x), & x \in \mathbb{R}, \\ u^n(t,x+1) = u^n(t,x), & t \geq 0, x \in \mathbb{R}. \end{cases} \tag{5.4}$$

Moreover, for all $n \geq 1$, $y_0^n = \mu(u_0^n) - u_{0,xx}^n = \mu_0 - u_{0,xx}^n \in H^1$ and

$$y_0^n = \mu(u_0^n) - u_{0,xx}^n = \phi_n * \mu(u_0) - \phi_n * u_{0,xx} = \phi_n * y_0 \geq 0.$$

Thus, by Lemma 5.1, we obtain the corresponding solution $u^n \in C(\mathbb{R}^+; H^3) \cap C^1(\mathbb{R}^+; H^2)$ to (5.4) with the initial data $u_0^n(x)$ and $y^n = \mu(u^n) - u_{xx}^n \geq 0$, $u^n = g * y^n \geq 0$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{S}$. Furthermore, combining Lemma 2.2, Lemmas 2.5-2.6, Lemma 5.1 and (5.3), we have:

$$\mu(u^n) = \mu_0^n e^{-\lambda t} = \mu_0 e^{-\lambda t}, \quad t \in [0,\infty), \tag{5.5}$$

$$\|u^n\|_{L^2} = \|u_0^n\|_{L^2} e^{-\lambda t} \leq \|u_0\|_{L^2} e^{-\lambda t} = \mu_1 e^{-\lambda t}, \quad t \in [0,\infty), \tag{5.6}$$

$$\|u^n(t,\cdot)\|_{L^\infty} \leq e^{-\lambda t}\left(\left(\frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_1\right)t + \|u_0\|_{H^1}\right), \tag{5.7}$$

$$\|u_x^n\|_{L^\infty(\mathbb{R}^+\times\mathbb{S})} \leq |\mu_0^n| = |\mu_0|, \tag{5.8}$$

$$|\mu_0| e^{-\lambda t} = \|y_0^n\|_{L^1} e^{-\lambda t} = \|y^n(t,\cdot)\|_{L^1} = \|u^n(t,\cdot)\|_{L^1}. \tag{5.9}$$

Lemma 5.7 *For any fixed $T > 0$, there exists a subsequence $\{u^{n_k}(t,x)\}$ of the sequence $\{u^n(t,x)\}$ and some function $u(t,x) \in L^\infty(\mathbb{R}^+;H^1) \cap H^1([0,T] \times \mathbb{S})$ such that*

$$u^{n_k} \rightharpoonup u \quad \text{in } H^1\bigl([0,T] \times \mathbb{S}\bigr) \text{ as } n_k \to \infty, \forall T > 0 \tag{5.10}$$

and

$$u^{n_k} \to u \quad \text{in } L^\infty\bigl([0,T] \times \mathbb{S}\bigr) \text{ as } n_k \to \infty. \tag{5.11}$$

Moreover, $u(t,x) \in C(\mathbb{R}^+ \times \mathbb{S})$.

Proof Firstly, we will prove that the sequence $\{u^n(t,x)\}$ is uniformly bounded in the space $H^1([0,T] \times \mathbb{S})$. By (5.6) and (5.8), we have

$$\|u^n\|^2_{L^2([0,T]\times\mathbb{S})} = \int_0^T \int_{\mathbb{S}} (u^n)^2\,dx\,dt = \int_0^T \|u^n\|^2_{L^2}\,dx \le \mu_1^2 T, \tag{5.12}$$

$$\|u^n_x\|_{L^2([0,T]\times\mathbb{S})} \le \|u^n_x\|_{L^\infty(\mathbb{R}^+\times\mathbb{S})} \le |\mu_0|. \tag{5.13}$$

Moreover, by (5.8) and (5.12), we obtain

$$\|u^n u^n_x\|_{L^2([0,T]\times\mathbb{S})} \le \|u^n\|_{L^2([0,T]\times\mathbb{S})}\|u^n_x\|_{L^\infty([0,T]\times\mathbb{S})} \le |\mu_0|\mu_1\sqrt{T}, \tag{5.14}$$

$$\begin{aligned}\|\partial_x g * (3\mu_0 e^{-\lambda t}u^n)\|_{L^2([0,T]\times\mathbb{S})} &\le \|\partial_x g\|_{L^2([0,T]\times\mathbb{S})}\|3\mu_0 e^{-\lambda t}u^n\|_{L^1([0,T]\times\mathbb{S})} \\ &\le \frac{T}{12}\cdot 3|\mu_0|\|u^n\|_{L^2([0,T]\times\mathbb{S})} \\ &\le \frac{T}{4}|\mu_0|\mu_1\sqrt{T}.\end{aligned} \tag{5.15}$$

Combining (5.12), (5.14)-(5.15) with (5.4), we know that $\{u^n_t(t,x)\}$ is uniformly bounded in $L^2([0,T] \times \mathbb{S})$. Thus, (5.12), (5.13) and this conclusion implies that

$$\int_0^T \int_{\mathbb{S}} \bigl((u^n)^2 + (u^n_x)^2 + (u^n_t)^2\bigr)\,dx\,dt \le K,$$

where $K = K(|\mu_0|, \mu_1, T, \lambda) \ge 0$. It follows that $\{u^n(t,x)\}$ is uniformly bounded in the space $H^1([0,T] \times \mathbb{S})$. Thus (5.10) holds for some $u \in H^1([0,T] \times \mathbb{S})$.

Observe that, for each $0 \le s, t \le T$,

$$\|u^n(t,\cdot) - u^n(s,\cdot)\|^2_{L^2} = \int_{\mathbb{S}} \left(\int_s^t \frac{\partial u^n}{\partial \tau}(\tau,x)\,d\tau\right)^2 dx \le |t-s| \int_0^T \int_{\mathbb{S}} (u^n_t)^2\,dx\,dt.$$

Note that $\{u^n(t,x)\}$ is uniformly bounded in $L^\infty([0,T];H^1)$, $\{u^n_t(t,x)\}$ is uniformly bounded in $L^2([0,T] \times \mathbb{S})$ and $H^1 \subset\subset C \subset L^\infty \subset L^2$, then (5.11) and $u(t,x) \in C(\mathbb{R}^+ \times \mathbb{S})$ is a consequence of Lemma 5.2. □

Proof of Theorem 5.1 Next, we will deal with u^n_x and $\partial_x g * (3\mu_0 e^{-\lambda t}u^n)$. By (5.5), (5.8)-(5.9), we have that for fixed $t \in [0,T]$ the sequence $u^{n_k}_x(t,\cdot) \in BV(\mathbb{S})$ satisfies

$$\mathbb{V}\bigl(u^{n_k}_x(t,\cdot)\bigr) = \|u^{n_k}_{xx}(t,\cdot)\|_{L^1} = \|\mu(u^{n_k}) - y^{n_k}\|_{L^1} \le \|\mu(u^{n_k})\|_{L^1} + \|y^{n_k}\|_{L^1} \le 2|\mu_0|$$

and

$$\left\|u_x^{n_k}(t,\cdot)\right\|_{L^\infty} \le \left\|u_x^{n_k}(t,x)\right\|_{L^\infty(\mathbb{R}^+\times\mathbb{S})} \le 2|\mu_0|.$$

Applying Lemma 5.3, we obtain that there exists a subsequence, denoted again by $\{u_x^{n_k}(t,\cdot)\}$, which converges at every point to some function $v(t,x)$ of finite variation with $\mathbb{V}(v(t,\cdot)) \le 2|\mu_0|$. Since for almost all $t \in [0,T]$, $u_x^{n_k}(t,\cdot) \to u_x(t,\cdot)$ in $\mathcal{D}'(\mathbb{S})$ in view of Lemma 5.7, it follows that $v(t,\cdot) = u_x(t,\cdot)$ for a.e. $t \in [0,T]$. So we have

$$u_x^{n_k}(t,\cdot) \to u_x(t,\cdot) \quad \text{a.e. on } [0,T]\times\mathbb{S} \text{ as } n_k \to \infty, \tag{5.16}$$

and for a.e. $t \in [0,T]$,

$$\mathbb{V}\big(u_x(t,\cdot)\big) = \left\|u_{xx}(t,\cdot)\right\|_{M(\mathbb{S})} \le 2|\mu_0|. \tag{5.17}$$

Therefore,

$$\begin{aligned}
&\left\|\partial_x g * \left(3\mu_0 e^{-\lambda t} u^{n_k}\right) - \partial_x g * \left(3\mu_0 e^{-\lambda t} u\right)\right\|_{L^\infty([0,T]\times\mathbb{S})} \\
&\quad \le \|\partial_x g\|_{L^1([0,T]\times\mathbb{S})} \left\|3\mu_0 e^{-\lambda t}\left(u^{n_k} - u\right)\right\|_{L^\infty([0,T]\times\mathbb{S})} \\
&\quad \le \frac{T}{4}\cdot 3|\mu_0| \left\|u^{n_k} - u\right\|_{L^\infty([0,T]\times\mathbb{S})}.
\end{aligned}$$

By (5.11), we have

$$\partial_x g * \left(3\mu_0 e^{-\lambda t} u^{n_k}\right) \to \partial_x g * \left(3\mu_0 e^{-\lambda t} u\right) \tag{5.18}$$

a.e. on $[0,T]\times\mathbb{S}$. The relations (5.11), (5.16) and (5.18) imply that u satisfies (2.7) in $\mathcal{D}'([0,T]\times\mathbb{S})$. Moreover, by (5.7)-(5.8), (5.11) and (5.16), we obtain $u \in L^\infty_{\text{loc}}(\mathbb{R}^+, W^{1,\infty})$ in view of T being arbitrary.

Now, we prove that $\mu(u) = \mu_0 e^{-\lambda t}$ and $(\mu(u) - u_{xx}(t,\cdot)) \in M^+$ is uniformly bounded on $\mathbb{S}$.

On the one hand, by (5.11), we have

$$\int_{\mathbb{S}} u^{n_k}(t,x)\,dx \to \int_{\mathbb{S}} u(t,x)\,dx = \mu(u) \quad \text{as } n_k \to \infty.$$

On the other hand,

$$\int_{\mathbb{S}} u^{n_k}(t,x)\,dx = \mu\left(u^{n_k}\right) = \mu_0 e^{-\lambda t}.$$

Obviously, $\mu(u) = \mu(u_0)e^{-\lambda t}$ by the uniqueness of limit.

Note that $L^1 \subset M$. By (5.17) and $\mu(u) = \mu_0 e^{-\lambda t}$, we have

$$\left\|\mu(u) - u_{xx}(t,\cdot)\right\|_M \le \left\|\mu(u)\right\|_{L^1} + \left\|u_{xx}(t,\cdot)\right\|_M \le 3|\mu_0|.$$

It follows that for all $t \in \mathbb{R}^+$, $(\mu(u) - u_{xx}(t,\cdot)) \in M$ is uniformly bounded on $\mathbb{S}$. For any fixed $T > 0$, in view of (5.11) and (5.16), we have for all $t \in [0,T]$,

$$\left[\mu\left(u^{n_k}\right) - u_{xx}^{n_k}(t,\cdot)\right] \to \left[\mu(u) - u_{xx}(t,\cdot)\right] \quad \text{in } \mathcal{D}'(\mathbb{S}) \text{ for } n_k \to \infty.$$

Since $\mu(u^{n_k}) - u_{xx}^{n_k}(t,\cdot) = y^{n_k}(t,\cdot) \geq 0$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{S}$, we have $(\mu(u) - u_{xx}(t,\cdot)) \in L^\infty_{\text{loc}}(\mathbb{R}^+, M^+)$.

Then we prove the uniqueness of global weak solutions.

Let $u, v \in L^\infty_{\text{loc}}(\mathbb{R}^+; W^{1,\infty}) \cap L^\infty(\mathbb{R}^+; H^1)$ be two global weak solutions of (2.7) with the initial data u_0. By $(\mu(u) - u_{xx}(t,\cdot)) \in M^+$ and $(\mu(v) - v_{xx}(t,\cdot)) \in M^+$ are uniformly bounded on $\mathbb{S}$, for fixed $T > 0$, we set

$$K(T) = \sup_{0 \leq t \leq T} \left\{ \left\| \mu(u) - u_{xx}(t,\cdot) \right\|_{M^+} + \left\| \mu(v) - v_{xx}(t,\cdot) \right\|_{M^+} \right\} < +\infty.$$

For all $(t,x) \in [0,T] \times \mathbb{S}$, it follows that

$$\left| u(t,x) \right| = \left| g * \left(\mu(u) - u_{xx}(t,x) \right) \right| \leq \|g\|_{L^\infty} \left\| \mu(u) - u_{xx}(t,x) \right\|_{M^+} \leq \frac{13}{12} K \tag{5.19}$$

and

$$\left| u_x(t,x) \right| = \left| g_x * \left(\mu(u) - u_{xx}(t,x) \right) \right| \leq \|g_x\|_{L^\infty} \left\| \mu(u) - u_{xx}(t,x) \right\|_{M^+} \leq \frac{13}{12} K. \tag{5.20}$$

Similarly, we get

$$\left| v(t,x) \right| \leq \frac{13}{12} K, \qquad \left| v_x(t,x) \right| \leq \frac{13}{12} K. \tag{5.21}$$

We define $w(t,x) = u(t,x) - v(t,x)$, $(t,x) \in [0,T] \times \mathbb{S}$. Since u, v are global weak solutions of (2.7), we have

$$\phi_n * u_t + \phi_n * (u u_x) + \phi_n * \partial_x g * \left(3\mu_0 e^{-\lambda t} u \right) + \lambda \phi_n * u = 0$$

and

$$\phi_n * v_t + \phi_n * (v v_x) + \phi_n * \partial_x g * \left(3\mu_0 e^{-\lambda t} v \right) + \lambda \phi_n * v = 0.$$

It follows that

$$\phi_n * w_t + \phi_n * (u w_x + w v_x) + \phi_n * \partial_x g * \left(3\mu_0 e^{-\lambda t} w \right) + \lambda \phi_n * w = 0.$$

By Lemma 5.4, a direct computation implies

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{S}} |\phi_n * w| \, dx \\
&\quad = \int_{\mathbb{S}} (\phi_n * w_t) \operatorname{sgn}(\phi_n * w) \, dx \\
&\quad = -\int_{\mathbb{S}} \left(\phi_n * (u w_x) \right) \operatorname{sgn}(\phi_n * w) \, dx - \int_{\mathbb{S}} \left(\phi_n * (w v_x) \right) \operatorname{sgn}(\phi_n * w) \, dx \\
&\qquad - 3\mu_0 e^{-\lambda t} \int_{\mathbb{S}} (\phi_n * \partial_x g * w) \operatorname{sgn}(\phi_n * w) \, dx \\
&\qquad - \lambda \int_{\mathbb{S}} (\phi_n * w) \operatorname{sgn}(\phi_n * w) \, dx.
\end{aligned} \tag{5.22}$$

Then we estimate the right-hand side of (5.22) term by term,

$$\begin{aligned}
&\left|\int_{\mathbb{S}}\bigl(\phi_n * (uw_x)\bigr)\operatorname{sgn}(\phi_n * w)\,dx\right| \\
&\quad\le \int_{\mathbb{S}}\bigl|\phi_n * (uw_x)\bigr|\,dx \\
&\quad\le \int_{\mathbb{S}}\bigl|(\phi_n * u)(\phi_n * w_x)\bigr|\,dx + \int_{\mathbb{S}}\bigl|\phi_n * (uw_x) - (\phi_n * u)(\phi_n * w_x)\bigr|\,dx \\
&\quad\le \|\phi_n * u\|_{L^\infty(\mathbb{S})}\int_{\mathbb{S}}|\phi_n * w_x|\,dx + \int_{\mathbb{S}}\bigl|\phi_n * (uw_x) - (\phi_n * u)(\phi_n * w_x)\bigr|\,dx \\
&\quad\le \frac{13}{12}K\int_{\mathbb{S}}|\phi_n * w_x|\,dx + \int_{\mathbb{S}}\bigl|\phi_n * (uw_x) - (\phi_n * u)(\phi_n * w_x)\bigr|\,dx,
\end{aligned}$$

here we used Young's inequality and (5.19),

$$\begin{aligned}
&\left|\int_{\mathbb{S}}\bigl(\phi_n * (wv_x)\bigr)\operatorname{sgn}(\phi_n * w)\,dx\right| \\
&\quad\le \int_{\mathbb{S}}\bigl|\phi_n * (wv_x)\bigr|\,dx \\
&\quad\le \int_{\mathbb{S}}\bigl|(\phi_n * w)(\phi_n * v_x)\bigr|\,dx + \int_{\mathbb{S}}\bigl|\phi_n * (wv_x) - (\phi_n * w)(\phi_n * v_x)\bigr|\,dx \\
&\quad\le \|\phi_n * v_x\|_{L^\infty(\mathbb{S})}\int_{\mathbb{S}}|\phi_n * w|\,dx + \int_{\mathbb{S}}\bigl|\phi_n * (wv_x) - (\phi_n * w)(\phi_n * v_x)\bigr|\,dx \\
&\quad\le \frac{13}{12}K\int_{\mathbb{S}}|\phi_n * w|\,dx + \int_{\mathbb{S}}\bigl|\phi_n * (wv_x) - (\phi_n * w)(\phi_n * v_x)\bigr|\,dx,
\end{aligned}$$

here we used Young's inequality and (5.21),

$$\begin{aligned}
&\left|\int_{\mathbb{S}}(\phi_n * \partial_x g * w)\operatorname{sgn}(\phi_n * w)\,dx\right| \\
&\quad\le \int_{\mathbb{S}}|\phi_n * \partial_x g * w|\,dx \\
&\quad\le \|g_x\|_{L^1(\mathbb{S})}\int_{\mathbb{S}}|\phi_n * w|\,dx \\
&\quad\le \frac{1}{4}\int_{\mathbb{S}}|\phi_n * w|\,dx.
\end{aligned}$$

Combining those three inequalities with (5.22), we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{S}}|\phi_n * w|\,dx \\
&\quad\le \left(\frac{13}{12}K + \frac{3}{4}|\mu_0| + \lambda\right)\int_{\mathbb{S}}|\phi_n * w|\,dx + \frac{13}{12}K\int_{\mathbb{S}}|\phi_n * w_x|\,dx + R_n(t).
\end{aligned}\tag{5.23}$$

By Lemmas 5.5-5.6, we get

$$R_n(t) \to 0 \quad \text{as } n \to \infty \quad \text{and} \quad \bigl|R_n(t)\bigr| \le C, \quad n \ge 1, t \in [0, T], \tag{5.24}$$

where C is a positive constant depending on K and the H^1-norms of $u(0)$ and $v(0)$. In the same way, convoluting (2.7) for u and v with $\phi_{n,x}$ and using Lemma 5.4, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}}|\phi_n * w_x|\,dx = &-\int_{\mathbb{S}}\bigl(\phi_n * \bigl(w_x(u_x+v_x)\bigr)\bigr)\operatorname{sgn}(\phi_{n,x} * w)\,dx \\ &-\int_{\mathbb{S}}\bigl(\phi_n * (uw_{xx})\bigr)\operatorname{sgn}(\phi_{n,x} * w)\,dx \\ &-\int_{\mathbb{S}}\bigl(\phi_n * (wv_{xx})\bigr)\operatorname{sgn}(\phi_{n,x} * w)\,dx \\ &-3\mu_0 e^{-\lambda t}\int_{\mathbb{S}}(\phi_{n,x} * \partial_x g * w)\operatorname{sgn}(\phi_{n,x} * w)\,dx \\ &-\lambda\int_{\mathbb{S}}(\phi_n * w_x)\operatorname{sgn}(\phi_{n,x} * w)\,dx. \qquad (5.25)\end{aligned}$$

Next, we estimate the right-hand side of (5.25) term by term,

$$\begin{aligned}&\left|\int_{\mathbb{S}}\bigl(\phi_n * \bigl(w_x(u_x+v_x)\bigr)\bigr)\operatorname{sgn}(\phi_{n,x} * w)\,dx\right| \\ &\quad\le \int_{\mathbb{S}}\bigl|\phi_n * \bigl(w_x(u_x+v_x)\bigr)\bigr|\,dx \\ &\quad\le \int_{\mathbb{S}}\bigl|(\phi_n * w_x)\bigl(\phi_n * (u_x+v_x)\bigr)\bigr|\,dx \\ &\qquad+\int_{\mathbb{S}}\bigl|\phi_n * \bigl(w_x(u_x+v_x)\bigr)-(\phi_n * w_x)\bigl(\phi_n * (u_x+v_x)\bigr)\bigr|\,dx \\ &\quad\le \bigl\|\phi_n * (u_x+v_x)\bigr\|_{L^\infty(\mathbb{S})}\int_{\mathbb{S}}|\phi_n * w_x|\,dx \\ &\qquad+\int_{\mathbb{S}}\bigl|\phi_n * \bigl(w_x(u_x+v_x)\bigr)-(\phi_n * w_x)\bigl(\phi_n * (u_x+v_x)\bigr)\bigr|\,dx \\ &\quad\le \frac{13}{6}K\int_{\mathbb{S}}|\phi_n * w_x|\,dx+\int_{\mathbb{S}}\bigl|\phi_n * \bigl(w_x(u_x+v_x)\bigr)-(\phi_n * w_x)\bigl(\phi_n * (u_x+v_x)\bigr)\bigr|\,dx,\end{aligned}$$

here we used Young's inequality and (5.20)-(5.21),

$$\begin{aligned}&-\int_{\mathbb{S}}\bigl(\phi_n * (uw_{xx})\bigr)\operatorname{sgn}(\phi_{n,x} * w)\,dx \\ &\quad= -\int_{\mathbb{S}}(\phi_n * u)(\phi_n * w_{xx})\operatorname{sgn}(\phi_{n,x} * w)\,dx \\ &\qquad-\int_{\mathbb{S}}\bigl(\phi_n * (uw_{xx})-(\phi_n * u)(\phi_n * w_{xx})\bigr)\operatorname{sgn}(\phi_{n,x} * w)\,dx \\ &\quad\le -\int_{\mathbb{S}}(\phi_n * u)\frac{d}{dx}|\phi_n * w_x|\,dx+\int_{\mathbb{S}}\bigl|\phi_n * (uw_{xx})-(\phi_n * u)(\phi_n * w_{xx})\bigr|\,dx \\ &\quad= \int_{\mathbb{S}}(\phi_n * u_x)|\phi_n * w_x|\,dx+\int_{\mathbb{S}}\bigl|\phi_n * (uw_{xx})-(\phi_n * u)(\phi_n * w_{xx})\bigr|\,dx \\ &\quad\le \frac{13}{12}K\int_{\mathbb{S}}|\phi_n * w_x|\,dx+\int_{\mathbb{S}}\bigl|\phi_n * (uw_{xx})-(\phi_n * u)(\phi_n * w_{xx})\bigr|\,dx,\end{aligned}$$

here we used Young's inequality and (5.20),

$$\begin{aligned}
&\left|\int_{\mathbb{S}} \big(\phi_n * (wv_{xx})\big) \operatorname{sgn}(\phi_{n,x} * w)\, dx\right| \\
&\quad \le \int_{\mathbb{S}} \big|\phi_n * (wv_{xx})\big|\, dx \\
&\quad \le \int_{\mathbb{S}} \big|(\phi_n * w)(\phi_n * v_{xx})\big|\, dx + \int_{\mathbb{S}} \big|\phi_n * (wv_{xx}) - (\phi_n * w)(\phi_n * v_{xx})\big|\, dx \\
&\quad \le \|\phi_n * w\|_{L^\infty(\mathbb{S})} \|\phi_n * v_{xx}\|_{L^1(\mathbb{S})} + \int_{\mathbb{S}} \big|\phi_n * (wv_{xx}) - (\phi_n * w)(\phi_n * v_{xx})\big|\, dx.
\end{aligned}$$

Note that $W^{1,1} \hookrightarrow L^\infty$, $\|f\|_{W^{1,1}} = \|f\|_{L^1} + \|f'\|_{L^1}$ and $\|\phi_n * v_{xx}\|_{L^1} \le \|v_{xx}\|_M$. It follows from (5.17) that

$$\begin{aligned}
&\left|\int_{\mathbb{S}} \big(\phi_n * (wv_{xx})\big) \operatorname{sgn}(\phi_{n,x} * w)\, dx\right| \\
&\quad \le 2|\mu_0| \int_{\mathbb{S}} |\phi_n * w|\, dx + 2|\mu_0| \int_{\mathbb{S}} |\phi_n * w_x|\, dx \\
&\qquad + \int_{\mathbb{S}} \big|\phi_n * (wv_{xx}) - (\phi_n * w)(\phi_n * v_{xx})\big|\, dx.
\end{aligned}$$

Now, we estimate the fourth term,

$$\begin{aligned}
&\left|\int_{\mathbb{S}} (\phi_{n,x} * \partial_x g * w) \operatorname{sgn}(\phi_{n,x} * w)\, dx\right| \\
&\quad \le \int_{\mathbb{S}} |\phi_{n,x} * \partial_x g * w|\, dx = \int_{\mathbb{S}} |\phi_n * \partial_x g * w_x|\, dx \\
&\quad \le \|g_x\|_{L^1(\mathbb{S})} \int_{\mathbb{S}} |\phi_n * w_x|\, dx \le \frac{1}{4} \int_{\mathbb{S}} |\phi_n * w_x|\, dx.
\end{aligned}$$

Combining the estimates with (5.25), we get

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{S}} |\phi_n * w_x|\, dx \\
&\quad \le 2|\mu_0| \int_{\mathbb{S}} |\phi_n * w|\, dx + \left(\frac{39}{12}K + \frac{11}{4}|\mu_0| + \lambda\right) \int_{\mathbb{S}} |\phi_n * w_x|\, dx + R_n(t),
\end{aligned} \tag{5.26}$$

where $R_n(t)$ satisfies (5.24).

Adding (5.23) and (5.26), we have

$$\frac{d}{dt} \int_{\mathbb{S}} \big(|\phi_n * w| + |\phi_n * w_x|\big)\, dx \le \left(\frac{13}{3}K + \frac{11}{4}|\mu_0| + \lambda\right) \int_{\mathbb{S}} \big(|\phi_n * w| + |\phi_n * w_x|\big)\, dx + R_n(t).$$

In view of Gronwall's inequality, we find

$$\begin{aligned}
&\int_{\mathbb{S}} \big(|\phi_n * w| + |\phi_n * w_x|\big)\, dx \\
&\quad \le e^{(\frac{13}{3}K + \frac{11}{4}|\mu_0| + \lambda)t} \left[\int_{\mathbb{S}} \big(|\phi_n * w| + |\phi_n * w_x|\big)(0,x)\, dx + \int_0^t R_n(s)\, ds\right].
\end{aligned}$$

Note that $w = u - v \in W^{1,1}$ and (5.24) holds. Letting $n \to \infty$ in the above inequality, we have

$$\int_{\mathbb{S}} \big(|\phi_n * w| + |\phi_n * w_x|\big)\,dx \le e^{(\frac{13}{3}K + \frac{11}{4}|\mu_0| + \lambda)t} \int_{\mathbb{S}} \big(|\phi_n * w| + |\phi_n * w_x|\big)(0,x)\,dx.$$

Since $w(0,x) = w_x(0,x) = 0$, we obtain $u(t,x) = v(t,x)$ for a.e. $(t,x) \in [0,T] \times \mathbb{S}$. In view of T is chosen arbitrarily, this completes the proof of uniqueness. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have equal contributions to each part of this article. All the authors read and approved the final manuscript.

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