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A fixed point theorem for smooth extension maps

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Abstract

Let X be a compact smooth n-manifold, with or without boundary, and let A be an $(n-1)$-dimensional smooth submanifold of the interior of X. Let $\phi: A \to A$ be a smooth map and $f: (X, A) \to (X, A)$ be a smooth map whose restriction to A is ϕ. If $p \in A$ is an isolated fixed point of f that is a transversal fixed point of ϕ, that is, the linear transformation $d\phi_p - I_A: T_pA \to T_pA$ is nonsingular, then the fixed point index of f at p satisfies the inequality $|i(X, f, p)| \leq 1$. It follows that if ϕ has k fixed points, all transverse, and the Lefschetz number $L(f) > k$, then there is at least one fixed point of f in $X \setminus A$. Examples demonstrate that these results do not hold if the maps are not smooth.

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1 Introduction

It has been known at least since the work of Shub and Sullivan in 1974 [1] that the values of the fixed point index of smooth maps are more restricted than they are for continuous functions in general. In [2] it is proved that, given integers r and s, there is a map $f: X \to X$ of a manifold with boundary ∂X that restricts to $f|_{\partial X} = \phi: \partial X \to \partial X$ and an isolated fixed point p of f such that the fixed point indices are $i(\partial X, \phi, p) = r$ and $i(X, f, p) = s$. On the other hand, it is proved in that paper that if $f: (X, \partial X) \to (X, \partial X)$ is smooth and p is a transverse fixed point of ϕ, then either $i(X, f, p) = 0$ or $i(X, f, p) = i(\partial X, \phi, p)$. A consequence of this result is that, under appropriate hypotheses on a smooth map f, it must have fixed points on $X \setminus \partial X$, the interior of the manifold X. Those same hypotheses are shown to be insufficient to imply the existence of such interior fixed points if the map f is not smooth. (See also [3, 4].)

We will consider a somewhat different setting, as follows. Let X be a compact smooth n-manifold, with or without boundary, and let A be a smooth $(n-1)$-dimensional submanifold of the interior of X. As in [2], we shall consider $f: (X, A) \to (X, A)$ to be an *extension* of its restriction $f|_A = \phi: A \to A$. Suppose that p is a transverse fixed point of ϕ, then a simple example will show that the relationship between $i(X, f, p)$ and $i(A, \phi, p)$ cannot be as close as it is when $A = \partial X$. However, we will prove that there is still a very strong restriction on the value of $i(X, f, p)$, namely, that $|i(f, X, p)| \leq 1$. As a consequence, we obtain a condition on the Lefschetz number $L(f)$ of f that implies the existence of fixed points of

f in $X \setminus A$. We demonstrate by an example that the same Lefschetz number condition is not sufficient to imply the existence of fixed points in $X \setminus A$ for maps $f : (X, A) \to (X, A)$ in general.

2 The index of fixed points of smooth extension maps

The index theorem of [2] is the following.

Theorem 1 *Let X be a compact smooth n-manifold with boundary ∂X. Given a smooth map $\phi : \partial X \to \partial X$ and a smooth map $f : (X, \partial X) \to (X, \partial X)$ extending ϕ, suppose that $p \in \partial X$ is an isolated fixed point of f and that $d\phi_p - I_{\partial X} : T_p(\partial X) \to T_p(\partial X)$ is a nonsingular linear transformation. Then either $i(X, f, p) = 0$ or $i(X, f, p) = i(\partial X, \phi, p)$.*

We will modify the proof of Theorem 1 that was given in [2] in order to obtain a result of this type in the setting of smooth extension maps on a pair consisting of an n-dimensional compact smooth manifold X and an $(n-1)$-dimensional smooth submanifold A of the interior of X. Although our index result is similar to Theorem 1, the following example demonstrates that we cannot expect that there will be as close a relationship between the indices of ϕ and of f as there is in Theorem 1. Let $f : (S^2, S^1) \to (S^2, S^1)$ where S^2 is viewed as the complex plane $\mathbb{C}$ compactified at infinity, S^1 is the unit circle, $f(z) = z^2$ for $z \in \mathbb{C}$ and $f(\infty) = \infty$. Let $\phi : S^1 \to S^1$ be the restriction of f. Then $i(S^1, \phi, 1) = -1$, whereas $i(S^2, f, 1) = 1$.

Theorem 2 *Let X be a smooth n-manifold, with or without boundary, and let A be an $(n-1)$-dimensional smooth submanifold of the interior of X. Let $\phi : A \to A$ be a smooth map with smooth extension $f : (X, A) \to (X, A)$. Suppose that $p \in A$ is an isolated fixed point of f and that $d\phi_p - I_A : T_p A \to T_p A$ is a nonsingular linear transformation. Then $|i(X, f, p)| \le 1$.*

Proof If the linear transformation $df_p - I : T_p X \to T_p X$ is nonsingular, then $i(X, f, p) = \pm 1$, see [5]. Therefore, we assume that the linear transformation $df_p - I : T_p X \to T_p X$ is singular. Note that determining the index of a map f at a fixed point p is a local problem, so we may choose a local coordinate system about p in which the smooth manifold X is identified with $\mathbb{R}^n$ such that p is the origin in $\mathbb{R}^n$ and the smooth submanifold A is identified with the subspace $\mathbb{R}^{n-1}$.

Let us define $G : \mathbb{R}^n \to \mathbb{R}^n$ by $G(\mathbf{x}) = f(\mathbf{x}) - \mathbf{x}$. To calculate the index of f at p, we need to determine the degree of $-G$ restricted to a sphere around $\mathbf{0}$. More specifically, for $\epsilon > 0$ sufficiently small, we may consider the map $E_\epsilon : S^{n-1} \to \mathbb{R}^n - \{\mathbf{0}\}$ defined by

$$E_\epsilon(\mathbf{x}) = -G(\epsilon \mathbf{x}).$$

Since G is a C^1 function with value $\mathbf{0}$ at the point p, we know from the definition of the derivative that

$$\left| G(\epsilon \mathbf{x}) - dG_p(\epsilon \mathbf{x}) \right| \le o(\epsilon)$$

uniformly over $\{\mathbf{x} \in A| : |\mathbf{x}| = 1\}$. Our goal is to define maps P_ϵ^σ related to E_ϵ such that P_ϵ^σ takes each of the upper and lower half-plane into either the upper or the lower half-plane

and that will allow us to calculate the index $i(X, f, p)$. However, we need to make use of a change of coordinate. Since $df_p - I = dG_p$ is singular but $dG_p|_A$ is not, we know that $dG_p(0, \ldots, 0, 1) \in \text{span}\{x_1, \ldots, x_{n-1}\}$. Now rotate the $x_1, \ldots, x_{n-1}$ coordinate system so that x_{n-1} points in the direction of $dG_p(0, \ldots, 0, 1)$. Choose new coordinates $y_1, \ldots, y_n$ so that the hyperplane $y_n = 0$ is the same as $x_n = 0$ and, for each $j = 1, \ldots, n-1$, the transformation dG_p takes the x_j unit vector to the y_i unit vector. Now we have

$$\{(x_1, \ldots, x_{n-1}, 0) \in \mathbb{R}^n\} = \{(y_1, \ldots, y_{n-1}, 0) \in \mathbb{R}^n\},$$
$$\{(x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n\} = \{(y_1, \ldots, y_{n-1}, y_n) \in \mathbb{R}^n\}$$

and the positive y_n axis is in the same half-space as the positive x_n axis. With this change of coordinates, dG_p, as a map from x coordinates into y coordinates, has the following form:

$$dG_p(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) = (\alpha_1, \ldots, \alpha_{n-1}, 0) + \alpha_n(0, \ldots, 0, B, 0)$$

for some constant B. Define

$$S_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\} \cap S^{n-1}, \qquad S_- = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \leq 0\} \cap S^{n-1}.$$

It is easy to check the following:

  (i) The point

$$\mathbf{x}^+ = \left(0, \ldots, 0, -\frac{B}{(1 + B^2)^{\frac{1}{2}}}, \frac{1}{(1 + B^2)^{\frac{1}{2}}}\right)$$

is the unique point of S_+ with the property that $dG_p(\mathbf{x}^+) = \mathbf{0}$. From the inverse function theorem, there is a small neighborhood U^+ of the point $\mathbf{x}^+$ such that $dG_p|_{S_+}$ is a nonsingular diffeomorphism of U^+ onto a neighborhood of $\mathbf{0}$ in the hyperplane $y_n = 0$ and $(dG_p|_{S_+})^{-1}(dG_p(U^+)) = U^+$.

 (ii) The point

$$\mathbf{x}^- = \left(0, \ldots, 0, -\frac{B}{(1 + B^2)^{\frac{1}{2}}}, -\frac{1}{(1 + B^2)^{\frac{1}{2}}}\right)$$

is the unique point of S_- with the property that $dG_p(\mathbf{x}^-) = \mathbf{0}$. Similarly, from the inverse function theorem, there is a small neighborhood U^- of the point $\mathbf{x}^-$ such that $dG_p|_{S_-}$ is a nonsingular diffeomorphism of U^- onto a neighborhood of $\mathbf{0}$ in the hyperplane $y_n = 0$ and $(dG_p|_{S_-})^{-1}(dG_p(U^-)) = U^-$.

Now consider the maps P_ϵ defined by

$$P_\epsilon(\mathbf{x}) = \epsilon^{-1} E_\epsilon(\mathbf{x}).$$

The map $P_\epsilon|_{S^{n-1}}$ converges in the C^1 topology to $-dG_p|_{S^{n-1}}$ because

$$d(P_\epsilon(\mathbf{x})) = d(\epsilon^{-1} E_\epsilon(\mathbf{x}))$$
$$= d(\epsilon^{-1}(-G(\epsilon \mathbf{x})))$$

$$= -\epsilon^{-1} d\big(G(\epsilon \mathbf{x})\big)$$

$$= -\epsilon^{-1} dG_p(\epsilon \mathbf{x}) \epsilon$$

$$= -dG_p(\epsilon \mathbf{x}).$$

Consequently, we have the following analogues of (i)-(ii) above.

(1) There is a unique point $\mathbf{x}_\epsilon^+ \in S_+$ such that the $y_1, \ldots, y_{n-1}$ coordinates of $P_\epsilon(\mathbf{x}_\epsilon^+)$ are all 0. Also, since p is an isolated fixed point of f, $P_\epsilon(\mathbf{x}_\epsilon^+) \neq \mathbf{0}$. In particular, the y_n coordinate of $P_\epsilon(\mathbf{x}_\epsilon^+)$ is nonzero. Hence we have

$$P_\epsilon(\mathbf{x}_\epsilon^+) = (0, \ldots, 0, y_n^{\epsilon^+}),$$

where $y_n^{\epsilon^+} \neq 0$.

(2) There is a unique point $\mathbf{x}_\epsilon^- \in S_-$ such that the $y_1, \ldots, y_{n-1}$ coordinates of $P_\epsilon(\mathbf{x}_\epsilon^-)$ are all 0. Also, since p is an isolated fixed point of f, $P_\epsilon(\mathbf{x}_\epsilon^-) \neq \mathbf{0}$. In particular, the y_n coordinate of $P_\epsilon(\mathbf{x}_\epsilon^-)$ is nonzero. Hence we have

$$P_\epsilon(\mathbf{x}_\epsilon^-) = (0, \ldots, 0, y_n^{\epsilon^-}),$$

where $y_n^{\epsilon^-} \neq 0$.

If $P_\epsilon(\mathbf{x}) = (y_1, y_2, \ldots, y_n)$, define a map $P_\epsilon^\sigma : S^{n-1} \to \mathbb{R}^n - \{\mathbf{0}\}$ by

$$P_\epsilon^\sigma = \begin{cases} (y_1, \ldots, y_{n-1}, \sigma^+|y_n|) & \text{if } \mathbf{x} \in S_+, \\ (y_1, \ldots, y_{n-1}, \sigma^-|y_n|) & \text{if } \mathbf{x} \in S_- \backslash \mathbb{R}^{n-1}, \end{cases}$$

where $\sigma^+ = y_n^{\epsilon^+}/|y_n^{\epsilon^+}| = \pm 1$ and $\sigma^- = y_n^{\epsilon^-}/|y_n^{\epsilon^-}| = \pm 1$.

Notice that the values of $P_\epsilon^\sigma |_{S_+}$ lie entirely in the half-space where $P_\epsilon(\mathbf{x}_\epsilon^+)$ is located with respect to the $y_1, \ldots, y_n$ coordinates and the values of $P_\epsilon^\sigma |_{S_-}$ lie entirely in the half-space where $P_\epsilon(\mathbf{x}_\epsilon^-)$ is located with respect to the $y_1, \ldots, y_n$ coordinates. Furthermore, P_ϵ and P_ϵ^σ are homotopic as maps into $\mathbb{R}^n - \{\mathbf{0}\}$ by the following homotopy:

$$H(\mathbf{x}, t) = t P_\epsilon(\mathbf{x}) + (1-t) P_\epsilon^\sigma(\mathbf{x}).$$

For $\mathbf{x} \in S_+$, either $(y_1, \ldots, y_{n-1}) \neq \mathbf{0}$ or $\mathbf{x} = \mathbf{x}_\epsilon^+$. If $(y_1, \ldots, y_{n-1}) \neq \mathbf{0}$, then $H(\mathbf{x}, t) \neq \mathbf{0}$ by definition. If $\mathbf{x} = \mathbf{x}_\epsilon^+$, then $H(\mathbf{x}, t) = P_\epsilon(\mathbf{x})$ which we know is never $\mathbf{0}$. A similar argument shows that $H(\mathbf{x}, t) \neq \mathbf{0}$ for $\mathbf{x} \in S_- \backslash \mathbb{R}^{n-1}$.

Since the map P_ϵ^σ takes S_+ into either the upper or lower half-planes with respect to the y coordinates and takes S_- into either the upper or lower half-planes with respect to the $y_1, \ldots, y_n$ coordinates, the map P_ϵ^σ is homotopic either to a constant map, the suspension of $P_\epsilon^\sigma |_{S^{n-2}} = P_\epsilon|_{S^{n-2}}$ or the suspension of $P_\epsilon^\sigma |_{S^{n-2}} = P_\epsilon|_{S^{n-2}}$ followed by a reflection about the hyperplane $y_n = 0$. This homotopy tells us that either P_ϵ has degree 0 or $\deg(P_\epsilon|_{S^{n-2}})$ or $-\deg(P_\epsilon|_{S^{n-2}})$.

Although P_ϵ is a map from x to y coordinates, we know that the x and y coordinates are related by a linear map, call it L, satisfying $L(x_i) = y_i$, for $i = 1, \ldots, n$. Thus, the map $L^{-1} \circ P_\epsilon$

takes $x_1, \ldots, x_n$ coordinates to $x_1, \ldots, x_n$ coordinates and

$$\deg(L^{-1} \circ P_\epsilon) = \deg(L^{-1}) \cdot \deg(P_\epsilon) = \deg(P_\epsilon).$$

Thus the index $i(X, f, p) = \deg(P_\epsilon)$ which is either 0 or ± 1. $\qquad \square$

3 Fixed point theorem for smooth extension maps

We can now use Theorem 2 to establish the existence of fixed points in $X \setminus A$.

Theorem 3 *Suppose that A is an $(n-1)$-dimensional smooth submanifold of the interior of a compact smooth n-manifold X. Given a smooth map $\phi : A \to A$ and a smooth map $f : (X, A) \to (X, A)$ extending ϕ, suppose that the fixed points of ϕ are $\{x_1, x_2, \ldots, x_k\}$, all of which are transversal, that is, the linear map $d\phi_{x_j} - I_A$ is nonsingular for each x_j. If the Lefschetz number $L(f) > k$, then there must be at least one fixed point in $X \setminus A$.*

Proof Suppose that f only has fixed points in A. This means the fixed point set of f is $\{x_1, \ldots, x_k\}$, the set of fixed points of ϕ. Then, by the Lefschetz-Hopf theorem [6], the Lefschetz number of f is

$$L(f) = \sum_{j=0}^{k} i(X, f, x_j) \leq k$$

because $i(X, f, x_j) \leq 1$ by Theorem 2. This is contrary to the assumption that $L(f) > k$, so f has fixed points in $X \setminus A$. $\qquad \square$

A consequence of this theorem is the following.

Corollary 4 *Let S^2 be the complex plane $\mathbb{C}$ compactified at infinity and S^1 be the unit circle. Suppose that $\phi : S^1 \to S^1$ is a smooth map defined by $\phi(\zeta) = \zeta^k$ for some $k \geq 2$ and that $f : (S^2, S^1) \to (S^2, S^1)$ is a smooth extension of ϕ. If f is homotopic to the suspension of ϕ, then there is at least one fixed point in $S^2 \setminus S^1$.*

Proof Since f is homotopic to the suspension of ϕ, then f is of degree k and thus $L(f) = 1 + k$. However, ϕ has only $k - 1$ fixed points. Note here that the $k - 1$ fixed points of ϕ on S^1 are all transversal so that the hypotheses of Theorem 3 hold. $\qquad \square$

The following example illustrates the fact that the corollary, and therefore Theorem 3, require the hypothesis that the map f is smooth, by exhibiting a non-smooth map $f : (S^2, S^1) \to (S^2, S^1)$ homotopic to the suspension of ϕ that has no fixed points on $S^2 \setminus S^1$.

Example 1 Let $S^2 = \mathbb{C} \cup \{\infty\}$ be the complex plane $\mathbb{C}$ compactified at infinity and S^1 be the unit circle. Let $\phi : S^1 \to S^1$ be defined by $\phi(\zeta) = \zeta^2$. Let $B_+^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ and $B_-^2 = \{z \in \mathbb{C} : |z| \geq 1\} \cup \{\infty\}$, so $S^2 = B_+^2 \cup B_-^2$ and $B_+^2 \cap B_-^2 = S^1$. For $z \in B_-^2 \setminus \{1\}$, there exist $\zeta \in S^1$ and $t \in [0, 1)$ such that $z = t1 + (1 - t)\zeta$. As in Lemma 6.1 of [7], define $f : B_+^2 \to B_+^2$ by setting $f(z) = t1 + (1 - t)\phi(\zeta)$ for $z \neq 1$ and $f(1) = 1$, then f has no fixed points in $B_+^2 \setminus S^1$. In order to extend f to a selfmap of B_-^2 so that there are no fixed points on $B_-^2 \setminus S^1$, we define $\rho : S^2 \to S^2$ by $\rho(z) = \rho(re^{i\theta}) = \frac{1}{r}e^{i\theta}$, $\rho(0) = \infty$ and $\rho(\infty) = 0$, so $\rho(B_-^2) = B_+^2$ and $\rho(B_+^2) = B_-^2$.

Then, for $z \in B^2_-$, let $f(z) = \rho f \rho(z)$. The only fixed point of f is $z = 1$ and thus there are no fixed points in $S^2 \setminus S^1$.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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