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Research Article

Blow-Up Solutions and Global Solutions to Discrete p-Laplacian Parabolic Equations

Soon-Yeong Chung[1] and Min-Jun Choi[2]

[1] *Department of Mathematics and Program of Integrated Biotechnology, Sogang University, Seoul 121-742, Republic of Korea*
[2] *Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea*

Correspondence should be addressed to Soon-Yeong Chung; sychung@sogang.ac.kr

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We discuss the conditions under which blow-up occurs for the solutions of discrete p-Laplacian parabolic equations on networks S with boundary ∂S as follows: $u_t(x,t) = \Delta_{p,\omega}u(x,t) + \lambda|u(x,t)|^{q-1}u(x,t)$, $(x,t) \in S \times (0,+\infty)$; $u(x,t) = 0$, $(x,t) \in \partial S \times (0,+\infty)$; $u(x,0) = u_0 \geq 0$, $x \in \overline{S}$, where $p > 1$, $q > 0$, $\lambda > 0$, and the initial data u_0 is nontrivial on S. The main theorem states that the solution u to the above equation satisfies the following: (i) if $0 < p-1 < q$ and $q > 1$, then the solution blows up in a finite time, provided $\overline{u}_0 > (\omega_0/\lambda)^{1/(q-p+1)}$, where $\omega_0 := \max_{x\in S}\sum_{y\in\overline{S}}\omega(x,y)$ and $\overline{u}_0 := \max_{x\in S} u_0(x)$; (ii) if $0 < q \leq 1$, then the nonnegative solution is global; (iii) if $1 < q < p-1$, then the solution is global. In order to prove the main theorem, we first derive the comparison principles for the solution of the equation above, which play an important role throughout this paper. Moreover, when the solution blows up, we give an estimate for the blow-up time and also provide the blow-up rate. Finally, we give some numerical illustrations which exploit the main results.

1. Introduction

In this paper, we discuss the blow-up property and global existence of solutions to the following discrete p-Laplacian parabolic equation:

$$\begin{aligned} u_t(x,t) &= \Delta_{p,w}u(x,t) + \lambda|u(x,t)|^{q-1}u(x,t), \\ &\qquad (x,t) \in S \times (0,+\infty), \\ u(x,t) &= 0, \quad (x,t) \in \partial S \times (0,+\infty), \\ u(x,0) &= u_0 \geq 0, \quad x \in \overline{S}, \end{aligned} \tag{1}$$

where $q > 0$, $p > 1$ and $\lambda > 0$.

The continuous case of this equation has been studied by many authors, assuming some conditions q, p, and λ, in order to get a blow-up solution or global solution (see [1–5]). For example, they consider the case $1 < p < 2$, $q > 0$, and $\lambda > 0$ in [2], the case $q > p-1 > 1$ in [5], the case $p = q > 2$ in [1, 3], the case $1 < p < 2$, $q > 0$, and $\lambda > 0$ in [4], respectively, and so on.

On the other hand, the long time behavior (extinction and positivity) of solutions to evolution p-Laplace equation with absorption on networks is studied in the paper [6, 7].

The goal of this paper is to give a condition on p, q, and λ for the solution to (1) to be blow-up or global. In fact, we prove the following as one of the main theorems.

Theorem 1. *Let u be a solution of (1). Then one has the following.*

(i) *If $0 < p-1 < q$ and $q > 1$, then the solution blows up in a finite time, provided $\overline{u}_0 > (\omega_0/\lambda)^{1/(q-p+1)}$, where $\omega_0 := \max_{x\in S}\sum_{y\in\overline{S}}\omega(x,y)$ and $\overline{u}_0 := \max_{x\in S}u_0(x)$.*

(ii) *If $0 < q \leq 1$, then the nonnegative solution is global.*

(iii) *If $1 < q < p-1$, then the solution is global.*

In order to prove the above theorem, we give comparison principles for the solutions of (1) in Section 2. Moreover, when the solutions to (1) blow up, we derive the blow-up rate as follows:

$$\begin{aligned}&[\lambda(q-1)(T-t)]^{-1/(q-1)}\\&\quad\le\max_{x\in S}u(x,t)\\&\quad\le\Big[\lambda(q-1)(T-t)-\omega_0\lambda^{(q-p+1)/(q-1)}\\&\qquad\times[(q-1)(T-t)]^{(2q-p)/(q-1)}\Big]^{-1/(q-1)},\end{aligned}\tag{2}$$

where $\omega_0 := \max_{x\in S}\sum_{y\in\overline{S}}\omega(x,y)$, and as a consequence

$$\lim_{t\to T^-}(T-t)^{1/(q-1)}\max_{x\in\overline{S}}u(x,t)=\left[\frac{1}{\lambda(q-1)}\right]^{1/(q-1)}.\tag{3}$$

We organized this paper as follows. In Section 2, we discuss the preliminary concepts on networks and the discrete version of comparison principles on networks. In Section 3, we are devoted to find out blow-up conditions of the solution and the blow-up rate with the blow-up time. Finally, in Section 4, we give some numerical illustrations to exploit the main results.

2. Preliminaries and Discrete Comparison Principles

In this section, we start with some definitions of graph theoretic notions frequently used throughout this paper (see [8–10], for more details).

For a graph $G = G(V, E)$, we mean finite sets V of *vertices* (or *nodes*) with a set E of two-element subsets of V (whose elements are called *edges*). The set of vertices and edges of a graph G are sometimes denoted by $V(G)$ and $E(G)$, or simply V and E, respectively. Conventionally, we denote by $x \in V$ or $x \in G$ the facts that x is a vertex in G.

A graph G is said to be *simple* if it has neither multiple edges nor loops, and G is said to be *connected* if, for every pair of vertices x and y, there exists a sequence (called a *path*) of vertices $x = x_0, x_1, \dots, x_{n-1}, x_n = y$, such that x_{j-1} and x_j are connected by an edge (called *adjacent*) for $j = 1, \dots, n$.

A graph $S = S(V', E')$ is said to be a *subgraph* of $G(V, E)$, if $V' \subset V$ and $E' \subset E$.

A *weight* on a graph G is a function $\omega : V\times V \to [0, +\infty)$ satisfying

(i) $\omega(x, x) = 0$, $x \in V$,

(ii) $\omega(x, y) = \omega(y, x)$ if $x \sim y$,

(iii) $\omega(x, y) = 0$ if and only if $x \nsim y$.

Here $x \sim y$ means that two vertices x and y are connected (adjacent) by an edge in E. A graph associated with a weight is said to be a *weight graph* or a *network*.

For a subgraph S of a graph $G(V, E)$, the (vertex) *boundary* ∂S of S is the set of all vertices $z \in V\setminus S$ but is adjacent to some vertex in S; that is,

$$\partial S := \{z\in V\setminus S \mid z\sim y \text{ for some } y\in S\}.\tag{4}$$

By $\overline{S}$, we denote a graph, whose vertices and edges are in both S and ∂S.

Throughout this paper, all subgraphs S and $\overline{S}$ in our concern are assumed to be simple and connected.

For a function $u : \overline{S} \to \mathbb{R}$, the *discrete p-Laplacian* $\Delta_{p,\omega}$ on S is defined by

$$\Delta_{p,\omega}u(x):=\sum_{y\in\overline{S}}|u(y)-u(x)|^{p-2}[u(y)-u(x)]\,\omega(x,y)\tag{5}$$

for $x \in S$.

The rest of this section is devoted to prove the comparison principle for the discrete p-Laplacian parabolic equation:

$$\begin{aligned}&u_t(x,t)=\Delta_{p,\omega}u(x,t)+\lambda|u(x,t)|^{q-1}u(x,t),\\&\qquad\qquad (x,t)\in S\times(0,+\infty),\\&u(x,t)=0,\quad (x,t)\in\partial S\times(0,+\infty),\\&u(x,0)=u_0\ge 0,\quad x\in\overline{S},\end{aligned}\tag{6}$$

where $\lambda > 0$, $q > 0$, $p > 1$, and the initial data u_0 is nontrivial on S, in order to study the blow-up occurrence and global existence which we begin in the next section.

Now, we state the comparison principles and some related corollaries.

Theorem 2. *Let $T > 0$ (T may be $+\infty$), $\lambda > 0$, $q \ge 1$, and $p > 1$. Suppose that real-valued functions $u(x,\cdot), v(x,\cdot) \in C[0,T)$ are differentiable in $(0,T)$ for each $x \in \overline{S}$ and satisfy*

$$\begin{aligned}&u_t(x,t)-\Delta_{p,\omega}u(x,t)-\lambda|u(x,t)|^{q-1}u(x,t)\\&\quad\ge v_t(x,t)-\Delta_{p,\omega}v(x,t)-\lambda|v(x,t)|^{q-1}v(x,t),\\&\qquad\qquad (x,t)\in S\times(0,T),\\&u(x,t)\ge v(x,t),\quad (x,t)\in\partial S\times[0,T),\\&u(x,0)\ge v(x,0),\quad x\in\overline{S}.\end{aligned}\tag{7}$$

Then $u(x,t) \ge v(x,t)$ for all $(x,t) \in \overline{S}\times[0,T)$.

Proof. Let $T' > 0$ be arbitrarily given with $T' < T$. Then, by the mean value theorem, for each $x \in S$ and $0 \le t \le T'$,

$$\begin{aligned}&|u(x,t)|^{q-1}u(x,t)-|v(x,t)|^{q-1}v(x,t)\\&\quad=q|\xi(x,t)|^{q-1}[u(x,t)-v(x,t)]\end{aligned}\tag{8}$$

for some $\xi(x,t)$ lying between $u(x,t)$ and $v(x,t)$. Then it follows from (7) that we have

$$\begin{aligned}&u_t(x,t)-\Delta_{p,\omega}u(x,t)-\lambda q|\xi(x,t)|^{q-1}u(x,t)\\&\quad\ge v_t(x,t)-\Delta_{p,\omega}v(x,t)-\lambda q|\xi(x,t)|^{q-1}v(x,t)\end{aligned}\tag{9}$$

for all $(x,t) \in S\times(0,T']$. Let $\tilde{u}, \tilde{v} : \overline{S}\times[0,T'] \to \mathbb{R}$ be the functions defined by

$$\tilde{u}(x,t):=e^{-2\lambda qLt}u(x,t),\qquad \tilde{v}(x,t):=e^{-2\lambda qLt}v(x,t),\tag{10}$$

where $L := \max_{|r|\le M}|r^{q-1}|$ and $M := \max_{x\in\overline{S},t\in[0,T']}\{|u(x,t)|, |v(x,t)|\}$.

Then inequality (9) can be written as

$$\begin{aligned}&\tilde{u}_t(x,t)-\tilde{v}_t(x,t)-e^{2q\lambda L(p-2)t}\left[\Delta_{p,\omega}\tilde{u}(x,t)-\Delta_{p,\omega}\tilde{v}(x,t)\right]\\&\quad+\lambda q\left[2L-\left|\xi(x,t)\right|^{q-1}\right][\tilde{u}(x,t)-\tilde{v}(x,t)]\ge 0\end{aligned}\tag{11}$$

for all $(x,t)\in S\times(0,T']$. Since $\overline{S}\times[0,T']$ is compact, there exists $(x_0,t_0)\in\overline{S}\times[0,T']$ such that

$$(\tilde{u}-\tilde{v})(x_0,t_0)=\min_{x\in\overline{S}}\min_{0\le t\le T'}(\tilde{u}-\tilde{v})(x,t).\tag{12}$$

Then we have only to show that $(\tilde{u}-\tilde{v})(x_0,t_0)\ge 0$. Suppose that $(\tilde{u}-\tilde{v})(x_0,t_0)<0$, on the contrary. Since $(\tilde{u}-\tilde{v})(x,t)\ge 0$ on both $\partial S\times[0,T']$ and $\overline{S}\times\{0\}$, we have $(x_0,t_0)\in S\times(0,T']$. Then we have

$$\tilde{u}_t(x_0,t_0)\le\tilde{v}_t(x_0,t_0),\qquad \Delta_{p,\omega}\tilde{u}(x_0,t_0)\ge\Delta_{p,\omega}\tilde{v}(x_0,t_0).\tag{13}$$

Since $|\xi^{q-1}(x,t)|\le\max_{|r|\le M}|r^{q-1}|=L$, we have

$$\left[2L-\left|\xi(x_0,t_0)\right|^{q-1}\right](\tilde{u}-\tilde{v})(x_0,t_0)\le L(\tilde{u}-\tilde{v})(x_0,t_0)<0.\tag{14}$$

Combining (13) and (14), we obtain

$$\begin{aligned}&\tilde{u}_t(x_0,t_0)-\tilde{v}_t(x_0,t_0)\\&\quad-e^{2q\lambda L(p-2)t_0}\left[\Delta_{p,\omega}\tilde{u}(x_0,t_0)-\Delta_{p,\omega}\tilde{v}(x_0,t_0)\right]\\&\quad+\lambda q\left[2L-\left|\xi(x_0,t_0)\right|^{q-1}\right][\tilde{u}(x_0,t_0)-\tilde{v}(x_0,t_0)]<0\end{aligned}\tag{15}$$

which contradicts (11). Therefore, $\tilde{u}(x,t)-\tilde{v}(x,t)\ge 0$ for all $(x,t)\in S\times(0,T']$ so that we get $u(x,t)\ge v(x,t)$ for all $(x,t)\in\overline{S}\times[0,T)$, since $T'<T$ is arbitrarily given. □

When $p\ge 2$, we obtain a strict comparison principle as follows.

Corollary 3 (strict comparison principle). *Let $T>0$ (T may be $+\infty$), $\lambda>0$, $q\ge 1$, and $p\ge 2$. Suppose that real-valued functions $u(x,\cdot),v(x,\cdot)\in C[0,T)$ are differentiable in $(0,T)$ for each $x\in\overline{S}$ and satisfy*

$$\begin{aligned}&u_t(x,t)-\Delta_{p,\omega}u(x,t)-\lambda|u(x,t)|^{q-1}u(x,t)\\&\quad\ge v_t(x,t)-\Delta_{p,\omega}v(x,t)-\lambda|v(x,t)|^{q-1}v(x,t),\\&\qquad\qquad(x,t)\in S\times(0,T),\\&u(x,t)\ge v(x,t),\quad(x,t)\in\partial S\times[0,T),\\&u(x,0)\ge v(x,0),\quad x\in\overline{S}.\end{aligned}\tag{16}$$

If $u_0(x^*)>v_0(x^*)$ for some $x^*\in S$, then $u(x,t)>v(x,t)$ for all $(x,t)\in S\times(0,T)$.

Proof. First, note that $u\ge v$ on $\overline{S}\times[0,T)$ by Theorem 2. Let $T'>0$ be arbitrarily given with $T'<T$ and let $\tau:\overline{S}\times[0,T']\to\mathbb{R}$ be a function defined by

$$\tau(x,t):=u(x,t)-v(x,t),\quad(x,t)\in\overline{S}\times\left[0,T'\right].\tag{17}$$

Then $\tau(x,t)\ge 0$ for all $(x,t)\in\overline{S}\times[0,T']$. Since $\tau(x^*,0)>0$ and $|u(x^*,t)|^{q-1}u(x^*,t)\ge|v(x^*,t)|^{q-1}v(x^*,t)$ for all $0<t\le T'$, we obtain from inequality (16) that

$$\tau_t(x^*,t)-\left[\Delta_{p,\omega}u(x^*,t)-\Delta_{p,\omega}v(x^*,t)\right]\ge 0,\tag{18}$$

for all $0<t\le T'$. Then, by the mean value theorem, for each $y\in\overline{S}$ and t with $0\le t\le T'$, it follows that

$$\begin{aligned}&\left|u(y,t)-u(x^*,t)\right|^{p-2}[u(y,t)-u(x^*,t)]\\&\quad-\left|v(y,t)-v(x^*,t)\right|^{p-2}[v(y,t)-v(x^*,t)]\\&=(p-1)\left|\eta(x^*,y,t)\right|^{p-2}[\tau(y,t)-\tau(x^*,t)],\end{aligned}\tag{19}$$

and $|\eta(x^*,y,t)|<2M$, where $M:=\max_{0<t\le T'}\{|u(x^*,t)|, |v(x^*,t)|\}$.

Then inequality (18) gives

$$\begin{aligned}&\tau_t(x^*,t)\\&\ge\sum_{y\in\overline{S}}(p-1)\left|\eta(x^*,y,t)\right|^{p-2}[\tau(y,t)-\tau(x^*,t)]\,\omega(x^*,y)\\&\ge -d(p-1)[2M]^{p-2}\tau(x^*,t),\end{aligned}\tag{20}$$

where $d=\sum_{y\in\overline{S}}\omega(x^*,y)$. This implies that

$$\tau(x^*,t)\ge\tau(x^*,0)e^{-d(p-1)[2M]^{p-2}t}>0,\quad t\in\left(0,T'\right].\tag{21}$$

Now, suppose that there exists $(x_0,t_0)\in S\times(0,T']$ such that

$$\tau(x_0,t_0)=\min_{x\in S,0<t\le T'}\tau(x,t)=0.\tag{22}$$

Then

$$\begin{aligned}&\tau_t(x_0,t_0)\le 0,\\&\Delta_{p,\omega}u(x_0,t_0)\ge\Delta_{p,\omega}v(x_0,t_0).\end{aligned}\tag{23}$$

Hence, inequality (18) gives

$$0\le\tau_t(x_0,t_0)-\left[\Delta_{p,\omega}u(x_0,t_0)-\Delta_{p,\omega}v(x_0,t_0)\right]\le 0.\tag{24}$$

Therefore,

$$\Delta_{p,\omega}u(x_0,t_0)=\Delta_{p,\omega}v(x_0,t_0);\tag{25}$$

that is,

$$\begin{aligned}&\sum_{y\in\overline{S}}\left\{\left|u(y,t_0)-u(x_0,t_0)\right|^{p-2}[u(y,t_0)-u(x_0,t_0)]\right.\\&\quad-\left|v(y,t_0)-v(x_0,t_0)\right|^{p-2}\\&\quad\left.\times[v(y,t_0)-v(x_0,t_0)]\right\}\omega(x_0,y)=0,\end{aligned}\tag{26}$$

which implies that $\tau(y, t_0) = 0$ for all $y \in \overline{S}$ with $y \sim x_0$. Now, for any $x \in \overline{S}$, there exists a path:

$$x_0 \sim x_1 \sim \cdots \sim x_{n-1} \sim x_n = x, \tag{27}$$

since $\overline{S}$ is connected. By applying the same argument as above inductively, we see that $\tau(x, t_0) = 0$ for every $x \in \overline{S}$. This gives a contradiction to (21). □

For the case $0 < q < 1$, it is well known that (6) may not have unique solution, in general, and the comparison principle in usual form as in Theorem 2 may not hold. Instead, with a strict condition on the parabolic boundary, we obtain a similar comparison principle as follows.

Theorem 4. *Let $T > 0$ (T may be $+\infty$), $\lambda > 0$, $q > 0$, and $p > 1$. Suppose that real-valued functions $u(x, \cdot), v(x, \cdot) \in C[0, T)$ are differentiable in $(0, T)$ for each $x \in \overline{S}$ and satisfy*

$$\begin{gathered} u_t(x,t) - \Delta_{p,\omega} u(x,t) - \lambda |u(x,t)|^{q-1} u(x,t) \\ \geq v_t(x,t) - \Delta_{p,\omega} v(x,t) - \lambda |v(x,t)|^{q-1} v(x,t), \\ (x,t) \in S \times (0,T), \\ u(x,t) > v(x,t), \quad (x,t) \in \partial S \times [0,T), \\ u(x,0) > v(x,0), \quad x \in \overline{S}. \end{gathered} \tag{28}$$

Then $u(x, t) \geq v(x, t)$ for all $(x, t) \in S \times (0, T)$.

Proof. Let $T' > 0$ and $\delta > 0$ be arbitrarily given with $T' < T$ and $0 < \delta < \min_{(x,t)\in\Gamma}[u(x, t) - v(x, t)]$, respectively, where $\Gamma := \{(x, t) \in \overline{S} \times [0, T'] \mid t = 0 \text{ or } x \in \partial S\}$ (called a parabolic boundary).

Now, let a function $\tau : \overline{S} \times (0, T'] \to \mathbb{R}$ be a function defined by

$$\tau(x,t) := [u(x,t) - v(x,t)] - \delta, \quad (x,t) \in \overline{S} \times \left(0, T'\right]. \tag{29}$$

Then $\tau(x, t) > 0$ on Γ. Now, we suppose that $\min_{x\in S, 0<t\leq T'} \tau(x, t) < 0$. Then there exists $(x_0, t_0) \in S \times (0, T']$ such that

(i) $\tau(x_0, t_0) = 0$,

(ii) $\tau(y, t_0) \geq \tau(x_0, t_0) = 0$, $y \in S$,

(iii) $\tau(x, t) > 0$, $(x, t) \in S \times (0, t_0)$.

Then

$$\tau_t\left(x_0, t_0\right) \leq 0 \tag{30}$$

and

$$\Delta_{p,\omega} u\left(x_0, t_0\right) \geq \Delta_{p,\omega} v\left(x_0, t_0\right), \tag{31}$$

since

$$u\left(y, t_0\right) - u\left(x_0, t_0\right) \geq v\left(y, t_0\right) - v\left(x_0, t_0\right). \tag{32}$$

Hence, (28) gives

$$\begin{aligned} 0 &\geq \tau_t\left(x_0, t_0\right) \\ &\geq \lambda \left[\left| u\left(x_0, t_0\right) \right|^{q-1} u\left(x_0, t_0\right) - \left| v\left(x_0, t_0\right) \right|^{q-1} v\left(x_0, t_0\right) \right] \\ &= \lambda \left[\left| v\left(x_0, t_0\right) + \delta \right|^{q-1} \left(v\left(x_0, t_0\right) + \delta \right) \right. \\ &\qquad \left. - \left| v\left(x_0, t_0\right) \right|^{q-1} v\left(x_0, t_0\right) \right] > 0, \end{aligned} \tag{33}$$

which leads to a contradiction. Hence, $\tau(x, t) \geq 0$ for all $(x, t) \in S \times (0, T']$ so that we have $u(x, t) \geq v(x, t)$ for all $(x, t) \in S \times (0, T)$, since δ and T' are arbitrary. □

Using the same method as in Corollary 3, we obtain a strict comparison principle as follows.

Corollary 5 (strict comparison principle). *Let $T > 0$ (T may be $+\infty$), $\lambda > 0$, $q > 0$, and $p \geq 2$. Suppose that real-valued functions $u(x, \cdot), v(x, \cdot) \in C[0, T)$ are differentiable in $(0, T)$ for each $x \in \overline{S}$ and satisfy*

$$\begin{gathered} u_t(x,t) - \Delta_{p,\omega} u(x,t) - \lambda |u(x,t)|^{q-1} u(x,t) \\ \geq v_t(x,t) - \Delta_{p,\omega} v(x,t) - \lambda |v(x,t)|^{q-1} v(x,t), \\ (x,t) \in S \times (0,T), \\ u(x,t) > v(x,t), \quad (x,t) \in \partial S \times [0,T), \\ u(x,0) > v(x,0), \quad x \in \overline{S}. \end{gathered} \tag{34}$$

If $u_0(x^*) > v_0(x^*)$ for some $x^* \in S$, then $u(x, t) > v(x, t)$ for all $(x, t) \in S \times (0, T)$.

3. Blow-Up and Blow-Up Estimates

In this section, we discuss the blow-up phenomena of the solutions to discrete reaction-diffusion equation defined on networks, which is a main part of this paper.

We first introduce the concept of the blow-up as follows.

Definition 6 (blow-up). One says that a solution u to an equation defined on a network $\overline{S}$ blows up in finite time T, if there exists $x \in S$ such that $|u(x, t)| \to +\infty$ as $t \nearrow T^-$.

According to the comparison principle in the previous section, we are guaranteed to get a solution to

$$\begin{gathered} u_t(x,t) = \Delta_{p,\omega} u(x,t) + \lambda u^q(x,t), \quad (x,t) \in S \times (0,\infty), \\ u(x,t) = 0, \quad (x,t) \in \partial S \times (0,\infty), \\ u(x,0) = u_0 \geq 0, \quad x \in \overline{S}, \end{gathered} \tag{35}$$

when $p > 1$, $q > 0$, $\lambda > 0$, and the initial data u_0 is nontrivial on S.

We now state the main theorem of this paper as follows.

Theorem 7. *Let u be a solution of (35). Then one has the following.*

(i) *If $0 < p-1 < q$ and $q > 1$, then the solution blows up in a finite time, provided $\overline{u}_0 > (\omega_0/\lambda)^{1/(q-p+1)}$, where $\omega_0 := \max_{x\in S}\sum_{y\in\overline{S}}\omega(x,y)$ and $\overline{u}_0 := \max_{x\in S}u_0(x)$.*

(ii) *If $0 < q \le 1$, then the nonnegative solution is global.*

(iii) *If $1 < q < p-1$, then the solution is global.*

Proof. First, we prove (i). We note that $u(x,t) \ge 0$, for all $(x,t) \in \overline{S}\times[0,+\infty)$, by Theorem 2. Assume that $0 < p-1 < q$, $q > 1$, and $\overline{u}_0 > (\omega_0/\lambda)^{1/(q-p+1)}$, where $\overline{u}_0 := \max_{x\in S}u_0(x)$. For each $t > 0$, let $x_t \in S$ be a node such that $u(x_t,t) := \max_{x\in S}u(x,t)$. In fact, we note that $\max_{x\in S}u(x,t)$ is differentiable, for almost all $t > 0$. Then (35) can be written as follows:

$$\begin{aligned} u_t(x_s,s) &= \Delta_{p,\omega}u(x_s,s) + \lambda u^q(x_s,s) \\ &= \sum_{y\in\overline{S}}|u(y,s)-u(x,s)|^{p-2} \\ &\qquad\times[u(y,s)-u(x_s,s)]\,\omega(x_s,y) \\ &\qquad+\lambda u^q(x_s,s) \\ &\ge -\omega_0 u^{p-1}(x_s,s)+\lambda u^q(x_s,s) \end{aligned} \tag{36}$$

for almost all $s > 0$. We need to show that $\max_{x\in S}u(x,t) > \overline{u}_0$, for all $t > 0$. Since $u(x,t) \ge 0$ on $S\times(0,\infty)$ and

$$\begin{aligned} &\lim_{s\to0^+}u_t(x,t) \\ &\quad=\lim_{s\to0^+}\Bigg[\sum_{y\in\overline{S}}|u(y,s)-u(x_s,s)|^{p-2} \\ &\qquad\times[u(y,s)-u(x_s,s)]\,\omega(x_s,y) \\ &\qquad+\lambda u^q(x_s,s)\Bigg] \\ &\quad\ge\lim_{s\to0^+}\left[-\omega_0u^{p-1}(x_s,s)+\lambda u^q(x_s,s)\right] \\ &\quad=-\omega_0\overline{u}_0^{p-1}+\lambda\overline{u}_0^q>0, \end{aligned} \tag{37}$$

$u(x_s,s)$ is increasing in some interval $(0,s_1)$. Suppose that there exists $s > 0$ somewhere at which $u(x_s,s) \le \overline{u}_0$. Then now take the interval $(0,s_1)$ to be maximal on which $u(x_s,s) > \overline{u}_0$, $s\in(0,s_1)$, and $u(x_s,s) = \overline{u}_0$. Then there exists $s^*\in(0,s_1)$ such that $u_t(x_{s^*},s^*) < 0$ but

$$\begin{aligned} 0 &> u_t(x_{s^*},s^*) \\ &= \sum_{y\in\overline{S}}|u(y,s^*)-u(x_{s^*},s^*)|^{p-2} \\ &\qquad\times[u(y,s^*)-u(x_{s^*},s^*)]\,\omega(x_{s^*},s^*)+\lambda u^q(x_{s^*},s^*) \\ &\ge -\omega_0u^{p-1}(x_{s^*},s^*)+\lambda u^q(x_{s^*},s^*)>0, \end{aligned} \tag{38}$$

which leads to a contradiction. Thus it follows that $u(x_s,s) > \overline{u}_0$, $s\in(0,+\infty)$.

Let $F : [\overline{u}_0,+\infty) \to (0,F(\overline{u}_0)]$ be a function defined by

$$F(y) := \int_y^{+\infty}\frac{ds}{-\omega_0s^{p-1}+\lambda s^q} < +\infty, \quad y \ge \overline{u}_0. \tag{39}$$

We note that $-\omega_0s^{p-1}+\lambda s^q > 0$, for $s \ge \overline{u}_0$, since $\overline{u}_0 > (\omega_0/\lambda)^{1/(q-p+1)}$.

Then F is a decreasing continuous function from $[\overline{u}_0,+\infty)$ onto $(0,F(\overline{u}_0)]$ with its inverse function G. Integrating (36) from 0 to t, we have

$$\begin{aligned} t &\le \int_0^t\frac{u_t(x_s,s)}{-\omega_0u^{p-1}(x_s,s)+\lambda u^q(x_s,s)}du \\ &= \int_{\overline{u}_0}^{u(x_t,t)}\frac{ds}{-\omega_0s^{p-1}+\lambda s^q}. \end{aligned} \tag{40}$$

This can be written as

$$F(u(x_t,t)) \le F(\overline{u}_0)-t \tag{41}$$

and, equivalently,

$$u(x_t,t) \ge G[F(\overline{u}_0)-t], \tag{42}$$

which implies that $u(x_t,t)$ blows up, as $t\to F(\overline{u}_0)$. □

Secondly, we prove (ii). Consider the following ODE problem:

$$\begin{aligned} \frac{d}{dt}z(t) &= \lambda z^q(t), \quad t>0, \\ z(0) &= \overline{u}_0+1. \end{aligned} \tag{43}$$

Then, we have

$$\begin{aligned} z(t) &= \left[(1-q)\lambda t+z^{1-q}(0)\right]^{1/(1-q)}, \quad q\ne1, \\ z(t) &= z(0)e^{\lambda t}, \quad q=1, \end{aligned} \tag{44}$$

for every $t \ge 0$.

Take $v(x,t) := z(t)$, for all $x\in\overline{S}$ and $t\ge0$. Then it is easy to see that $v(x,t) > u(x,t)$, $(x,t)\in\partial S\times(0,+\infty)$, $v(x,0) = z(0) > \overline{u}_0$, $x\in\overline{S}$, and

$$\begin{aligned} &v_t(x,t)-\Delta_{p,\omega}v(x,t)-\lambda v^q(x,t) \\ &\quad=\frac{d}{dt}z(t)-\lambda z^q(t)=0. \end{aligned} \tag{45}$$

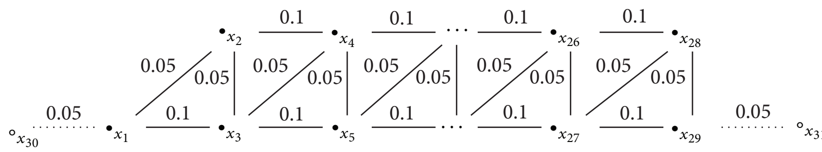
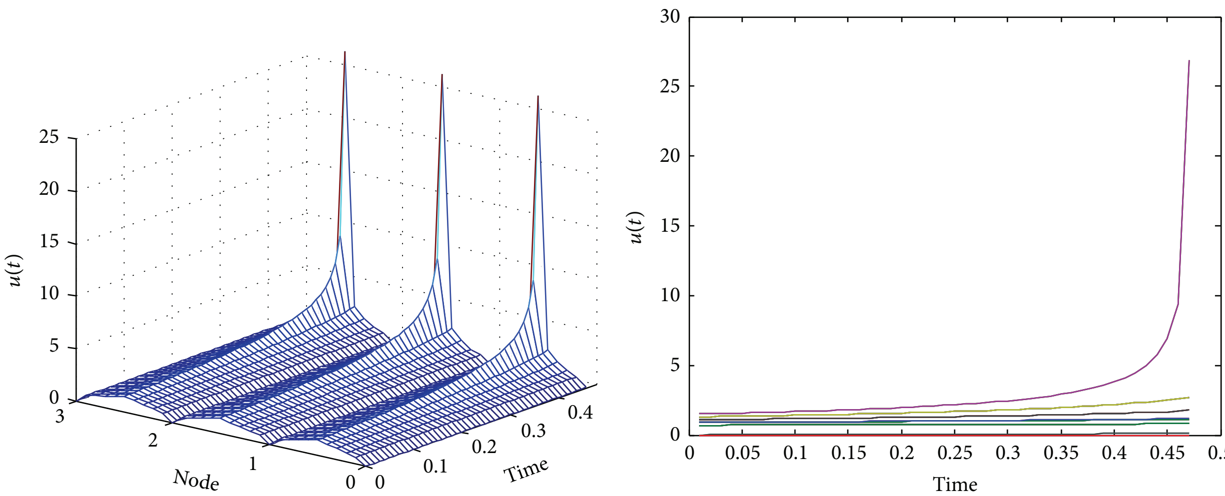


FIGURE 1: Graph $\overline{S}$.



(a) (b)

FIGURE 2: Behavior of each node for $q = 3$ and $p = 2.5$.

Thus, $0 \le u(x,t) \le v(x,t) = z(t)$ for every $(x,t) \in \overline{S}\times(0,+\infty)$ by Theorem 4. This implies that u must be global.

Finally, we prove (iii). Consider the following eigenvalue problem:

$$-\Delta_{p,\omega}\phi(x) = \lambda_1 |\phi(x)|^{p-2}\phi(x), \quad x \in S, \qquad \phi(x) = 0, \quad x \in \partial S. \tag{46}$$

Note that it is well known that $\lambda_1 > 0$ and $\phi(x) > 0$, for all $x \in S$ (see [11, 12]).

Now, take $v(x,t) := k\phi(x)$, $x \in \overline{S}$, $t \ge 0$. Choosing $k > 0$ so large that $k\phi(x) > \overline{u}_0$ and $k\phi(x) > (\lambda/\lambda_1)^{1/(p-1-q)}$, then we see that $v(x,0) = k\phi(x) \ge u_0(x) = u(x,0)$, $x \in S$, and

$$\begin{aligned} &v_t(x,t) - \Delta_{p,\omega}v(x,t) - \lambda v^q(x,t) \\ &\quad = \lambda_1 (k\phi(x))^{p-1} - \lambda(k\phi(x))^q \ge 0. \end{aligned} \tag{47}$$

Therefore, $0 \le u(x,t) \le v(x,t) = k\phi(x)$ for every $(x,t) \in \overline{S}\times(0,+\infty)$ by Theorems 2 and 4, which is required.

Remark 8. (i) When the solution blows up in the above, the blow-up time T can be estimated as

$$\frac{\overline{u}_0^{1-q}}{\lambda(q-1)} \le T \le F(\overline{u}_0) := \int_{\overline{u}_0}^{+\infty} \frac{ds}{-\omega_0 s^{p-1} + \lambda s^q}. \tag{48}$$

In fact, the first inequality is derived as follows. By the definition of maximum function $u(x_t,t)$, (35) gives

$$u_t(x_s,s) \le \lambda u^q(x_s,s), \tag{49}$$

for almost all $s > 0$. Then integrating both sides, we have

$$t \ge \int_0^t \frac{u_t(x_s,s)}{\lambda u^q(x_s,s)}ds = \int_{\overline{u}_0}^{u(x_t,t)} \frac{ds}{\lambda s^q}, \tag{50}$$

so that we obtain $T \ge \int_{\overline{u}_0}^{+\infty}(ds/\lambda s^q) = \overline{u}_0^{1-q}/\lambda(q-1)$, by taking the limit as $t \to T^-$.

(ii) In the above, if $\overline{u}_0 := \max_{x\in S}u_0(x)$ is not sufficiently large, then the solution may be global. This can be seen in the numerical examples in Section 4.

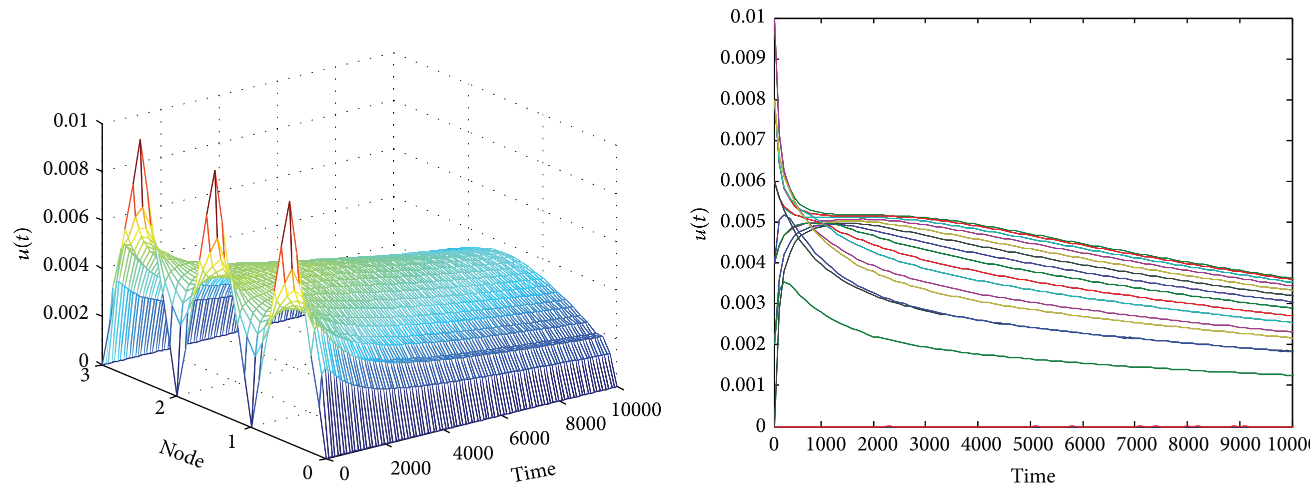
(iii) In the above, the case where $1 < p - 1 = q$ was not discussed. As a matter of fact, the solution to (35) in this case may blow up or not, depending on the magnitude of the parameter λ. Each case is illustrated in Section 4. A full argument will be discussed in a forthcoming paper.

We now derive the lower bound, the upper bound, and the blow-up rate for the maximum function of blow-up solutions.

Theorem 9. *Let u be a solution of (35), which blows up at a finite time T, $q > p - 1 > 0$, and $q > 1$. Then one has the following.*

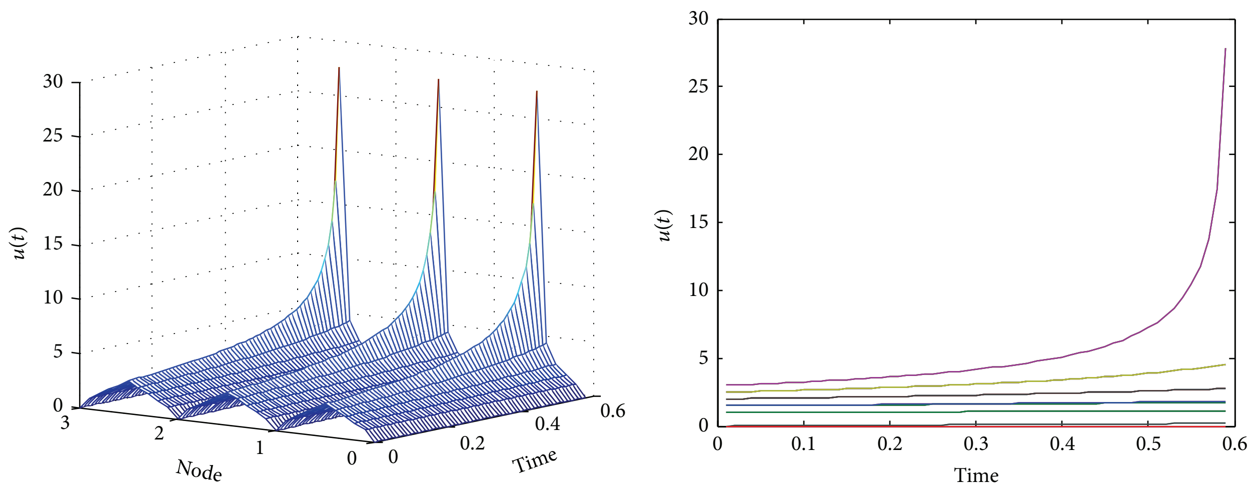
(i) *The lower bound is*

$$\max_{x\in S} u(x,t) \ge [\lambda(q-1)(T-t)]^{-1/(q-1)}, \quad 0 < t < T. \tag{51}$$



(a) (b)

FIGURE 3: Behavior of each node for $q = 3$ and $p = 2.5$.



(a) (b)

FIGURE 4: Behavior of each node for $q = 3$ and $p = 1.5$.

(ii) *The upper bound is*

$$\max_{x\in S} u(x,t) \le \left[\lambda(q-1)(T-t) - \alpha(T-t)^{(2q-p)/(q-1)}\right]^{-1/(q-1)}, \quad 0 < t < T, \tag{52}$$

where $\alpha := \omega_0 \lambda^{(q-p+1)/(q-1)}(q-1)^{(2q-p)/(q-1)}$ *and* $\omega_0 = \max_{x\in S}\sum_{y\in\bar{S}}\omega(x,y)$.

(iii) *The blow-up rate is*

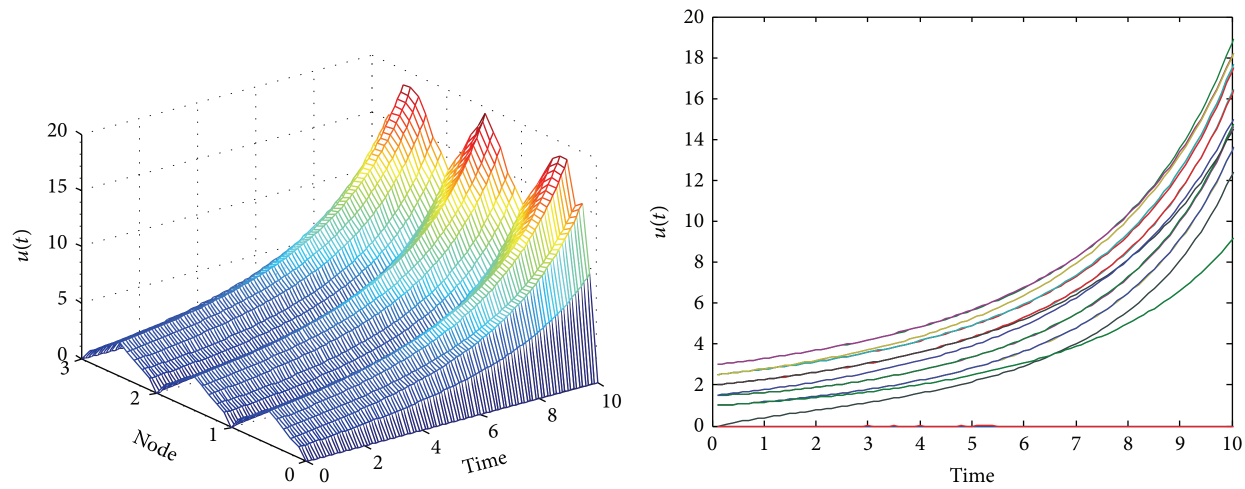
$$\lim_{t\to T^-}(T-t)^{1/(q-1)}\max_{x\in\bar{S}} u(x,t) = \left[\frac{1}{\lambda(q-1)}\right]^{1/(q-1)}, \quad 0 < t < T. \tag{53}$$

Proof. First, we prove (i). As in the previous theorem, let $x_t \in S$ be a node such that $u(x_t,t) := \max_{x\in S} u(x,t)$, for each $t > 0$. Then it follows from (35) that

$$u_t(x_s,s) \le \lambda u^q(x_s,s), \tag{54}$$

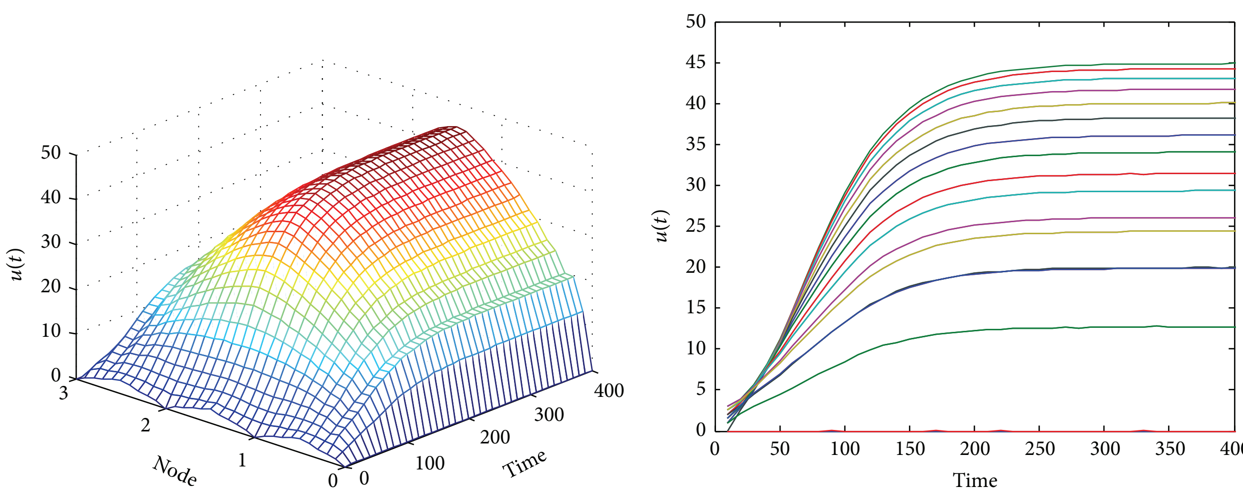
for almost all $s > 0$. Then integrating from t to T, we get

$$\begin{aligned} \lambda(T-t) &\ge \int_t^T \frac{u_t(x_s,s)}{u^q(x_s,s)}ds \\ &= \int_{u(x_t,t)}^{+\infty}\frac{ds}{s^q} \\ &= \frac{1}{q-1}u^{1-q}(x_t,t). \end{aligned} \tag{55}$$



(a) (b)

Figure 5: Behavior of each node for $q = 1.5$ and $p = 3$.



(a) (b)

Figure 6: Behavior of each node for $q = 0.5$ and $p = 3$.

Hence, we obtain

$$u(x_t, t) \geq [\lambda(q-1)(T-t)]^{-1/(q-1)}, \quad 0 < t < T. \tag{56}$$

Next, we prove (ii). Since the solution u is positive, we get

$$\begin{aligned} u_t(x_s, s) &\geq -\sum_{y\in\bar{S}} u^{p-1}(x_s, s)\,\omega(x_s, y) + \lambda u^q(x_s, s) \\ &\geq -\omega_0 u^{p-1}(x_s, s) + \lambda u^q(x_s, s) \\ &= u^q(x_s, s)\left[\lambda - \omega_0 u^{p-1-q}(x_s, s)\right], \end{aligned} \tag{57}$$

for almost all $s > 0$ and $\omega_0 = \max_{x\in S}\sum_{y\in\bar{S}}\omega(x, y)$. Then, it follows from (i) (*lower bound*) that we have

$$u_t(x_s, s) \geq u^q(x_s, s)\left[\lambda - \omega_0\left[\lambda(q-1)(T-t)\right]^{(q-p+1)/(q-1)}\right]. \tag{58}$$

Integrating from t to T, we get

$$u(x_t, t) \leq \left[\lambda(q-1)(T-t) - \alpha(T-t)^{(2q-p)/(q-1)}\right]^{-1/(q-1)}, \quad 0 < t < T, \tag{59}$$

where $\alpha := \omega_0\lambda^{(q-p+1)/(q-1)}(q-1)^{(2q-p)/(q-1)}$.

Finally, (iii) can be easily obtained by (i) and (ii). □

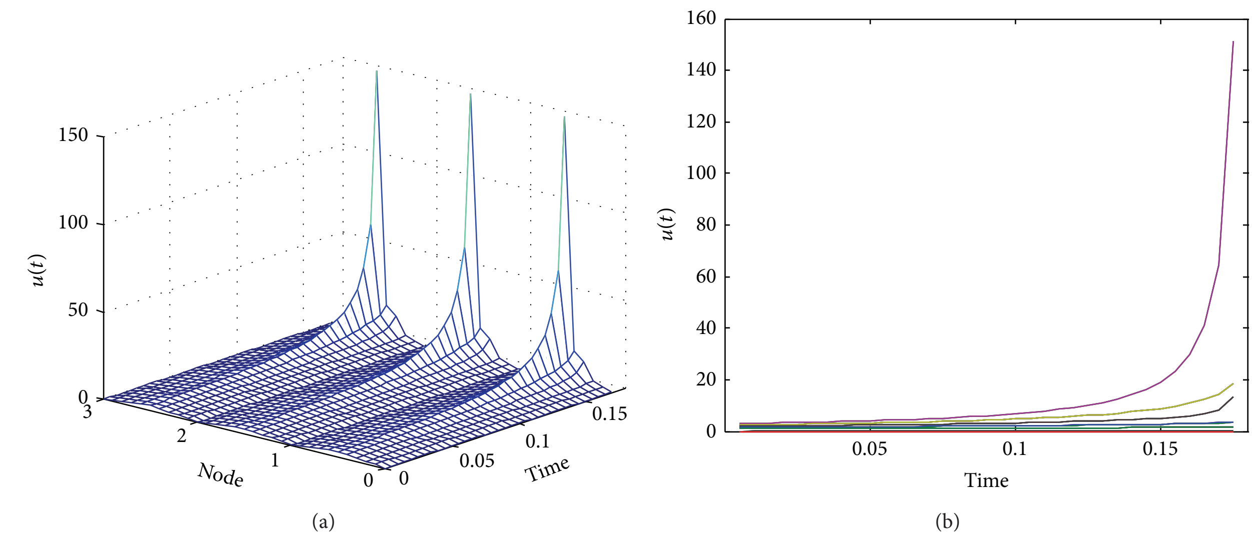


FIGURE 7: Behavior of each node for $q = 2$ and $p = 3$.

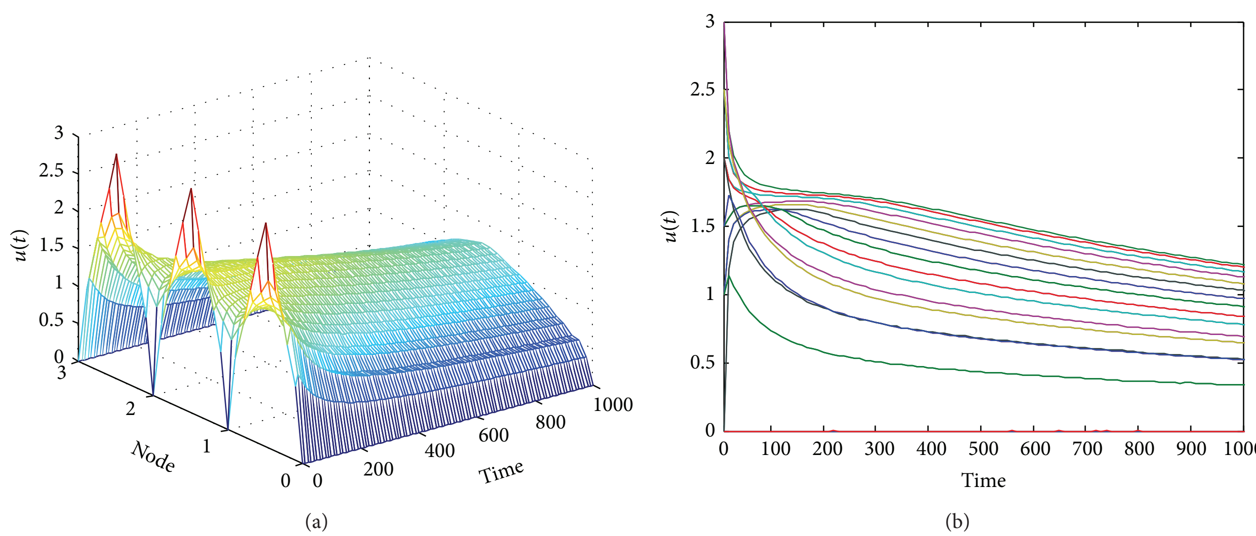


FIGURE 8: Behavior of each node for $q = 2$ and $p = 3$.

4. Examples and Numerical Illustrations

In this section, we show numerical illustrations to exploit our results in the previous section.

Now, consider a graph $S = \{x_1, \ldots, x_{29}\}$ with the boundary $\partial S = \{x_{30}, x_{31}\}$ and the weight

$$\omega\left(x_i, x_j\right) = \begin{cases} 0.05, & j = i + 1, \\ 0.1, & j = i + 2, \\ 0.05, & i = 1,\ j = 30,\ i = 29,\ j = 31, \\ 0, & \text{otherwise}, \end{cases} \tag{60}$$

where $i = 1, \ldots, 28$ (see Figure 1). Then, we note that $\omega_0 := \max_{x \in S} \sum_{y \in \overline{S}} \omega(x, y) = 0.3$.

Example 1 ($1 < p - 1 < q$). For the graph $\overline{S}$ (see Figure 1), consider $q = 3$, $p = 2.5$, $\lambda = 0.5$, and the initial data u_0 given by Table 1.

Then $1 < p - 1 = 1.5 < q = 3$ and $\max_{x \in S} u_0(x) = 1.5 > (\omega_0/\lambda)^{1/(q-p+1)} \fallingdotseq 0.711$. Then Figure 2 shows that the solution to (35) blows up and the computed blow-up time T is estimated as $T \fallingdotseq 0.4617817$ and

$$0.444 \fallingdotseq \int_{1.5}^{+\infty} \frac{ds}{0.5 s^3} \leq T \leq \int_{1.5}^{+\infty} \frac{ds}{-0.3 s^{1.5} + 0.5 s^3} \fallingdotseq 0.553. \tag{61}$$

On the other hand, consider a small initial data u_0 given by Table 2.

Then $\max_{x \in S} u_0(x) = 0.01 \ngtr (\omega_0/\lambda)^{1/(q-p+1)} \fallingdotseq 0.711$ and Figure 3 shows that the solution to (35) is global.

TABLE 1: Initial data of u.

Node i	$u_0(x_i)$
1	0.7
2	0.9
3	1.1
4	1.3
5	1.5
6	1.3
7	1.1
8	0.9
9	0.7
10	0
11	0.7
12	0.9
13	1.1
14	1.3
15	1.5
16	1.3
17	1.1
18	0.9
19	0.7
20	0
21	0.7
22	0.9
23	1.1
24	1.3
25	1.5
26	1.3
27	1.1
28	0.9
29	0.7
30	0
31	0

TABLE 2: Initial data of u.

Node i	$u_0(x_i)$
1	0.002
2	0.004
3	0.006
4	0.008
5	0.01
6	0.008
7	0.006
8	0.004
9	0.002
10	0
11	0.002
12	0.004
13	0.006
14	0.008
15	0.01
16	0.008
17	0.006
18	0.004
19	0.002
20	0
21	0.002
22	0.004
23	0.006
24	0.008
25	0.01
26	0.008
27	0.006
28	0.004
29	0.002
30	0
31	0

Example 2 ($0 < p-1 < 1 < q$). For the graph $\overline{S}$ (see Figure 1), consider $q = 3$, $p = 1.5$, $\lambda = 0.1$, and the initial data u_0 given by Table 3.

Then $0 < p-1 = 0.5 < q = 3$ and $\max_{x\in S} u_0(x) = 3 > (\omega_0/\lambda)^{1/(q-p+1)} \fallingdotseq 1.55$. Then Figure 4 shows that the solution to (35) blows up and the computed blow-up time T is estimated as $T \fallingdotseq 0.5864884$ and

$$0.555 \fallingdotseq \int_3^{+\infty} \frac{ds}{0.1s^3} \le T \le \int_3^{+\infty} \frac{ds}{-0.3s^{p-1} + 0.1s^q} \fallingdotseq 0.610. \quad (62)$$

Example 3 ($1 < q < p-1$). For the graph $\overline{S}$ (see Figure 1), consider $q = 1.5$, $p = 3$, $\lambda = 0.1$, and the initial data u_0 given by Table 3 in Example 2. Then $1 < q = 1.5 < p-1 = 2$ and Figure 5 shows that the solution to (35) is global.

Example 4 ($0 < q \le 1$). For the graph $\overline{S}$ (see Figure 1), consider $q = 0.5$, $p = 3$, $\lambda = 0.1$, and the initial data u_0 given by Table 3 in Example 2. Then $0 < q = 0.5 \le 1$ and Figure 6 shows that the solution to (35) is global.

Example 5 ($1 < p-1 = q$). For the graph $\overline{S}$ (see Figure 1), consider $q = 2$, $p = 3$, $\lambda = 2$, and the initial data u_0 given by Table 3 in Example 2. Then $1 < q = p-1 = 2$ and Figure 7 shows that the solution to (35) blows up.

On the contrary, when $\lambda = 0.00001$, the solution to (35) is global, as seen in Figure 8.

5. Conclusion

We discuss the conditions under which blow-up occurs for the solutions of discrete p-Laplacian parabolic equations on networks S with boundary ∂S:

$$u_t(x,t) = \Delta_{p,\omega} u(x,t) + \lambda |u(x,t)|^{q-1} u(x,t), \quad (x,t) \in S \times (0,+\infty),$$

Table 3: Initial data of u.

Node i	$u_0(x_i)$
1	1
2	1.5
3	2
4	2.5
5	3
6	2.5
7	2
8	1.5
9	1
10	0
11	1
12	1.5
13	2
14	2.5
15	3
16	2.5
17	2
18	1.5
19	1
20	0
21	1
22	1.5
23	2
24	2.5
25	3
26	2.5
27	2
28	1.5
29	1
30	0
31	0

$$u(x,t) = 0, \quad (x,t) \in \partial S \times (0,+\infty),$$
$$u(x,0) = u_0 \geq 0, \quad x \in \overline{S}, \tag{63}$$

where $p > 1$, $q > 0$, $\lambda > 0$, and the initial data u_0 is nontrivial on S.

The main theorem states that the solution u to the above equation satisfies the following:

(i) if $0 < p-1 < q$ and $q > 1$, then the solution blows up in a finite time, provided $\overline{u}_0 > (\omega_0/\lambda)^{1/(q-p+1)}$, where $\omega_0 := \max_{x\in S}\sum_{y\in\overline{S}}\omega(x,y)$ and $\overline{u}_0 := \max_{x\in S}u_0(x)$;

(ii) if $0 < q \leq 1$, then the nonnegative solution is global;

(iii) if $1 < q < p-1$, then the solution is global.

In addition, we give an estimate for the blow-up time and the blow-up rate for the blow-up solution. Finally, we give some numerical illustrations which exploit the main results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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