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## Research Article

# Strong Convergence Properties for Asymptotically Almost Negatively Associated Sequence

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By applying the moment inequality for asymptotically almost negatively associated (in short AANA) random sequence and truncated method, we get the three series theorems for AANA random variables. Moreover, a strong convergence property for the partial sums of AANA random sequence is obtained. In addition, we also study strong convergence property for weighted sums of AANA random sequence.

## 1. Introduction

A finite family of random variables  $\{X_k, 1 \leq k \leq n, n \geq 2\}$  is said to be negatively associated (in short NA) if for every pair of disjoint subsets  $A_1, A_2$  of  $\{1, 2, \dots, n\}$

$$\text{Cov}(f(X_i : i \in A_1), g(X_j : j \in A_2)) \leq 0, \quad (1.1)$$

whenever  $f, g$  are coordinate-wise nondecreasing such that the covariance exists. An infinite sequence of random variables  $\{X_n, n \geq 1\}$  is said to be NA if every finite subfamily is NA.

The notion of NA was first introduced by Block et al. (1982) [1]. Joag-Dev and Proschan (1983) [2] showed that many well-known multivariate distributions possess the NA property. By inspecting the proof of maximal inequality for NA random variables in Matuła [3], Chandra and Ghosal discovered that one can also allow negative correlations provided they are small. Primarily motivated by this, Chandra and Ghosal [4, 5] introduced the following dependence.

*Definition 1.1.* A sequence  $\{X_n, n \geq 1\}$  of random variables is said to be asymptotically almost negatively associated, if there exists a nonnegative sequence  $q(n) \rightarrow 0$  as  $n \rightarrow \infty$  such that

$$\text{Cov}(f(X_n), g(X_{n+1}, X_{n+2}, \dots, X_{n+k})) \leq q(n) [\text{Var } f(X_n) \text{Var } g(X_{n+1}, X_{n+2}, \dots, X_{n+k})]^{1/2}, \quad (1.2)$$

for all  $n, k \geq 1$  and for all coordinatewise nondecreasing continuous functions  $f$  and  $g$  whenever the variances exit.

Obviously, the family of AANA sequences contain NA (in particular, independent) sequences (with  $q(n) = 0, n \geq 1$ ) and some more sequences of random variables which are not much deviated from being NA. An example of an AANA sequence which is not NA was introduced by Chandra and Ghosal [4].

Since the notion of AANA sequence was introduced by Chandra and Ghosal [4], the AANA properties have aroused wide interest because of numerous applications in reliability theory, percolation theory, and multivariate statistical analysis. In the past decades, a lot of effort was dedicated to proving the limit theorems of AANA random variables; we can refer to [4–10]. Hence, extending the limit properties of AANA random variables has very important significance in the theory and application.

In this paper, we mainly study the strong convergence property for the partial sums of AANA random variables; furthermore the strong convergence property for weighted sums of AANA random variables is also obtained.

Throughout the paper, let  $I(A)$  be the indicator function of the set  $A$ , and let  $X^c = -cI(X < -c) + XI(|X| \leq c) + cI(X > c)$  for some  $c > 0$ . The  $a_n = O(b_n)$  denotes that there exists a positive constant  $C$  such that  $|a_n/b_n| \leq C$ . The symbol  $C$  represents a positive constant which may be different in various places. The main results of this paper are dependent on the following lemmas.

**Lemma 1.2** (Yuan and An [6]). *Let  $\{X_n, n \geq 1\}$  be a sequence of AANA random variables with mixing coefficients  $\{q(n), n \geq 1\}$ , and let  $f_1, f_2, \dots$  be all nondecreasing (or nonincreasing) functions; then  $\{f_n(X_n), n \geq 1\}$  is still a sequence of AANA random variables with mixing coefficients  $\{q(n), n \geq 1\}$ .*

**Lemma 1.3** (Wang et al. [7]). *For  $1 < p \leq 2$ , let  $\{X_n, n \geq 1\}$  be a sequence of AANA random variables with mixing coefficients  $\{q(n), n \geq 1\}$  and  $EX_n = 0$  for each  $n \geq 1$ . If  $\sum_{n=1}^{\infty} q^2(n) < \infty$ , then there exists a positive constant  $C_p$  depending only on  $p$  such that*

$$E\left(\max_{1 \leq i \leq n} |S_i|^p\right) \leq C_p \sum_{i=1}^n E|X_i|^p, \quad (1.3)$$

for all  $n \geq 1$  where  $S_i = \sum_{j=1}^i X_j$ ,  $C_p = 2^p [2^{2-p}p + (6p)^p (\sum_{n=1}^{\infty} q^2(n))^{p/q}]$ , and  $q = p/(p-1)$  is the dual number of  $p$ .

**Lemma 1.4** (Wu [11]). *Let  $\{X_n, n \geq 1\}$  be a sequence of random variables. For each  $n \geq 1$ , there exists a random variable  $X$  such that*

$$P(|X_n| \geq x) \leq CP(|X| \geq x) \quad (1.4)$$

then, for any  $r > 0$ ,  $x > 0$ , the following two statements hold:

$$\begin{aligned} E|X_n|^r I(|X_n| \leq x) &\leq C[E|X|^r I(|X| \leq x) + x^r P(|X| > x)], \\ E|X_n|^r I(|X_n| > x) &\leq C[E|X|^r I(|X| > x)]. \end{aligned} \quad (1.5)$$

**Lemma 1.5** (Sung [12]). Let  $\phi(x)$  be a positive increasing function on  $(0, +\infty)$  satisfying  $\phi(x) \uparrow \infty$  as  $n \rightarrow \infty$ , and let  $\psi(x)$  be the inverse function of  $\phi(x)$ . If  $\psi(x)$  and  $\phi(x)$  satisfy, respectively,

$$\psi(n) \sum_{i=1}^n \frac{1}{\psi(i)} = O(n), \quad E[\phi(|X|)] < \infty, \quad (1.6)$$

then

$$\sum_{i=1}^{\infty} \frac{1}{\psi(n)} E|X| I(|X| > \psi(n)) < \infty. \quad (1.7)$$

## 2. Strong Convergence for the Partial Sums of AANA Random Variables

**Theorem 2.1.** Let  $\{X_n, n \geq 1\}$  be a sequence of AANA random variables with  $\sum_{n=1}^{\infty} q^2(n) < \infty$ , if the following assumptions holds:

$$\sum_{n=1}^{\infty} P(|X_n| > c) < \infty, \quad \sum_{n=1}^{\infty} EX_n^c < \infty, \quad \sum_{n=1}^{\infty} \text{Var } X_n^c < \infty; \quad (2.1)$$

then  $\sum_{n=1}^{\infty} X_n$  almost surely convergence.

*Remark 2.2.* The proof of Theorem 2.1 is similar to the proof of Theorem 4.3.4 in [11], and by Lemmas 1.2 and 1.3, we omit it.

**Theorem 2.3.** Let  $\{X_n, n \geq 1\}$  be a sequence of AANA random variables with  $\sum_{n=1}^{\infty} q^2(n) < \infty$ .

Assume that  $\{g_n(x), n \geq 1\}$  is a sequence of even functions in  $\mathbb{R}^1$ , for each  $n \geq 1$ ,  $g_n(x)$  is a positive nondecreasing function in  $(0, +\infty)$  and satisfies one of the following conditions:

- (i) for  $x \in (0, 1]$  there exists a constant  $\alpha > 0$  such that  $g_n(x) \geq \alpha x$ ;
- (ii) for  $x \in (0, 1]$ , there exists a constant  $r \in (1, 2]$  and  $\alpha > 0$  such that  $g_n(x) \geq \alpha x^r$ ; however, for  $x \in (1, \infty)$ ,  $g_n(x) \geq \alpha x$ , furthermore assume that  $EX_n = 0$ , for each  $n \geq 1$ .

Let  $\{a_n, n \geq 1\}$  be a constant sequence satisfying  $0 < a_n \uparrow \infty$  such that

$$\sum_{n=1}^{\infty} E g_n \left( \frac{X_n}{a_n} \right) < \infty, \quad (2.2)$$

then  $\sum_{n=1}^{\infty} (X_n/a_n)$  almost surely convergence, and further it follows from the "Kronecker lemma" that

$$a_n^{-1} \sum_{k=1}^n X_k \longrightarrow 0 \text{ a.s., as } n \longrightarrow \infty. \quad (2.3)$$

*Proof.* For each  $n \geq 1$ , denote  $X_n^{a_n} \triangleq -a_n I(X_n < -a_n) + X_n I(|X_n| \leq a_n) + a_n I(X_n > a_n)$ .

By Lemma 1.2, we can see that, for fixed  $n \geq 1$ ,  $\{X_n^{a_n}\}$  is still a sequence of AANA random variables. To verify the Theorem 2.3, for  $c = 1$  we only need to prove the convergence of three series of (2.1) under condition (i) or (ii). The proof of Theorem 2.3 includes the following three steps.

(1) We prove  $\sum_{n=1}^{\infty} P(|X_n/a_n| > 1) < \infty$  under condition (i) or (ii).

For each  $n \geq 1$ , if  $g_n(x)$  satisfies condition (i), noting that  $g_n(x)$  is a positive nondecreasing even function in  $(0, +\infty)$ , it is obvious that

$$P\left(\left|\frac{X_n}{a_n}\right| > 1\right) = EI\left(\left|\frac{X_n}{a_n}\right| > 1\right) \leq \alpha^{-1} E g_n\left(\frac{X_n}{a_n}\right). \quad (2.4)$$

By (2.2), we can get

$$\sum_{n=1}^{\infty} P\left(\left|\frac{X_n}{a_n}\right| > 1\right) \leq \alpha^{-1} \sum_{n=1}^{\infty} E g_n\left(\frac{X_n}{a_n}\right) < \infty. \quad (2.5)$$

If  $g_n(x)$  satisfies condition (ii), it is easy to prove that (2.5) also holds when  $|X_n| > a_n > 0$ .

(2) Next we will show  $\sum_{n=1}^{\infty} E|X_n^{a_n}/a_n| < \infty$ .

If  $g_n(x)$  satisfies condition (i), it follows that

$$\begin{aligned} \left|E \frac{X_n^{a_n}}{a_n}\right| &= \left|-EI(X_n < -a_n) + E \frac{X_n}{a_n} I(|X_n| \leq a_n) + EI(X_n > a_n)\right| \\ &\leq EI(|X_n| > a_n) + \left|E \frac{X_n}{a_n} I(|X_n| \leq a_n)\right| \\ &\leq \alpha^{-1} E g_n\left(\frac{X_n}{a_n}\right) + \left|\int_{|X_n| \leq a_n} \frac{X_n}{a_n} dP\right| \\ &\leq 2\alpha^{-1} E g_n\left(\frac{X_n}{a_n}\right). \end{aligned} \quad (2.6)$$

On the other hand, if condition (ii) holds, according to  $EX_n = 0$ , for each  $n \geq 1$ , we have

$$\begin{aligned} \left| E \frac{X_n^{a_n}}{a_n} \right| &\leq EI(|X_n| > a_n) + \left| E \frac{X_n}{a_n} I(|X_n| \leq a_n) \right| \\ &= EI(|X_n| > a_n) + \left| E \frac{X_n}{a_n} I(|X_n| > a_n) \right| \\ &\leq 2\alpha^{-1} E g_n \left( \frac{X_n}{a_n} \right). \end{aligned} \quad (2.7)$$

Hence, it follows from (2.2) that

$$\sum_{n=1}^{\infty} E \left| \frac{X_n^{a_n}}{a_n} \right| < 2\alpha^{-1} \sum_{n=1}^{\infty} E g_n \left( \frac{X_n}{a_n} \right) < \infty. \quad (2.8)$$

(3) Finally we prove  $\sum_{n=1}^{\infty} E(X_n^{a_n}/a_n)^2 < \infty$ .

If  $g_n(x)$  satisfies condition (i), for each  $n \geq 1$ , it is easy to show that by the  $C_r$ -inequality

$$\begin{aligned} E \left( \frac{X_n^{a_n}}{a_n} \right)^2 &= E \left| -I(X_n < -a_n) + \frac{X_n}{a_n} I(|X_n| \leq a_n) + I(X_n > a_n) \right|^2 \\ &\leq 3E \left[ I(|X_n| > a_n) + \left[ \frac{X_n}{a_n} \right]^2 I(|X_n| \leq a_n) \right] \\ &\leq C\alpha^{-1} E g_n \left( \frac{X_n}{a_n} \right) + CE \left| \frac{X_n}{a_n} \right| I(|X_n| \leq a_n) \\ &\leq C\alpha^{-1} E g_n \left( \frac{X_n}{a_n} \right). \end{aligned} \quad (2.9)$$

If condition (ii) holds, according to the  $C_r$ -inequality, for each  $n \geq 1$ , we get

$$\begin{aligned} E \left( \frac{X_n^{a_n}}{a_n} \right)^2 &= E \left| -I(X_n < -a_n) + \frac{X_n}{a_n} I(|X_n| \leq a_n) + I(X_n > a_n) \right|^2 \\ &\leq 3E \left[ I(|X_n| > a_n) + \left( \frac{X_n}{a_n} \right)^2 I(|X_n| \leq a_n) \right] \\ &\leq C\alpha^{-1} E g_n \left( \frac{X_n}{a_n} \right) + CE \left| \frac{X_n}{a_n} \right|^r I(|X_n| \leq a_n) \\ &\leq C\alpha^{-1} E g_n \left( \frac{X_n}{a_n} \right). \end{aligned} \quad (2.10)$$

Therefore, it also follows from (2.2) that

$$\sum_{n=1}^{\infty} E \left( \frac{X_n^{a_n}}{a_n} \right)^2 < C \alpha^{-1} \sum_{n=1}^{\infty} E g_n \left( \frac{X_n}{a_n} \right) < \infty. \quad (2.11)$$

The proof of the Theorem 2.3 is completed by (2.5), (2.8), and (2.11).  $\square$

**Corollary 2.4.** *Let  $\{X_n, n \geq 1\}$  be a sequence of AANA random variables with  $\sum_{n=1}^{\infty} q^2(n) < \infty$ , and let  $\{a_n, n \geq 1\}$  be a constant sequence satisfying  $0 < a_n \uparrow \infty$ . For  $\theta \in (0, 1]$ , let  $g_n(x) = |x|^\theta / (1 + |x|^\theta)$ , and if  $\{X_n/a_n, n \geq 1\}$  satisfies (2.2), then  $a_n^{-1} \sum_{k=1}^n X_k \rightarrow 0$  a.s., as  $n \rightarrow \infty$ .*

*Proof.* It is easy to check that  $\{g_n(x), n \geq 1\}$  is a sequence of even functions in  $R^1$ , for each  $n \geq 1$ ,  $g_n(x)$  is a positive nondecreasing function in  $(0, +\infty)$ , and the following condition holds:

$$g_n(x) \geq \frac{1}{2} x^\theta \geq \frac{1}{2} x, \quad 0 < x \leq 1, \quad 0 < \theta \leq 1. \quad (2.12)$$

$\square$

### 3. Strong Convergence for the Weighted Sums of AANA Random Variables

**Theorem 3.1.** *Let  $\{X_n, n \geq 1\}$  be a different distribution sequence of AANA random variables with  $\sum_{n=1}^{\infty} q^2(n) < \infty$  and  $EX_n = 0$ , for each  $n \geq 1$ . There exists a random variable  $X$  satisfying  $E|X|^r < \infty, 0 < r \leq 2$ , such that*

$$P(|X_n| > x) \leq CP(|X| > x), \quad n \geq 1, \quad x > 0. \quad (3.1)$$

Assume that the following conditions hold for the constant arrays  $\{a_{ni}, n \geq 1, 1 \leq i \leq n\}$ .

(i)  $\max_{1 \leq i \leq n} |a_{ni}| = O(\psi^{-1}(n))$ ; (ii) for some constant  $\delta > 0$ ,  $\sum_{i=1}^n |a_{ni}|^r = O(n^{-1} \log^{-1-\delta} n)$ , where  $\phi(x), \psi(x)$  satisfy Lemma 1.5; then

$$T_n = \sum_{i=1}^n a_{ni} X_i \xrightarrow{a.s.} 0, \quad n \rightarrow \infty. \quad (3.2)$$

*Proof.* Let  $Y_i = -\psi(n)I(X_i < -\psi(n)) + X_i I(|X_i| \leq \psi(n)) + \psi(n)I(X_i > \psi(n))$ ,  $\bar{Y}_i = Y_i - EY_i$ :

$$T_n = \sum_{i=1}^n a_{ni} (X_i - Y_i) + \sum_{i=1}^n a_{ni} \bar{Y}_i + \sum_{i=1}^n a_{ni} EY_i \triangleq T_{n1} + T_{n2} + T_{n3}. \quad (3.3)$$

It suffices to prove that  $T_{ni} \rightarrow 0$  a.s., as  $n \rightarrow \infty, i = 1, 2, 3$ . We will estimate each of these terms separately.

To verify  $T_{n1} \rightarrow 0$  a.s., as  $n \rightarrow \infty$ , we can get from (3.1) and  $E\phi(|X|) < \infty$  that

$$\begin{aligned}
\sum_{n=1}^{\infty} P(X_i \neq Y_i) &= \sum_{n=1}^{\infty} P(|X_i| > \psi(n)) \\
&\leq C \sum_{n=1}^{\infty} P(|X| > \psi(n)) \\
&= C \sum_{n=1}^{\infty} P(\phi|X| > n) \\
&\leq CE\phi(|X|) < \infty.
\end{aligned} \tag{3.4}$$

Hence, by the Borel-Cantelli Lemma it is obvious that  $T_{n1} \rightarrow 0$  a.s., as  $n \rightarrow \infty$ .

Next we will show that  $T_{n2} \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. For any  $\varepsilon > 0$ ,  $0 < r \leq 2$ , note that  $E|X|^r < \infty$ , and it follows from the Markov inequality, Lemma 1.2, Lemma 1.3,  $C_r$ -inequality, and Lemma 1.5 that

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\sum_{i=1}^n a_{ni} \bar{Y}_i > \varepsilon\right) &\leq C \sum_{n=1}^{\infty} E \left| \sum_{i=1}^n a_{ni} \bar{Y}_i \right|^r \leq C \sum_{n=1}^{\infty} \sum_{i=1}^n E |a_{ni} \bar{Y}_i|^r \\
&\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n |a_{ni}|^r [E|X_i|^r I(|X_i| \leq \psi(n)) + \psi^r(n) EI(|X_i| > \psi(n))] \\
&\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n |a_{ni}|^r (E|X|^r I(|X| \leq \psi(n)) + \psi^r(n) EI(|X| > \psi(n))) \\
&\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n |a_{ni}|^r (EX|^r I(|X| \leq \psi(n)) + E|X|^r) \\
&\leq C \sum_{n=1}^{\infty} \sum_{i=1}^n |a_{ni}|^r \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{n \log^{1+\delta} n} < \infty.
\end{aligned} \tag{3.5}$$

the last series converges using condition (ii), and by Borel-Cantelli lemma we get  $T_{n2} \rightarrow 0$  a.s., as  $n \rightarrow \infty$ .

Finally we will prove that  $T_{n3} \rightarrow 0$  a.s., as  $n \rightarrow \infty$ . Note that  $EX_n = 0$ ; for each  $n \geq 1$ , it is easy to show that by Lemma 1.5, Lemma 1.4, and the Kronecker lemma

$$\begin{aligned}
\left| \sum_{i=1}^n E a_{ni} Y_i \right| &\leq \left| \sum_{i=1}^n E a_{ni} X_i I(|X_i| \leq \psi(n)) \right| + \left| \sum_{i=1}^n a_{ni} \psi(n) EI(|X_i| > \psi(n)) \right| \\
&\leq \left| \sum_{i=1}^n E a_{ni} X_i I(|X_i| > \psi(n)) \right| + \left| \sum_{i=1}^n a_{ni} \psi(n) EI(|X_i| > \psi(n)) \right| \\
&\leq C \sum_{i=1}^n E |a_{ni} X_i| I(|X_i| > \psi(i))
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{i=1}^n E|a_{ni}X|I(|X| > \psi(i)) \\
&\leq \frac{1}{\psi(n)} \sum_{i=1}^n E|X|I(|X| > \psi(i)) \rightarrow 0, n \rightarrow \infty.
\end{aligned}
\tag{3.6}$$

The proof of Theorem 3.1 is completed.  $\square$

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