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Research Article

Hybrid Algorithms for Solving Variational Inequalities, Variational Inclusions, Mixed Equilibria, and Fixed Point Problems

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We present a hybrid iterative algorithm for finding a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of solutions of a finite family of variational inequalities for inverse strong monotone mappings, the set of fixed points of an infinite family of nonexpansive mappings, and the set of solutions of a variational inclusion in a real Hilbert space. Furthermore, we prove that the proposed hybrid iterative algorithm has strong convergence under some mild conditions imposed on algorithm parameters. Here, our hybrid algorithm is based on Korpelevič's extragradient method, hybrid steepest-descent method, and viscosity approximation method.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C is a nonempty closed convex subset of H, and P_C is the metric projection of H onto C. Let $S : C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $A : C \to H$ is called ρ-Lipschitzian if there exists a constant $\rho \ge 0$ such that

$$\|Ax - Ay\| \le \rho\|x - y\|, \quad \forall x, y \in C. \tag{1}$$

In particular, if $\rho = 1$, then A is called a nonexpansive mapping [1]; if $\rho \in [0, 1)$, then A is called ρ-contraction.

Recall that a mapping $A : C \to H$ is called

(i) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \eta\|x - y\|^2, \quad \forall x, y \in C; \tag{2}$$

(ii) α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{3}$$

It is obvious that if A is α-inverse strongly monotone, then A is monotone and $1/\alpha$-Lipschitz continuous. In addition, a mapping V is called strongly positive on H if there exists a constant $\mu > 0$ such that

$$\langle Vx, x\rangle \ge \mu\|x\|^2, \quad \forall x \in H. \tag{4}$$

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, let $A : H \to H$ be a nonlinear mapping, and let $\Theta : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [2] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \ge 0, \quad \forall y \in C. \tag{5}$$

We denote the set of solutions of GMEP (5) by $\mathrm{GMEP}(\Theta, \varphi, A)$. The GMEP (5) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games, and others. The GMEP is further considered and studied in [3–8].

We present some special cases of GMEP (5) as follows.

If $\varphi = 0$, then GMEP (5) reduces to the generalized equilibrium problem (GEP) which is to find $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C, \tag{6}$$

which was introduced and studied by S. Takahashi and W. Takahashi [9]. The set of solutions of GEP is denoted by $\mathrm{GEP}(\Theta, A)$.

If $A = 0$, then GMEP (5) reduces to the mixed equilibrium problem (MEP) which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C, \tag{7}$$

which was considered and studied in [10]. The set of solutions of MEP is denoted by $\mathrm{MEP}(\Theta, \varphi)$.

If $\varphi = 0$, $A = 0$, then GMEP (5) reduces to the equilibrium problem (EP) which is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{8}$$

The set of solutions of EP is denoted by $\mathrm{EP}(\Theta)$. It is worth pointing out that the EP is a unified model of several problems, for instance, variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems, and so forth.

Throughout this paper, it is assumed as in [2] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)–(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $\Theta(x, x) = 0$ for all $x \in C$,

(A2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$,

(A3) Θ is upper hemicontinuous, that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y), \tag{9}$$

(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$,

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \backslash D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0, \tag{10}$$

(B2) C is a bounded set.

Let B be a single-valued mapping of C into H and R a multivalued mapping with $D(R) = C$. Consider the following variational inclusion: find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{11}$$

We denote by $\mathrm{I}(B, R)$ the solution set of the variational inclusion (11). In particular, if $B = R = 0$, then $\mathrm{I}(B, R) = C$. If $B = 0$, then problem (11) becomes the inclusion problem introduced by Rockafellar [11]. It is known that problem (11) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, equilibria, and game theory.

In 1998, Huang [12] studied problem (11) in the case where R is maximal monotone and B is strongly monotone and Lipschitz continuous with $D(R) = C = H$. Subsequently, Zeng et al. [13] further studied problem (11) in the case which is more general than Huang's one [12]. Moreover, the authors [13] obtained the same strong convergence conclusion as in Huang's result [12]. In addition, the authors also gave the geometric convergence rate estimate for approximate solutions. Also, various types of iterative algorithms for solving variational inclusions have been further studied and developed; for more details, refer to [14–17] and the references therein.

On the other hand, consider the following variational inequality problem (VIP): find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{12}$$

The solution set of VIP (12) is denoted by $\mathrm{VI}(C, A)$.

In 1976, Korpelevič [18] proposed an iterative algorithm for solving the VIP (12) in Euclidean space $\mathbf{R}^n$:

$$\begin{aligned} y_n &= P_C(x_n - \tau A x_n), \\ x_{n+1} &= P_C(x_n - \tau A y_n), \quad \forall n \geq 0, \end{aligned} \tag{13}$$

with $\tau > 0$ being a given number, which is known as the extragradient method (see also [19]). The literature on the VIP is vast and Korpelevič's extragradient method has received great attention given by many authors, who improved it in various ways; see, for example, [2, 5, 6, 8, 17, 20–28] and references therein, to name but a few.

VIP (12) was first discussed by Lions [29] and now is well known; there are a lot of different approaches towards solving VIP (12) in finite-dimensional and infinite-dimensional spaces, and the research is intensively continued. VIP (12) has many applications in computational mathematics, mathematical physics, operations research, mathematical economics, optimization theory, and other fields; see, for example, [30–33]. It is well known that, if A is a strongly monotone and Lipschitz-continuous mapping on C, then VIP (12) has a unique solution. Not only the existence and uniqueness of solutions are important topics in the study of VIP (12) but also how to actually find a solution of VIP (12) is important.

Let C be a nonempty closed convex subset of a real Banach space X. Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C and let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a self-mapping W_n on C as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n) I, \\ U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1}) I, \\ &\vdots \\ U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k) I, \\ U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1}) I, \\ &\vdots \\ U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2) I, \\ W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1) I. \end{aligned} \tag{14}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$; see [34].

In 2008, Ceng and Yao [35] introduced and analyzed the following relaxed viscosity approximation method for finding a common fixed point of an infinite family of nonexpansive mappings in a strictly convex and reflexive Banach space.

Theorem 1 (see [35, Theorem 3.2]). *Let X be a strictly convex and reflexive Banach space with a uniformly Gateaux differentiable norm, let C be a nonempty closed convex subset of X, let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that the common fixed point set $\cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$, and let $f : C \to C$ be a ρ-contraction with the contraction coefficient $\rho \in (1/2, 1)$. Let $\{\lambda\}_{n=1}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0,1)$. For any given $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n-\beta_n) x_n + \alpha_n f(y_n) + \beta_n W_n y_n, \\ y_n &= (1-\gamma_n) x_n + \gamma_n W_n x_n, \quad \forall n \geq 1, \end{aligned} \tag{15}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are two sequences in $(0,1)$ with $\alpha_n + \beta_n \leq 1 (n \geq 1)$, $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$, and W_n is the W-mapping defined by (14)*. Assume that*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(ii) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$ *and* $\limsup_{n\to\infty} \gamma_n < 1$.

Then there hold the following:

(I) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;

(II) *the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $q \in \cap_{n=1}^{\infty} \mathrm{Fix}(T_n)$, provided $\lim_{n\to\infty} \gamma_n = 0$ and $\beta_n \equiv \beta$ for some fixed $\beta \in (0,1)$, which is the unique solution of the* VIP:

$$\langle (I-f) q, J(q-p) \rangle, \quad \forall p \in \cap_{n=1}^{\infty} \mathrm{Fix}(T_n), \tag{16}$$

where J is the normalized duality mapping of X.

Furthermore, let $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N} \in (0,1]$, $n \geq 1$. Given the nonexpansive mappings $S_1, S_2, \ldots, S_N$ on H, Atsushiba and Takahashi [36] defined, for each $n \geq 1$, mappings $U_{n,1}, U_{n,2}, \ldots, U_{n,N}$ by

$$\begin{aligned} U_{n,1} &= \lambda_{n,1} S_1 + (1-\lambda_{n,1}) I, \\ U_{n,2} &= \lambda_{n,2} S_n U_{n,1} + (1-\lambda_{n,2}) I, \\ U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1}) I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1} S_{N-1} U_{n,N-2} + (1-\lambda_{n,N-1}) I, \\ W_n := U_{n,N} &= \lambda_{n,N} S_N U_{n,N-1} + (1-\lambda_{n,N}) I. \end{aligned} \tag{17}$$

The W_n is called the W-mapping generated by $S_1, \ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Note that the nonexpansivity of S_i implies the nonexpansivity of W_n.

In 2008, Colao et al. [37] introduced and studied an iterative method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a finite family of nonexpansive mappings in a real Hilbert space H. Subsequently, combining Yamada's hybrid steepest-descent method [38] and Colao et al.'s hybrid viscosity approximation method [37], Ceng et al. [7] proposed and analyzed the following hybrid iterative method for finding a common element of the set of solutions of GMEP (5) *and the set of fixed points of a finite family of nonexpansive mappings $\{S_i\}_{i=1}^{N}$.*

Theorem 2 (see [7, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying assumptions (A1)–(A4) and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with restriction (B1) or (B2). Let the mapping $A : H \to H$ be δ-inverse strongly monotone, and let $\{S_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings on H such that $\cap_{i=1}^{N} \mathrm{Fix}(S_i) \cap$* GMEP$(\Theta, \varphi, A) \neq \emptyset$. *Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$ and $f : H \to H$ a ρ-Lipschitzian mapping with constant $\rho \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)}$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$, $\{\gamma_n\}$ is a sequence in $(0, 2\delta]$, and $\{\lambda_{n,i}\}_{i=1}^{N}$ is a sequence in $[a,b]$ with $0 < a \leq b < 1$. For every $n \geq 1$, let W_n be the W-mapping generated by $S_1, \ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Given $x_1 \in H$ arbitrarily, suppose that the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by*

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle A x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n \mu F) W_n u_n, \quad \forall n \geq 1, \tag{18}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ *and the finite family of sequences* $\{\lambda_{n,i}\}_{i=1}^{N}$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\delta$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$;

(iv) $\lim_{n\to\infty}(\lambda_{n+1,i} - \lambda_{n,i}) = 0$ *for all* $i = 1, 2, \ldots, N$.

Then, both $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $x^* \in \cap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, \varphi, A)$, *where* $x^* = P_{\cap_{i=1}^{N}\mathrm{Fix}(S_i)\cap\mathrm{GMEP}(\Theta,\varphi,A)}(I - \mu F + \gamma f)x^*$ *is a unique solution of the variational inequality:*

$$\langle(\mu F - \gamma f) x^*, x^* - x\rangle \leq 0, \quad \forall x \in \cap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, \varphi, A). \tag{19}$$

On the other hand, whenever $C = H$ *a real Hilbert space, Yao et al. [4] very recently introduced and analyzed an iterative algorithm for finding a common element of the set of solutions of* GMEP (5), *the set of solutions of the variational inclusion* (11), *and the set of fixed points of an infinite family of nonexpansive mappings.*

Theorem 3 (see [4, Theorem 3.2]). *Let* $\varphi : H \to \mathbf{R}$ *be a lower semicontinuous and convex function and let* $\Theta : H \times H \to \mathbf{R}$ *be a bifunction satisfying conditions (A1)–(A4) and (B1). Let* V *be a strongly positive bounded linear operator with coefficient* $\mu > 0$ *and let* $R : H \to 2^H$ *be a maximal monotone mapping. Let the mappings* $A, B : H \to H$ *be* α*-inverse strongly monotone and* β*-inverse strongly monotone, respectively. Let* $f : H \to H$ *be a* ρ*-contraction. Let* $r > 0$, $\gamma > 0$, *and* $\lambda > 0$ *be three constants such that* $r < 2\alpha$, $\lambda < 2\beta$, *and* $0 < \gamma < \mu/\rho$. *Let* $\{\lambda\}_{n=1}^{\infty}$ *be a sequence of positive numbers in* $(0, b]$ *for some* $b \in (0, 1)$ *and* $\{T_n\}_{n=1}^{\infty}$ *an infinite family of nonexpansive self-mappings on* H *such that* $\Omega := \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{I}(B, R) \neq \emptyset$. *For arbitrarily given* $x_1 \in H$, *let the sequence* $\{x_n\}$ *be generated by*

$$\begin{aligned} &\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle y - u_n, Ax_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ &x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1-\beta_n) I - \alpha_n V] \times W_n J_{R,\lambda}(u_n - \lambda B u_n), \quad \forall n \geq 1, \end{aligned} \tag{20}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are two real sequences in* $[0, 1]$ *and* W_n *is the* W*-mapping defined by* (14) *(with* $X = H$ *and* $C = H$*). Assume that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Then, the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega(\gamma f(x^*) + (I - V)x^*)$ *is a unique solution of the* VIP:

$$\langle(\gamma f - V) x^*, y - x^*\rangle \leq 0, \quad \forall y \in \Omega. \tag{21}$$

Motivated and inspired by the above facts, in this paper, we introduce and analyze a hybrid iterative algorithm for finding a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of solutions of a finite family of variational inequalities for inverse strong monotone mappings, the set of fixed points of an infinite family of nonexpansive mappings, and the set of solutions of the variational inclusion (11) in a real Hilbert space. Furthermore, it is proven that the proposed hybrid iterative algorithm is strongly convergent under some mild conditions imposed on algorithm parameters. Here our hybrid algorithm is based on Korpelevič's extragradient method, hybrid steepest-descent method, and viscosity approximation method. The results obtained in this paper improve and extend the corresponding results announced by many others.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; that is,

$$\omega_w(x_n) := \left\{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \left\{x_{n_i}\right\} \text{ of } \{x_n\}\right\}. \tag{22}$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C}\|x - y\| =: d(x, C). \tag{23}$$

Some important properties of projections are gathered in the following proposition.

Proposition 4 (see [31, 39]). *For given* $x \in H$ *and* $z \in C$:

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0, \forall y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \forall y \in H$.

Consequently, P_C *is nonexpansive and monotone.*

If A is an α-inverse strongly monotone mapping (α-ism) of C into H, then it is obvious that A is $1/\alpha$-Lipschitzian. We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I-\lambda A)u-(I-\lambda A)v\|^2 &= \|(u-v)-\lambda(Au-Av)\|^2 \\ &= \|u-v\|^2-2\lambda\langle Au-Av,u-v\rangle+\lambda^2\|Au-Av\|^2 \\ &\le \|u-v\|^2+\lambda(\lambda-2\alpha)\|Au-Av\|^2.\end{aligned} \tag{24}$$

So, if $\lambda \le 2\alpha$, then $I-\lambda A$ is a nonexpansive mapping from C to H.

It is also easy to see that a projection P_C is 1-ism. Inverse strongly monotone (also referred to as cocoercive) operators have been applied widely in solving practical problems in various fields.

A set-valued mapping $R : D(R) \subset H \to 2^H$ is called monotone if, for all $x, y \in D(R)$, $f \in R(x)$ and $g \in R(y)$ imply

$$\langle f-g, x-y\rangle \ge 0. \tag{25}$$

A set-valued mapping R is called maximal monotone if R is monotone and $(I+\lambda R)D(R) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(R)$ the graph of R. It is known that a monotone mapping R is maximal if and only if, for $(x, f) \in H\times H$, $\langle f-g, x-y\rangle \ge 0$ for every $(y, g) \in G(R)$ implies $f \in R(x)$.

Let $A : C \to H$ be a monotone, k-Lipschitzian mapping and let $N_C v$ be the normal cone to C at $v \in C$; that is,

$$N_C v = \{w \in H : \langle v-u, w\rangle \ge 0, \ \forall u \in C\}. \tag{26}$$

Defining

$$Tv = \begin{cases} Av + N_C v, & if \ v \in C, \\ \emptyset, & if \ v \notin C, \end{cases} \tag{27}$$

then, T is maximal monotone and $0 \in Tv$ if and only if $v \in$ VI (C, A); see [11].

Assume that $R : D(R) \subset H \to 2^H$ is a maximal monotone mapping. Then, for $\lambda > 0$, associated with R, the resolvent operator $J_{R,\lambda}$ can be defined as

$$J_{R,\lambda}x = (I+\lambda R)^{-1}x, \quad \forall x \in H. \tag{28}$$

In terms of Huang [12] (see also [13]), there holds the following property for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 5. *$J_{R,\lambda}$ is single-valued and firmly nonexpansive; that is,*

$$\langle J_{R,\lambda}x - J_{R,\lambda}y, x-y\rangle \ge \|J_{R,\lambda}x - J_{R,\lambda}y\|^2, \quad \forall x, y \in H. \tag{29}$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 6. *Let C be a nonempty closed convex subset of H and $A : C \to H$ a monotone mapping. In the context of the* VIP *(12), there holds the following relation:*

$$u \in \mathrm{VI}(C, A) \Longleftrightarrow u = P_C(u-\lambda Au), \quad for \ some \ \lambda > 0. \tag{30}$$

Lemma 7 (see [17]). *Let R be a maximal monotone mapping with $D(R) = C$. Then, for any given $\lambda > 0$, $u \in C$ is a solution of problem (11) if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda}(u-\lambda Bu). \tag{31}$$

Lemma 8 (see [13]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous, and single-valued mapping. Then, for each $z \in H$, the equation $z \in (B+\lambda R)x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 9 (see [17]). *Let R be a maximal monotone mapping with $D(R) = C$ and $B : C \to H$ a monotone, continuous, and single-valued mapping. Then $(I+\lambda(R+B))C = H$ for each $\lambda > 0$. In this case, $R+B$ is maximal monotone.*

Lemma 10 ([40], see also [10]). *Assume that $\Theta : C\times C \to$ **R** satisfies (A1)–(A4) and let $\varphi : C \to$ **R** be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$\begin{aligned}T_r^{(\Theta,\varphi)}(x) = \Big\{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) \\ + \frac{1}{r}\langle y-z, z-x\rangle \ge 0, \ \forall y \in C\Big\}\end{aligned} \tag{32}$$

for all $x \in H$. Then the following hold:

(i) *for each $x \in H$, $T_r^{(\Theta,\varphi)}(x) \ne \emptyset$;*

(ii) *$T_r^{(\Theta,\varphi)}$ is single-valued;*

(iii) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\right\|^2 \le \langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x-y\rangle; \tag{33}$$

(iv) *$\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;*

(v) *$\mathrm{MEP}(\Theta, \varphi)$ is closed and convex.*

Proposition 11 (see [5, Proposition 2.1]). *Let C, H, Θ, φ, and $T_r^{(\Theta,\varphi)}$ be as in Lemma 10. Then the following inequality holds:*

$$\left\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\right\|^2 \le \frac{s-t}{s}\left\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\right\rangle, \tag{34}$$

for all $s, t > 0$ and $x \in H$.

Remark 12. From the conclusion of Proposition 11, it immediately follows that

$$\left\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\right\| \le \frac{|s-t|}{s}\left\|T_s^{(\Theta,\varphi)}x - x\right\| \tag{35}$$

for all $s, t > 0$ and $x \in H$.

Lemma 13 (see [41]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ a sequence in $[0, 1]$ with*

$$0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1. \tag{36}$$

Suppose that $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ *for each* $n \geq 1$ *and*

$$\limsup_{n\to\infty} \left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\right) \leq 0. \quad (37)$$

Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

We have the following crucial lemmas concerning the W-mapping defined by (14).

Lemma 14 (see [42, Lemma 3.2]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Lemma 15 (see [42, Lemma 3.3]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0,1)$. Then, $\mathrm{Fix}(W) = \cap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.*

Remark 16. Using Lemma 14, we can define the mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C. \quad (38)$$

Such a W is called the W-mapping generated by the sequences $\{T_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$. As pointed out in [43], if $\{x_n\}$ is a bounded sequence in C, then we have

$$\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0. \quad (39)$$

Throughout this paper, we always assume that $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers in $(0, b]$ for some $b \in (0,1)$.

Lemma 17 (see [44]). *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the conditions*

$$s_{n+1} \leq (1-\alpha_n) s_n + \alpha_n \beta_n, \quad \forall n \geq 1, \quad (40)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *or equivalently,*

$$\prod_{n=1}^{\infty} (1-\alpha_n) := \lim_{n\to\infty} \prod_{k=1}^{n} (1-\alpha_k) = 0, \quad (41)$$

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$, *or* $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 18 (see [39, demiclosedness principle]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C with $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

The following lemma is an immediate consequence of an inner product.

Lemma 19. *In a real Hilbert space H, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H. \quad (42)$$

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to C$, we define the mapping $T^{\lambda} : C \to H$ by

$$T^{\lambda}x := Tx - \lambda\mu F(Tx), \quad \forall x \in C, \quad (43)$$

where $F : C \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on C; that is, F satisfies the conditions

$$\|Fx - Fy\| \leq \kappa\|x-y\|, \qquad \langle Fx - Fy, x - y\rangle \geq \eta\|x-y\|^2 \quad (44)$$

for all $x, y \in C$.

Lemma 20 (see [44, Lemma 3.1]). *T^{λ} is a contraction provided $0 < \mu < 2\eta/\kappa^2$; that is,*

$$\left\|T^{\lambda}x - T^{\lambda}y\right\| \leq (1-\lambda\tau)\|x-y\|, \quad \forall x, y \in C, \quad (45)$$

where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)} \in (0,1]$.

Remark 21. (i) Since F is κ-Lipschitzian and η-strongly monotone on C, we get $0 < \eta \leq \kappa$. Hence, whenever $0 < \mu < 2\eta/\kappa^2$, we have

$$\begin{aligned} 0 &\leq (1-\mu\eta)^2 = 1 - 2\mu\eta + \mu^2\eta^2 \\ &\leq 1 - 2\mu\eta + \mu^2\kappa^2 \\ &< 1 - 2\mu\eta + \frac{2\eta}{\kappa^2}\mu\kappa^2 = 1, \end{aligned} \quad (46)$$

which implies

$$0 < 1 - \sqrt{1 - 2\mu\eta + \mu^2\kappa^2} \leq 1. \quad (47)$$

So, $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)} \in (0,1]$.

(ii) In Lemma 20, put $F = (1/2)I$ and $\mu = 2$. Then we know that $\kappa = \eta = 1/2$, $0 < \mu = 2 < 2\eta/\kappa^2 = 4$ and

$$\begin{aligned} \tau &= 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)} \\ &= 1 - \sqrt{1 - 2\left(2\times\frac{1}{2} - 2\times\left(\frac{1}{2}\right)^2\right)} = 1. \end{aligned} \quad (48)$$

3. A Strong Convergence Theorem

In this section, we will prove a strong convergence theorem for a hybrid iterative algorithm for finding a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of solutions of a finite family of variational inequalities for inverse strong monotone mappings, the set of fixed points of an infinite family of nonexpansive mappings, and the set of solutions of the variational inclusion (11) in a real Hilbert space.

Theorem 22. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let M, N be two integers. Let Θ_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \ldots, M\}$. Let $B_k : H \to H$ and $A_i : C \to H$ be μ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $k \in \{1, 2, \ldots, M\}$, $i \in \{1, 2, \ldots, N\}$. Let $F : C \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$ and let $f : H \to H$ be a ρ-Lipschitzian mapping with constant $\rho \geq 0$. Let $R : C \to 2^H$ be a maximal monotone mapping and let the mapping $B : C \to H$ be β-inverse strongly monotone. Let $0 < \lambda < 2\beta$, $0 < \mu < 2\eta/\kappa^2$, and $0 \leq \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$ and $\{T_n\}_{n=1}^{\infty}$ an infinite family of nonexpansive self-mappings on C such that $\Omega := \cap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, B_k) \cap \cap_{i=1}^{N} \mathrm{VI}(C, A_i) \cap \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{I}(B, R) \neq \emptyset$. For arbitrarily given $x_1 \in H$, let the sequence $\{x_n\}$ be generated by*

$$\begin{aligned} u_n &= T_{r_{M,n}}^{(\Theta_M, \varphi_M)} (I - r_{M,n} B_M) T_{r_{M,n}}^{(\Theta_{M-1}, \varphi_{M-1})} (I - r_{M-1,n} B_{M-1}) \\ &\quad \times \cdots T_{r_{1,n}}^{(\Theta_1, \varphi_1)} (I - r_{1,n} B_1) x_n, \\ v_n &= P_C (I - \lambda_{N,n} A_N) P_C (I - \lambda_{N-1,n} A_{N-1}) \\ &\quad \times \cdots P_C (I - \lambda_{2,n} A_2) P_C (I - \lambda_{1,n} A_1) u_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n) I - \alpha_n \mu F] \\ &\quad \times W_n J_{R,\lambda} (v_n - \lambda B v_n), \quad \forall n \geq 1, \end{aligned} \tag{49}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0, 1]$ and W_n is the W-mapping defined by (14). Assume that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ *and* $\lim_{n\to\infty} |\lambda_{i,n+1} - \lambda_{i,n}| = 0$ *for all* $i \in \{1, 2, \ldots, N\}$;

(iv) $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ *and* $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$ *for all* $k \in \{1, 2, \ldots, M\}$.

Assume that either (B1) or (B2) holds. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega(I - \mu F + \gamma f)x^*$ is a unique solution of the VIP:

$$\langle (\mu F - \gamma f) x^*, x^* - y \rangle \leq 0, \quad \forall y \in \Omega. \tag{50}$$

Proof. Let $Q = P_\Omega$. Note that $F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$ and $f : H \to H$ is a ρ-Lipschitzian mapping with constant $\rho \geq 0$. Then, we have

$$\begin{aligned} &\|(I - \mu F)x - (I - \mu F)y\|^2 \\ &\quad = \|x - y\|^2 - 2\mu\langle x - y, Fx - Fy \rangle + \mu^2 \|Fx - Fy\|^2 \\ &\quad \leq (1 - 2\mu\eta + \mu^2\kappa^2) \|x - y\|^2 \\ &\quad = (1 - \tau)^2 \|x - y\|^2, \end{aligned} \tag{51}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$, and hence

$$\begin{aligned} &\|Q(I - \mu F + \gamma f)x - Q(I - \mu F + \gamma f)y\| \\ &\quad \leq \|(I - \mu F + \gamma f)x - (I - \mu F + \gamma f)y\| \\ &\quad \leq \|(I - \mu F)x - (I - \mu F)y\| + \gamma \|f(x) - f(y)\| \\ &\quad \leq (1 - \tau)\|x - y\| + \gamma\rho \|x - y\| \\ &\quad = (1 - (\tau - \gamma\rho)) \|x - y\|, \quad \forall x, y \in C. \end{aligned} \tag{52}$$

Since $0 \leq \gamma\rho < \tau \leq 1$, it is known that $1 - (\tau - \gamma\rho) \in [0, 1)$. Therefore, $Q(I - \mu F + \gamma f)$ is a contraction of C into itself, which implies that there exists a unique element $x^* \in C$ such that $x^* = Q(I - \mu F + \gamma f)x^* = P_\Omega(I - \mu F + \gamma f)x^*$.

We divide the remainder of the proof into several steps.

Step 1. Let us show that $\{x_n\}$ is bounded.

Indeed, taking into account the control conditions (i) and (ii), we may assume, without loss of generality, that $\alpha_n \leq 1 - \beta_n$ for all $n \geq 1$. Put

$$\begin{aligned} \Delta_n^k &= T_{r_{k,n}}^{(\Theta_k, \varphi_k)} (I - r_{k,n} B_M) \\ &\quad \times T_{r_{k-1,n}}^{(\Theta_{k-1}, \varphi_{k-1})} (I - r_{k-1,n} B_{k-1}) \cdots T_{r_{1,n}}^{(\Theta_1, \varphi_1)} (I - r_{1,n} B_1) x_n \end{aligned} \tag{53}$$

for all $k \in \{1, 2, \ldots, M\}$ and $n \geq 1$,

$$\begin{aligned} \Lambda_n^i &= P_C (I - \lambda_{i,n} A_i) P_C (I - \lambda_{i-1,n} A_{i-1}) \\ &\quad \times \cdots P_C (I - \lambda_{2,n} A_2) P_C (I - \lambda_{1,n} A_1) \end{aligned} \tag{54}$$

for all $i \in \{1, 2, \ldots, N\}$ and $n \geq 1$, and $\Delta_n^0 = \Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta_n^M x_n$

and $v_n = \Lambda_n^N u_n$. Take $p \in \Omega$ arbitrarily. Then from (24) and Lemma 10 we have

$$\begin{aligned}\|u_n - p\| &= \Big\| T_{r_{M,n}}^{(\Theta_M,\varphi_M)} (I - r_{M,n} B_M) \Delta_n^{M-1} x_n \\ &\quad - T_{r_{M,n}}^{(\Theta_M,\varphi_M)} (I - r_{M,n} B_M) \Delta_n^{M-1} p \Big\| \\ &\le \left\| (I - r_{M,n} B_M) \Delta_n^{M-1} x_n - (I - r_{M,n} B_M) \Delta_n^{M-1} p \right\| \\ &\le \left\| \Delta_n^{M-1} x_n - \Delta_n^{M-1} p \right\| \\ &\ \ \vdots \\ &\le \left\| \Delta_n^0 x_n - \Delta_n^0 p \right\| \\ &= \|x_n - p\|. \end{aligned} \tag{55}$$

Similarly, we have

$$\begin{aligned}&\|v_n - p\| \\ &\quad = \left\| P_C (I - \lambda_{N,n} A_N) \Lambda_n^{N-1} u_n - P_C (I - \lambda_{N,n} A_N) \Lambda_n^{N-1} p \right\| \\ &\quad \le \left\| (I - \lambda_{N,n} A_N) \Lambda_n^{N-1} u_n - (I - \lambda_{N,n} A_N) \Lambda_n^{N-1} p \right\| \\ &\quad \le \left\| \Lambda_n^{N-1} u_n - \Lambda_n^{N-1} p \right\| \\ &\quad\ \ \vdots \\ &\quad \le \left\| \Lambda_n^0 u_n - \Lambda_n^0 p \right\| \\ &\quad = \|u_n - p\|. \end{aligned} \tag{56}$$

Combining (55) and (56), we have

$$\|v_n - p\| \le \|x_n - p\|. \tag{57}$$

Since the mapping $B : C \to H$ is β-inverse strongly monotone with $0 < \lambda < 2\beta$, we have

$$\|(I - \lambda B)x - (I - \lambda B)y\|^2 \le \|x - y\|^2 + \lambda (\lambda - 2\beta) \|Bx - By\|^2. \tag{58}$$

It is clear that, if $0 \le \lambda \le 2\beta$, then $I - \lambda B$ is nonexpansive. Set $y_n = J_{R,\lambda}(v_n - \lambda B v_n)$ for each $n \ge 1$. It follows that

$$\begin{aligned}\|y_n - p\| &= \|J_{R,\lambda}(v_n - \lambda B v_n) - J_{R,\lambda}(p - \lambda B p)\| \\ &\le \|(v_n - \lambda B v_n) - (p - \lambda B p)\| \\ &\le \|v_n - p\|, \end{aligned} \tag{59}$$

which, together with (57), yields

$$\|y_n - p\| \le \|x_n - p\|. \tag{60}$$

Utilizing Lemma 20, from (49) we obtain

$$\begin{aligned}&\|x_{n+1} - p\| \\ &\quad = \|\alpha_n (\gamma f(x_n) - \mu F p) + \beta_n (x_n - p) + ((1 - \beta_n) I - \alpha_n \mu F) \\ &\qquad \times W_n y_n - ((1 - \beta_n) I - \alpha_n \mu F) W_n p\| \\ &\quad \le \alpha_n \|\gamma f(x_n) - \mu F p\| + \beta_n \|x_n - p\| \\ &\qquad + \|((1 - \beta_n) I - \alpha_n \mu F) W_n y_n - ((1 - \beta_n) I - \alpha_n \mu F) W_n p\| \\ &\quad = \alpha_n \|\gamma f(x_n) - \mu F p\| + \beta_n \|x_n - p\| + (1 - \beta_n) \\ &\qquad \times \left\| \left(I - \frac{\alpha_n}{1 - \beta_n} \mu F\right) W_n y_n - \left(I - \frac{\alpha_n}{1 - \beta_n} \mu F\right) W_n p \right\| \\ &\quad \le (1 - \beta_n) \left(1 - \frac{\alpha_n \tau}{1 - \beta_n}\right) \|y_n - p\| + \beta_n \|x_n - p\| \\ &\qquad + \alpha_n \|\gamma f(x_n) - \mu F p\| \\ &\quad = (1 - \beta_n - \alpha_n \tau) \|y_n - p\| + \beta_n \|x_n - p\| \\ &\qquad + \alpha_n \|\gamma f(x_n) - \mu F p\| \\ &\quad \le (1 - \alpha_n \tau) \|x_n - p\| + \alpha_n \gamma \|f(x_n) - f(p)\| \\ &\qquad + \alpha_n \|\gamma f(p) - \mu F p\| \\ &\quad \le (1 - \alpha_n \tau) \|x_n - p\| + \alpha_n \gamma \rho \|x_n - p\| \\ &\qquad + \alpha_n \|\gamma f(p) - \mu F p\| \\ &\quad = (1 - \alpha_n (\tau - \gamma \rho)) \|x_n - p\| + \alpha_n \|\gamma f(p) - \mu F p\| \\ &\quad = (1 - \alpha_n (\tau - \gamma \rho)) \|x_n - p\| + \alpha_n (\tau - \gamma \rho) \frac{\|\gamma f(p) - \mu F p\|}{\tau - \gamma \rho} \\ &\quad \le \max \left\{ \|x_n - p\|, \frac{\|\gamma f(p) - \mu F p\|}{\tau - \gamma \rho} \right\}. \end{aligned} \tag{61}$$

By induction, we get

$$\|x_n - p\| \le \max \left\{ \|x_1 - p\|, \frac{\|\gamma f(p) - \mu F p\|}{\tau - \gamma \rho} \right\}, \quad \forall n \ge 1. \tag{62}$$

Therefore, $\{x_n\}$ is bounded and hence $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, $\{W_n y_n\}$, $\{F W_n y_n\}$, and $\{f(x_n)\}$ are also bounded.

Step 2. Let us show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Indeed, define $x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$ for each $n \ge 1$. Then from the definition of z_n we obtain

$$\begin{aligned}&z_{n+1} - z_n \\ &\quad = \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + [(1-\beta_{n+1})I - \alpha_{n+1}\mu F] W_{n+1} y_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n \gamma f(x_n) + [(1-\beta_n)I - \alpha_n \mu F] W_n y_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}} \gamma f(x_{n+1}) - \frac{\alpha_n}{1-\beta_n} \gamma f(x_n) + W_{n+1} y_{n+1} \\
&\quad - W_n y_n + \frac{\alpha_n}{1-\beta_n} \mu F W_n y_n - \frac{\alpha_{n+1}}{1-\beta_{n+1}} \mu F W_{n+1} y_{n+1} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}} [\gamma f(x_{n+1}) - \mu F W_{n+1} y_{n+1}] \\
&\quad + \frac{\alpha_n}{1-\beta_n} [\mu F W_n y_n - \gamma f(x_n)] \\
&\quad + W_{n+1} y_{n+1} - W_{n+1} y_n + W_{n+1} y_n - W_n y_n.
\end{aligned}
\tag{63}
$$

It follows that

$$
\begin{aligned}
&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\
&\quad \le \frac{\alpha_{n+1}}{1-\beta_{n+1}} (\gamma \|f(x_{n+1})\| + \mu \|F W_{n+1} y_{n+1}\|) \\
&\qquad + \frac{\alpha_n}{1-\beta_n} (\mu \|F W_n y_n\| + \gamma \|f(x_n)\|) \\
&\qquad + \|W_{n+1} y_{n+1} - W_{n+1} y_n\| + \|W_{n+1} y_n - W_n y_n\| \\
&\qquad - \|x_{n+1} - x_n\| \\
&\quad \le \frac{\alpha_{n+1}}{1-\beta_{n+1}} (\gamma \|f(x_{n+1})\| + \mu \|F W_{n+1} y_{n+1}\|) \\
&\qquad + \frac{\alpha_n}{1-\beta_n} (\mu \|F W_n y_n\| + \gamma \|f(x_n)\|) + \|W_{n+1} y_n - W_n y_n\| \\
&\qquad + \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|.
\end{aligned}
\tag{64}
$$

Utilizing (14) and the nonexpansivity of T_i and $U_{n,i}$, we obtain that for each $n \ge 1$

$$
\begin{aligned}
\|W_{n+1} y_n - W_n y_n\| &= \|\lambda_1 T_1 U_{n+1,1} y_n - \lambda_1 T_1 U_{n,1} y_n\| \\
&\le \lambda_1 \|U_{n+1,1} y_n - U_{n,1} y_n\| \\
&= \lambda_1 \|\lambda_2 T_2 U_{n+1,2} y_n - \lambda_2 T_2 U_{n,2} y_n\| \\
&\le \lambda_1 \lambda_2 \|U_{n+1,2} y_n - U_{n,2} y_n\| \\
&\;\;\vdots \\
&\le \left(\prod_{i=1}^{n} \lambda_i\right) \|U_{n+1,n+1} y_n - U_{n,n+1} y_n\| \\
&\le \widetilde{M} \prod_{i=1}^{n} \lambda_i,
\end{aligned}
\tag{65}
$$

where $\sup_{n\ge1} \|U_{n+1,n+1} y_n - U_{n,n+1} y_n\| \le \widetilde{M}$ for some $\widetilde{M} > 0$. Note that

$$
\begin{aligned}
\|y_{n+1} - y_n\| &= \|J_{R,\lambda}(v_{n+1} - \lambda B v_{n+1}) - J_{R,\lambda}(v_n - \lambda B v_n)\| \\
&\le \|(v_{n+1} - \lambda B v_{n+1}) - (v_n - \lambda B v_n)\| \\
&\le \|v_{n+1} - v_n\|.
\end{aligned}
\tag{66}
$$

Note that

$$
\begin{aligned}
&\|v_{n+1} - v_n\| \\
&\quad = \|\Lambda_{n+1}^N u_{n+1} - \Lambda_n^N u_n\| \\
&\quad = \|P_C(I - \lambda_{N,n+1} A_N) \Lambda_{n+1}^{N-1} u_{n+1} \\
&\qquad - P_C(I - \lambda_{N,n} A_N) \Lambda_n^{N-1} u_n\| \\
&\quad \le \|P_C(I - \lambda_{N,n+1} A_N) \Lambda_{n+1}^{N-1} u_{n+1} \\
&\qquad - P_C(I - \lambda_{N,n} A_N) \Lambda_{n+1}^{N-1} u_{n+1}\| \\
&\qquad + \|P_C(I - \lambda_{N,n} A_N) \Lambda_{n+1}^{N-1} u_{n+1} \\
&\qquad - P_C(I - \lambda_{N,n} A_N) \Lambda_n^{N-1} u_n\| \\
&\quad \le \|(I - \lambda_{N,n+1} A_N) \Lambda_{n+1}^{N-1} u_{n+1} - (I - \lambda_{N,n} A_N) \Lambda_{n+1}^{N-1} u_{n+1}\| \\
&\qquad + \|(I - \lambda_{N,n} A_N) \Lambda_{n+1}^{N-1} u_{n+1} - (I - \lambda_{N,n} A_N) \Lambda_n^{N-1} u_n\| \\
&\quad \le |\lambda_{N,n+1} - \lambda_{N,n}| \|A_N \Lambda_{n+1}^{N-1} u_{n+1}\| \\
&\qquad + \|\Lambda_{n+1}^{N-1} u_{n+1} - \Lambda_n^{N-1} u_n\| \\
&\quad \le |\lambda_{N,n+1} - \lambda_{N,n}| \|A_N \Lambda_{n+1}^{N-1} u_{n+1}\| + |\lambda_{N-1,n+1} - \lambda_{N-1,n}| \\
&\qquad \times \|A_{N-1} \Lambda_{n+1}^{N-2} u_{n+1}\| + \|\Lambda_{n+1}^{N-2} u_{n+1} - \Lambda_n^{N-2} u_n\| \\
&\qquad \vdots \\
&\quad \le |\lambda_{N,n+1} - \lambda_{N,n}| \|A_N \Lambda_{n+1}^{N-1} u_{n+1}\| + |\lambda_{N-1,n+1} - \lambda_{N-1,n}| \\
&\qquad \times \|A_{N-1} \Lambda_{n+1}^{N-2} u_{n+1}\| + \cdots + |\lambda_{1,n+1} - \lambda_{1,n}| \\
&\qquad \times \|A_1 \Lambda_{n+1}^0 u_{n+1}\| + \|\Lambda_{n+1}^0 u_{n+1} - \Lambda_n^0 u_n\| \\
&\quad \le \widetilde{M}_0 \sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \|u_{n+1} - u_n\|,
\end{aligned}
\tag{67}
$$

where $\sup_{n\ge1} \{\sum_{i=1}^{N} \|A_i \Lambda_{n+1}^{i-1} u_{n+1}\|\} \le \widetilde{M}_0$ for some $\widetilde{M}_0 > 0$. Also, utilizing Lemma 10 and Proposition 11 we deduce that

$$
\begin{aligned}
&\|u_{n+1} - u_n\| \\
&\quad = \|\Delta_{n+1}^M x_{n+1} - \Delta_n^M x_n\|
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n+1}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right.\\
&\qquad \left. - T_{r_{M,n}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n}B_M\right) \Delta_{n}^{M-1} x_{n} \right\| \\
&\le \left\| T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n+1}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right.\\
&\qquad \left. - T_{r_{M,n}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right\| \\
&\quad + \left\| T_{r_{M,n}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right.\\
&\qquad \left. - T_{r_{M,n}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n}B_M\right) \Delta_{n}^{M-1} x_{n} \right\| \\
&\le \left\| T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n+1}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right.\\
&\qquad \left. - T_{r_{M,n}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n+1}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right\| \\
&\quad + \left\| T_{r_{M,n}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n+1}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right.\\
&\qquad \left. - T_{r_{M,n}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right\| \\
&\quad + \left\| \left(I - r_{M,n}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} - \left(I - r_{M,n}B_M\right) \Delta_{n}^{M-1} x_{n} \right\| \\
&\le \frac{\left|r_{M,n+1} - r_{M,n}\right|}{r_{M,n+1}} \left\| T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n+1}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right.\\
&\qquad \left. - \left(I - r_{M,n+1}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right\| \\
&\quad + \left|r_{M,n+1} - r_{M,n}\right| \left\| B_M \Delta_{n+1}^{M-1} x_{n+1} \right\| + \left\| \Delta_{n+1}^{M-1} x_{n+1} - \Delta_{n}^{M-1} x_{n} \right\| \\
&= \left|r_{M,n+1} - r_{M,n}\right| \left[\left\| B_M \Delta_{n+1}^{M-1} x_{n+1} \right\| + \frac{1}{r_{M,n+1}} \right.\\
&\qquad \times \left\| T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n+1}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right.\\
&\qquad \left.\left. - \left(I - r_{M,n+1}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right\| \right] \\
&\quad + \left\| \Delta_{n+1}^{M-1} x_{n+1} - \Delta_{n}^{M-1} x_{n} \right\| \\
&\vdots \\
&\le \left|r_{M,n+1} - r_{M,n}\right| \left[\left\| B_M \Delta_{n+1}^{M-1} x_{n+1} \right\| + \frac{1}{r_{M,n+1}} \right.\\
&\qquad \times \left\| T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n+1}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right.\\
&\qquad \left.\left. - \left(I - r_{M,n+1}B_M\right) \Delta_{n+1}^{M-1} x_{n+1} \right\| \right] \\
&\quad + \cdots + \left|r_{1,n+1} - r_{1,n}\right| \left[\left\| B_1 \Delta_{n+1}^{0} x_{n+1} \right\| + \frac{1}{r_{1,n+1}} \right.\\
&\qquad \times \left\| T_{r_{1,n+1}}^{(\Theta_1,\varphi_1)} \left(I - r_{1,n+1}B_1\right) \Delta_{n+1}^{0} x_{n+1} \right.\\
&\qquad \left.\left. - \left(I - r_{1,n+1}B_1\right) \Delta_{n+1}^{0} x_{n+1} \right\| \right] \\
&\quad + \left\| \Delta_{n+1}^{0} x_{n+1} - \Delta_{n}^{0} x_{n} \right\| \\
&\le \widetilde{M}_1 \sum_{k=1}^{M} \left|r_{k,n+1} - r_{k,n}\right| + \left\| x_{n+1} - x_n \right\|,
\end{aligned}
\tag{68}
$$

where $\widetilde{M}_1 > 0$ is a constant such that for each $n \ge 1$

$$
\begin{aligned}
\sum_{k=1}^{M} &\left[\left\| B_k \Delta_{n+1}^{k-1} x_{n+1} \right\| \right.\\
&+ \frac{1}{r_{k,n+1}} \left\| T_{r_{k,n+1}}^{(\Theta_k,\varphi_k)} \left(I - r_{k,n+1}B_k\right) \Delta_{n+1}^{k-1} x_{n+1} \right.\\
&\left.\left. - \left(I - r_{k,n+1}B_k\right) \Delta_{n+1}^{k-1} x_{n+1} \right\| \right] \le \widetilde{M}_1.
\end{aligned}
\tag{69}
$$

Combining (64)–(68), we get

$$
\begin{aligned}
&\left\| z_{n+1} - z_n \right\| - \left\| x_{n+1} - x_n \right\| \\
&\quad \le \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left(\gamma \left\| f\left(x_{n+1}\right) \right\| + \mu \left\| F W_{n+1} y_{n+1} \right\| \right) \\
&\qquad + \frac{\alpha_n}{1-\beta_n} \left(\mu \left\| F W_n y_n \right\| + \gamma \left\| f\left(x_n\right) \right\| \right) + \left\| W_{n+1} y_n - W_n y_n \right\| \\
&\qquad + \left\| y_{n+1} - y_n \right\| - \left\| x_{n+1} - x_n \right\| \\
&\quad \le \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left(\gamma \left\| f\left(x_{n+1}\right) \right\| + \mu \left\| F W_{n+1} y_{n+1} \right\| \right) \\
&\qquad + \frac{\alpha_n}{1-\beta_n} \left(\mu \left\| F W_n y_n \right\| + \gamma \left\| f\left(x_n\right) \right\| \right) + \widetilde{M} \prod_{i=1}^{n} \lambda_i \\
&\qquad + \left\| v_{n+1} - v_n \right\| - \left\| x_{n+1} - x_n \right\| \\
&\quad \le \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left(\gamma \left\| f\left(x_{n+1}\right) \right\| + \mu \left\| F W_{n+1} y_{n+1} \right\| \right) \\
&\qquad + \frac{\alpha_n}{1-\beta_n} \left(\mu \left\| F W_n y_n \right\| + \gamma \left\| f\left(x_n\right) \right\| \right) + \widetilde{M} \prod_{i=1}^{n} \lambda_i \\
&\qquad + \widetilde{M}_0 \sum_{i=1}^{N} \left| \lambda_{i,n+1} - \lambda_{i,n} \right| + \left\| u_{n+1} - u_n \right\| - \left\| x_{n+1} - x_n \right\| \\
&\quad \le \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left(\gamma \left\| f\left(x_{n+1}\right) \right\| + \mu \left\| F W_{n+1} y_{n+1} \right\| \right) \\
&\qquad + \frac{\alpha_n}{1-\beta_n} \left(\mu \left\| F W_n y_n \right\| + \gamma \left\| f\left(x_n\right) \right\| \right) + \widetilde{M} \prod_{i=1}^{n} \lambda_i \\
&\qquad + \widetilde{M}_0 \sum_{i=1}^{N} \left| \lambda_{i,n+1} - \lambda_{i,n} \right| + \widetilde{M}_1 \sum_{k=1}^{M} \left| r_{k,n+1} - r_{k,n} \right|.
\end{aligned}
\tag{70}
$$

Consequently, it follows from (70), $\{\lambda_n\} \subset (0,b]$, and conditions (i)–(iv) that

$$\limsup_{n\to\infty}\left(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\right)\leq 0. \tag{71}$$

Hence, by Lemma 13 we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{72}$$

Consequently

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \tag{73}$$

Step 3. Let us show that $\|Bv_n-Bp\|\to 0$, $\|B_k\Delta_n^{k-1}x_n-B_kp\|\to 0$, and $\|A_i\Lambda_n^{i-1}u_n-A_ip\|\to 0$, $k\in\{1,2,\ldots,M\}$, $i\in\{1,2,\ldots,N\}$.

Indeed, we can rewrite (49) as follows:

$$x_{n+1}=\alpha_n(\gamma f(x_n)-\mu FW_ny_n)+\beta_n(x_n-W_ny_n)+W_ny_n. \tag{74}$$

It follows that

$$\begin{aligned}\|x_n-W_ny_n\|&\leq\|x_n-x_{n+1}\|+\|x_{n+1}-W_ny_n\|\\&\leq\|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-\mu FW_ny_n\|\\&\quad+\beta_n\|x_n-W_ny_n\|;\end{aligned} \tag{75}$$

that is,

$$\begin{aligned}\|x_n-W_ny_n\|\leq{}&\frac{1}{1-\beta_n}\|x_n-x_{n+1}\|\\&+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-\mu FW_ny_n\|.\end{aligned} \tag{76}$$

This, together with $\alpha_n\to 0$ and (73), implies that

$$\lim_{n\to\infty}\|x_n-W_ny_n\|=0. \tag{77}$$

Also, from (24) it follows that for all $i\in\{1,2,\ldots,N\}$ and $k\in\{1,2,\ldots,M\}$

$$\begin{aligned}&\|v_n-p\|\\&=\left\|\Lambda_n^Nu_n-p\right\|^2\leq\left\|\Lambda_n^iu_n-p\right\|^2\\&=\left\|P_C(I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n-P_C(I-\lambda_{i,n}A_i)p\right\|^2\\&\leq\left\|(I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n-(I-\lambda_{i,n}A_i)p\right\|^2\\&\leq\left\|\Lambda_n^{i-1}u_n-p\right\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|^2\\&\leq\|u_n-p\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|^2\\&\leq\|x_n-p\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|^2,\\&\|u_n-p\|\\&=\left\|\Delta_n^Mx_n-p\right\|^2\leq\left\|\Delta_n^kx_n-p\right\|^2\\&=\left\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}B_k)\Delta_n^{k-1}x_n-T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}B_k)p\right\|^2\\&\leq\left\|(I-r_{k,n}B_k)\Delta_n^{k-1}x_n-(I-r_{k,n}B_k)p\right\|^2\\&\leq\left\|\Delta_n^{k-1}x_n-p\right\|^2+r_{k,n}(r_{k,n}-2\mu_k)\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\|^2\\&\leq\|x_n-p\|^2+r_{k,n}(r_{k,n}-2\mu_k)\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\|^2.\end{aligned} \tag{78}$$

So, from (57), (58), and (78), it follows that

$$\begin{aligned}\|y_n-p\|^2&=\|J_{R,\lambda}(v_n-\lambda Bv_n)-J_{R,\lambda}(p-\lambda Bp)\|^2\\&\leq\|(v_n-\lambda Bv_n)-(p-\lambda Bp)\|^2\\&\leq\|v_n-p\|^2+\lambda(\lambda-2\beta)\|Bv_n-Bp\|^2\\&\leq\|u_n-p\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|^2\\&\quad+\lambda(\lambda-2\beta)\|Bv_n-Bp\|^2\\&\leq\|x_n-p\|^2+r_{k,n}(r_{k,n}-2\mu_k)\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\|^2\\&\quad+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|^2\\&\quad+\lambda(\lambda-2\beta)\|Bv_n-Bp\|^2.\end{aligned} \tag{79}$$

By (49) and Lemma 20, we obtain

$$\begin{aligned}&\|x_{n+1}-p\|^2\\&=\|\alpha_n(\gamma f(x_n)-\mu Fp)+\beta_n(x_n-W_ny_n)\\&\quad+(I-\alpha_n\mu F)W_ny_n-(I-\alpha_n\mu F)p\|^2\\&\leq\|\beta_n(x_n-W_ny_n)+(I-\alpha_n\mu F)W_ny_n-(I-\alpha_n\mu F)p\|^2\\&\quad+2\alpha_n\langle\gamma f(x_n)-\mu Fp,x_{n+1}-p\rangle\\&\leq[\beta_n\|x_n-W_ny_n\|\\&\quad+\|(I-\alpha_n\mu F)W_ny_n-(I-\alpha_n\mu F)W_np\|]^2\\&\quad+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|\\&\leq[\beta_n\|x_n-W_ny_n\|+(1-\alpha_n\tau)\|y_n-p\|]^2\\&\quad+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|\end{aligned}$$

$$= (1-\alpha_n\tau)^2\|y_n-p\|^2 + \beta_n^2\|x_n-W_ny_n\|^2 + 2(1-\alpha_n\tau)\beta_n\|y_n-p\|\|x_n-W_ny_n\| + 2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|. \tag{80}$$

From (79) and (80), it follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|y_n-p\|^2 + \beta_n^2\|x_n-W_ny_n\|^2 + 2(1-\alpha_n\tau)\beta_n\|y_n-p\| \\ &\quad\times\|x_n-W_ny_n\| + 2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\| \\ &\le \|x_n-p\|^2 + r_{k,n}(r_{k,n}-2\mu_k)\left\|B_k\Delta_n^{k-1}x_n - B_kp\right\|^2 \\ &\quad + \lambda_{i,n}(\lambda_{i,n}-2\eta_i)\left\|A_i\Lambda_n^{i-1}u_n - A_ip\right\|^2 + \lambda(\lambda-2\beta) \\ &\quad\times\|Bv_n-Bp\|^2 + \beta_n^2\|x_n-W_ny_n\|^2 \\ &\quad + 2(1-\alpha_n\tau)\beta_n\|y_n-p\|\|x_n-W_ny_n\| \\ &\quad + 2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|,\end{aligned} \tag{81}$$

and so

$$\begin{aligned}&r_{k,n}(2\mu_k-r_{k,n})\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\|^2 + \lambda_{i,n}(2\eta_i-\lambda_{i,n}) \\ &\qquad\times\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|^2 + \lambda(2\beta-\lambda)\|Bv_n-Bp\|^2 \\ &\quad\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \beta_n^2\|x_n-W_ny_n\|^2 \\ &\qquad + 2(1-\alpha_n\tau)\beta_n\|y_n-p\|\|x_n-W_ny_n\| \\ &\qquad + 2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\| \\ &\quad\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\| \\ &\qquad + \beta_n^2\|x_n-W_ny_n\|^2 + 2(1-\alpha_n\tau)\beta_n\|y_n-p\| \\ &\qquad\times\|x_n-W_ny_n\| + 2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|.\end{aligned} \tag{82}$$

Since $\{\lambda_{i,n}\}\subset[a_i,b_i]\subset(0,2\eta_i)$ and $\{r_{k,n}\}\subset[e_k,f_k]\subset(0,2\mu_k)$ for all $i\in\{1,2,\dots,N\}$ and $k\in\{1,2,\dots,M\}$, by (73), (77), and (82) we conclude immediately that

$$\lim_{n\to\infty}\|Bv_n-Bp\|=0,\qquad \lim_{n\to\infty}\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|=0,$$
$$\lim_{n\to\infty}\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\|=0, \tag{83}$$

for all $i\in\{1,2,\dots,N\}$ and $k\in\{1,2,\dots,M\}$.

Step 4. Let us show that $\|x_n-Wx_n\|\to 0$.

Indeed, by Lemma 10 (iii) we obtain that for each $k\in\{1,2,\dots,M\}$

$$\begin{aligned}&\left\|\Delta_n^kx_n-p\right\|^2 \\ &= \left\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}B_k)\Delta_n^{k-1}x_n - T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}B_k)p\right\|^2 \\ &\le \langle (I-r_{k,n}B_k)\Delta_n^{k-1}x_n-(I-r_{k,n}B_k)p, \Delta_n^kx_n-p\rangle \\ &= \frac12\Big(\left\|(I-r_{k,n}B_k)\Delta_n^{k-1}x_n-(I-r_{k,n}B_k)p\right\|^2 + \left\|\Delta_n^kx_n-p\right\|^2 \\ &\qquad - \left\|(I-r_{k,n}B_k)\Delta_n^{k-1}x_n-(I-r_{k,n}B_k)p-(\Delta_n^kx_n-p)\right\|^2\Big) \\ &\le \frac12\Big(\left\|\Delta_n^{k-1}x_n-p\right\|^2 + \left\|\Delta_n^kx_n-p\right\|^2 \\ &\qquad - \left\|\Delta_n^{k-1}x_n-\Delta_n^kx_n-r_{k,n}(B_k\Delta_n^{k-1}x_n-B_kp)\right\|^2\Big),\end{aligned} \tag{84}$$

which implies that

$$\begin{aligned}\left\|\Delta_n^kx_n-p\right\|^2 &\le \left\|\Delta_n^{k-1}x_n-p\right\|^2 \\ &\quad - \left\|\Delta_n^{k-1}x_n-\Delta_n^kx_n-r_{k,n}(B_k\Delta_n^{k-1}x_n-B_kp)\right\|^2 \\ &= \left\|\Delta_n^{k-1}x_n-p\right\|^2 \\ &\quad - \left\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\right\|^2 - r_{k,n}^2\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\|^2 \\ &\quad + 2r_{k,n}\langle\Delta_n^{k-1}x_n-\Delta_n^kx_n, B_k\Delta_n^{k-1}x_n-B_kp\rangle \\ &\le \left\|\Delta_n^{k-1}x_n-p\right\|^2 - \left\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\right\|^2 \\ &\quad + 2r_{k,n}\left\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\right\|\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\| \\ &\le \|x_n-p\|^2 - \left\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\right\|^2 \\ &\quad + 2r_{k,n}\left\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\right\|\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\|.\end{aligned} \tag{85}$$

Also, by Proposition 4 (iii), we obtain that for each $i\in\{1,2,\dots,N\}$

$$\begin{aligned}&\left\|\Lambda_n^iu_n-p\right\|^2 \\ &= \left\|P_C(I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n - P_C(I-\lambda_{i,n}A_i)p\right\|^2 \\ &\le \langle (I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n-(I-\lambda_{i,n}A_i)p, \Lambda_n^iu_n-p\rangle\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\left\|(I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n-(I-\lambda_{i,n}A_i)p\right\|^2+\left\|\Lambda_n^i u_n-p\right\|^2\\
&\qquad -\left\|(I-\lambda_{i,n}A_i)\Lambda_n^{i-1}u_n-(I-\lambda_{i,n}A_i)p-(\Lambda_n^i u_n-p)\right\|^2\Big)\\
&\le \frac{1}{2}\Big(\left\|\Lambda_n^{i-1}u_n-p\right\|^2+\left\|\Lambda_n^i u_n-p\right\|^2\\
&\qquad -\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n-\lambda_{i,n}(A_i\Lambda_n^{i-1}u_n-A_ip)\right\|^2\Big)\\
&\le \frac{1}{2}\Big(\left\|u_n-p\right\|^2+\left\|\Lambda_n^i u_n-p\right\|^2\\
&\qquad -\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n-\lambda_{i,n}(A_i\Lambda_n^{i-1}u_n-A_ip)\right\|^2\Big),
\end{aligned}
\tag{86}
$$

which implies that

$$
\begin{aligned}
&\left\|\Lambda_n^i u_n-p\right\|^2\\
&\le \left\|u_n-p\right\|^2-\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n-\lambda_{i,n}(A_i\Lambda_n^{i-1}u_n-A_ip)\right\|^2\\
&= \left\|u_n-p\right\|^2-\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|^2-\lambda_{i,n}^2\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|^2\\
&\quad +2\lambda_{i,n}\langle\Lambda_n^{i-1}u_n-\Lambda_n^i u_n, A_i\Lambda_n^{i-1}u_n-A_ip\rangle\\
&\le \left\|u_n-p\right\|^2-\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|^2\\
&\quad +2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|.
\end{aligned}
\tag{87}
$$

Since $J_{R,\lambda}$ is 1-inverse strongly monotone, we have

$$
\begin{aligned}
&\left\|y_n-p\right\|^2\\
&\quad= \left\|J_{R,\lambda}(v_n-\lambda Bv_n)-J_{R,\lambda}(p-\lambda Bp)\right\|^2\\
&\quad\le \langle v_n-\lambda Bv_n-(p-\lambda Bp), y_n-p\rangle\\
&\quad= \frac{1}{2}\Big(\left\|v_n-\lambda Bv_n-(p-\lambda Bp)\right\|^2+\left\|y_n-p\right\|^2\\
&\qquad\quad -\left\|v_n-\lambda Bv_n-(p-\lambda Bp)-(y_n-p)\right\|^2\Big)\\
&\quad\le \frac{1}{2}\Big(\left\|v_n-p\right\|^2+\left\|y_n-p\right\|^2-\left\|v_n-y_n-\lambda(Bv_n-Bp)\right\|^2\Big)\\
&\quad= \frac{1}{2}\Big(\left\|v_n-p\right\|^2+\left\|y_n-p\right\|^2-\left\|v_n-y_n\right\|^2\\
&\qquad\quad +2\lambda\langle Bv_n-Bp, v_n-y_n\rangle-\lambda^2\left\|Bv_n-Bp\right\|^2\Big),
\end{aligned}
\tag{88}
$$

which implies that

$$
\left\|y_n-p\right\|^2\le\left\|v_n-p\right\|^2-\left\|v_n-y_n\right\|^2+2\lambda\langle Bv_n-Bp, v_n-y_n\rangle.
\tag{89}
$$

Thus, from (85)–(89) we get

$$
\begin{aligned}
&\left\|y_n-p\right\|^2\\
&\quad\le \left\|v_n-p\right\|^2-\left\|v_n-y_n\right\|^2+2\lambda\left\|Bv_n-Bp\right\|\left\|v_n-y_n\right\|\\
&\quad= \left\|\Lambda_n^N u_n-p\right\|^2-\left\|v_n-y_n\right\|^2+2\lambda\left\|Bv_n-Bp\right\|\left\|v_n-y_n\right\|\\
&\quad\le \left\|\Lambda_n^i u_n-p\right\|^2-\left\|v_n-y_n\right\|^2+2\lambda\left\|Bv_n-Bp\right\|\left\|v_n-y_n\right\|\\
&\quad\le \left\|u_n-p\right\|^2-\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|^2\\
&\qquad +2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|-\left\|v_n-y_n\right\|^2\\
&\qquad +2\lambda\left\|Bv_n-Bp\right\|\left\|v_n-y_n\right\|\\
&\quad= \left\|\Delta_n^M x_n-p\right\|^2-\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|^2\\
&\qquad +2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|-\left\|v_n-y_n\right\|^2\\
&\qquad +2\lambda\left\|Bv_n-Bp\right\|\left\|v_n-y_n\right\|\\
&\quad\le \left\|\Delta_n^k x_n-p\right\|^2-\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|^2\\
&\qquad +2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|-\left\|v_n-y_n\right\|^2\\
&\qquad +2\lambda\left\|Bv_n-Bp\right\|\left\|v_n-y_n\right\|\\
&\quad\le \left\|x_n-p\right\|^2-\left\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\right\|^2+2r_{k,n}\left\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\right\|\\
&\qquad \times\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\|-\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|^2\\
&\qquad +2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|\\
&\qquad -\left\|v_n-y_n\right\|^2+2\lambda\left\|Bv_n-Bp\right\|\left\|v_n-y_n\right\|.
\end{aligned}
\tag{90}
$$

Substituting (90) into (80), we have

$$
\begin{aligned}
&\left\|x_{n+1}-p\right\|^2\\
&\le (1-\alpha_n\tau)^2\left\|y_n-p\right\|^2+\beta_n^2\left\|x_n-W_ny_n\right\|^2\\
&\quad +2(1-\alpha_n\tau)\beta_n\left\|y_n-p\right\|\left\|x_n-W_ny_n\right\|\\
&\quad +2\alpha_n\left\|\gamma f(x_n)-\mu Fp\right\|\left\|x_{n+1}-p\right\|\\
&\le (1-\alpha_n\tau)^2\Big\{\left\|x_n-p\right\|^2-\left\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\right\|^2\\
&\qquad\qquad +2r_{k,n}\left\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\right\|\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\|\\
&\qquad\qquad -\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|^2+2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|\\
&\qquad\qquad \times\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|-\left\|v_n-y_n\right\|^2\\
&\qquad\qquad +2\lambda\left\|Bv_n-Bp\right\|\left\|v_n-y_n\right\|\Big\}+\beta_n^2\left\|x_n-W_ny_n\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2(1-\alpha_n\tau)\beta_n\|y_n-p\|\|x_n-W_ny_n\| \\
&+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\| \\
&\le \|x_n-p\|^2-(1-\alpha_n\tau)^2\left\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\right\|^2 \\
&+2r_{k,n}\left\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\right\|\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\| \\
&-(1-\alpha_n\tau)^2\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|^2+2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\| \\
&\times\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|-(1-\alpha_n\tau)^2\|v_n-y_n\|^2 \\
&+2\lambda\|Bv_n-Bp\|\|v_n-y_n\|+\beta_n^2\|x_n-W_ny_n\|^2 \\
&+2(1-\alpha_n\tau)\beta_n\|y_n-p\|\|x_n-W_ny_n\| \\
&+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|;
\end{aligned} \tag{91}
$$

that is,

$$
\begin{aligned}
&(1-\alpha_n\tau)^2 \\
&\times\left[\left\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\right\|^2+\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|^2+\|v_n-y_n\|^2\right] \\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2r_{k,n}\left\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\right\| \\
&\times\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\|+2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\| \\
&\times\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\|+2\lambda\|Bv_n-Bp\|\|v_n-y_n\| \\
&+\beta_n^2\|x_n-W_ny_n\|^2+2(1-\alpha_n\tau)\beta_n\|y_n-p\|\|x_n-W_ny_n\| \\
&+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\| \\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\| \\
&+2r_{k,n}\left\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\right\|\left\|B_k\Delta_n^{k-1}x_n-B_kp\right\| \\
&+2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|\left\|A_i\Lambda_n^{i-1}u_n-A_ip\right\| \\
&+2\lambda\|Bv_n-Bp\|\|v_n-y_n\|+\beta_n^2\|x_n-W_ny_n\|^2 \\
&+2(1-\alpha_n\tau)\beta_n\|y_n-p\|\|x_n-W_ny_n\| \\
&+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|.
\end{aligned} \tag{92}
$$

So, from $\alpha_n \to 0$, (73), (77), and (83) we immediately get

$$
\begin{gathered}
\lim_{n\to\infty}\|v_n-y_n\|=0, \qquad \lim_{n\to\infty}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^i u_n\right\|, \\
\lim_{n\to\infty}\left\|\Delta_n^{k-1}x_n-\Delta_n^k x_n\right\|=0,
\end{gathered} \tag{93}
$$

for all $i\in\{1,2,\ldots,N\}$ and $k\in\{1,2,\ldots,M\}$. Note that

$$
\begin{aligned}
\|x_n-u_n\| &= \left\|\Delta_n^0x_n-\Delta_n^M x_n\right\| \\
&\le \left\|\Delta_n^0x_n-\Delta_n^1x_n\right\|+\left\|\Delta_n^1x_n-\Delta_n^2x_n\right\| \\
&\quad+\cdots+\left\|\Delta_n^{M-1}x_n-\Delta_n^M x_n\right\|,
\end{aligned} \tag{94}
$$

$$
\begin{aligned}
\|u_n-v_n\| &= \left\|\Lambda_n^0u_n-\Lambda_n^N u_n\right\| \\
&\le \left\|\Lambda_n^0u_n-\Lambda_n^1u_n\right\|+\left\|\Lambda_n^1u_n-\Lambda_n^2u_n\right\| \\
&\quad+\cdots+\left\|\Lambda_n^{N-1}u_n-\Lambda_n^N u_n\right\|.
\end{aligned} \tag{95}
$$

Thus, from (93) we have

$$
\lim_{n\to\infty}\|x_n-u_n\|=0, \qquad \lim_{n\to\infty}\|u_n-v_n\|=0. \tag{96}
$$

It is easy to see that as $n\to\infty$

$$
\|x_n-y_n\|\le\|x_n-u_n\|+\|u_n-v_n\|+\|v_n-y_n\|\longrightarrow 0. \tag{97}
$$

Also, observe that

$$
\begin{aligned}
\|W_ny_n-y_n\| &\le \|W_ny_n-x_n\|+\|x_n-u_n\| \\
&\quad+\|u_n-v_n\|+\|v_n-y_n\|.
\end{aligned} \tag{98}
$$

Hence, we have

$$
\lim_{n\to\infty}\|W_ny_n-y_n\|=0. \tag{99}
$$

Since

$$
\|Wy_n-y_n\|\le\|Wy_n-W_ny_n\|+\|W_ny_n-y_n\|, \tag{100}
$$

it follows from Remark 16 that

$$
\lim_{n\to\infty}\|Wy_n-y_n\|=0. \tag{101}
$$

This, together with $\|x_n-y_n\|\to 0$, implies that

$$
\lim_{n\to\infty}\|x_n-Wx_n\|=0. \tag{102}
$$

Step 5. Let us show that $\limsup_{n\to\infty}\langle(\gamma f-\mu F)x^*, x_n-x^*\rangle\le 0$ where $x^*=P_\Omega(I-\mu F+\gamma f)x^*$.

Indeed, as previously noted, it is known that $P_\Omega(I-\mu F+\gamma f)$ is contractive and so $P_\Omega(I-\mu F+\gamma f)$ has a unique fixed point, denoted by $x^*\in C$. This implies that $x^*=P_\Omega(I-\mu F+\gamma f)x^*$.

First, we show that $\omega_w(x_n)\subset\Omega$. As a matter of fact, we note that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$
\begin{aligned}
&\limsup_{n\to\infty}\langle(\gamma f-\mu F)x^*, x_n-x^*\rangle \\
&\quad=\lim_{j\to\infty}\langle(\gamma f-\mu F)x^*, x_{n_j}-x^*\rangle.
\end{aligned} \tag{103}
$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ which converges weakly to w. Without loss of generality, we may assume that $x_{n_j}\rightharpoonup w$. Note that $\|Wx_n-x_n\|\to 0$.

Then, by the demiclosedness principle for nonexpansive mappings, we obtain $w \in \mathrm{Fix}(W) = \cap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Furthermore, from (93) and (96), we have that $\Delta_{n_i}^k x_{n_i} \rightharpoonup w$, $\Lambda_{n_i}^m u_{n_i} \rightharpoonup w$, $u_{n_i} \rightharpoonup w$, $v_{n_i} \rightharpoonup w$, and $y_{n_i} \rightharpoonup w$, where $k \in \{1, 2, \ldots, M\}$ and $m \in \{1, 2, \ldots, N\}$.

Now we prove that $w \in \cap_{m=1}^{N} \mathrm{VI}(C, A_m)$. Let

$$T_m v = \begin{cases} A_m v + N_C v, & v \in C, \\ \emptyset, & v \notin C, \end{cases} \tag{104}$$

where $m \in \{1, 2, \ldots, N\}$. Let $(v, u) \in G(T_m)$. Since $u - A_m v \in N_C v$ and $\Lambda_n^m u_n \in C$, we have

$$\langle v - \Lambda_n^m u_n, u - A_m v \rangle \geq 0. \tag{105}$$

On the other hand, from $\Lambda_n^m u_n = P_C(I - \lambda_{m,n} A_m)\Lambda_n^{m-1} u_n$ and $v \in C$, we have

$$\langle v - \Lambda_n^m u_n, \Lambda_n^m u_n - \left(\Lambda_n^{m-1} u_n - \lambda_{m,n} A_m \Lambda_n^{m-1} u_n\right) \rangle \geq 0, \tag{106}$$

and hence

$$\langle v - \Lambda_n^m u_n, \frac{\Lambda_n^m u_n - \Lambda_n^{m-1} u_n}{\lambda_{m,n}} + A_m \Lambda_n^{m-1} u_n \rangle \geq 0. \tag{107}$$

Therefore, we have

$$\begin{aligned} &\left\langle v - \Lambda_{n_i}^m u_{n_i}, u \right\rangle \\ &\quad \geq \langle v - \Lambda_{n_i}^m u_{n_i}, A_m v \rangle \\ &\quad \geq \langle v - \Lambda_{n_i}^m u_{n_i}, A_m v \rangle \\ &\qquad - \left\langle v - \Lambda_{n_i}^m u_{n_i}, \frac{\Lambda_{n_i}^m u_{n_i} - \Lambda_{n_i}^{m-1} u_{n_i}}{\lambda_{m,n_i}} + A_m \Lambda_{n_i}^{m-1} u_{n_i} \right\rangle \\ &\quad = \langle v - \Lambda_{n_i}^m u_{n_i}, A_m v - A_m \Lambda_{n_i}^m u_{n_i} \rangle \\ &\qquad + \langle v - \Lambda_{n_i}^m u_{n_i}, A_m \Lambda_{n_i}^m u_{n_i} - A_m \Lambda_{n_i}^{m-1} u_{n_i} \rangle \\ &\qquad - \langle v - \Lambda_{n_i}^m u_{n_i}, \frac{\Lambda_{n_i}^m u_{n_i} - \Lambda_{n_i}^{m-1} u_{n_i}}{\lambda_{m,n_i}} \rangle \\ &\quad \geq \langle v - \Lambda_{n_i}^m u_{n_i}, A_m \Lambda_{n_i}^m u_{n_i} - A_m \Lambda_{n_i}^{m-1} u_{n_i} \rangle \\ &\qquad - \langle v - \Lambda_{n_i}^m u_{n_i}, \frac{\Lambda_{n_i}^m u_{n_i} - \Lambda_{n_i}^{m-1} u_{n_i}}{\lambda_{m,n_i}} \rangle. \end{aligned} \tag{108}$$

From (93) and since A_m is Lipschitzian, we obtain that $\lim_{n\to\infty} \|A_m \Lambda_n^m u_n - A_m \Lambda_n^{m-1} u_n\| = 0$. From $\Lambda_{n_i}^m u_{n_i} \rightharpoonup w$, $\{\lambda_{m,n}\} \subset [a_m, b_m] \subset (0, 2\eta_m)$, $\forall m \in \{1, 2, \ldots, N\}$, and (93), we have

$$\langle v - w, u \rangle \geq 0. \tag{109}$$

Since T_m is maximal monotone, we have $w \in T_m^{-1} 0$ and hence $w \in \mathrm{VI}(C, A_m)$, $m = 1, 2, \ldots, N$, which implies that $w \in \cap_{m=1}^{N} \mathrm{VI}(C, A_m)$.

Next we prove that $w \in \cap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, B_k)$. Since $\Delta_n^k x_n = T_{r_{k,n}}^{(\Theta_k, \varphi_k)}(I - r_{k,n} B_k)\Delta_n^{k-1} x_n$, $n \geq 1$, $k \in \{1, 2, \ldots, M\}$, we have

$$\begin{aligned} &\Theta_k\left(\Delta_n^k x_n, y\right) + \varphi_k(y) - \varphi_k\left(\Delta_n^k x_n\right) + \langle B_k \Delta_n^{k-1} x_n, y - \Delta_n^k x_n \rangle \\ &\quad + \frac{1}{r_{k,n}} \langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1} x_n \rangle \geq 0. \end{aligned} \tag{110}$$

By (A2), we have

$$\begin{aligned} &\varphi_k(y) - \varphi_k\left(\Delta_n^k x_n\right) + \langle B_k \Delta_n^{k-1} x_n, y - \Delta_n^k x_n \rangle \\ &\quad + \frac{1}{r_{k,n}} \langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1} x_n \rangle \geq \Theta_k\left(y, \Delta_n^k x_n\right). \end{aligned} \tag{111}$$

Let $z_t = ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in C$. This implies that $z_t \in C$. Then, we have

$$\begin{aligned} &\langle z_t - \Delta_n^k x_n, B_k z_t \rangle \\ &\quad \geq \varphi_k\left(\Delta_n^k x_n\right) - \varphi_k(z_t) + \langle z_t - \Delta_n^k x_n, B_k z_t \rangle \\ &\qquad - \langle z_t - \Delta_n^k x_n, B_k \Delta_n^{k-1} x_n \rangle \\ &\qquad - \langle z_t - \Delta_n^k x_n, \frac{\Delta_n^k x_n - \Delta_n^{k-1} x_n}{r_{k,n}} \rangle + \Theta_k\left(z_t, \Delta_n^k x_n\right) \\ &\quad = \varphi_k\left(\Delta_n^k x_n\right) - \varphi_k(z_t) + \langle z_t - \Delta_n^k x_n, B_k z_t - B_k \Delta_n^k x_n \rangle \\ &\qquad + \langle z_t - \Delta_n^k x_n, B_k \Delta_n^k x_n - B_k \Delta_n^{k-1} x_n \rangle \\ &\qquad - \langle z_t - \Delta_n^k x_n, \frac{\Delta_n^k x_n - \Delta_n^{k-1} x_n}{r_{k,n}} \rangle + \Theta_k\left(z_t, \Delta_n^k x_n\right). \end{aligned} \tag{112}$$

By (93) and the fact that B_k is Lipschitzian, we have $\|B_k \Delta_n^k x_n - B_k \Delta_n^{k-1} x_n\| \to 0$ as $n \to \infty$. In addition, by the monotonicity of B_k, we obtain $\langle z_t - \Delta_n^k x_n, B_k z_t - B_k \Delta_n^k x_n \rangle \geq 0$. Then, by (A4) and (112) we obtain

$$\langle z_t - w, B_k z_t \rangle \geq \varphi_k(w) - \varphi_k(z_t) + \Theta_k(z_t, w). \tag{113}$$

Utilizing (A1), (A4), and (113), we obtain

$$\begin{aligned} 0 &= \Theta_k(z_t, z_t) + \varphi_k(z_t) - \varphi_k(z_t) \\ &\leq t\Theta_k(z_t, y) + (1-t)\Theta_k(z_t, w) + t\varphi_k(y) \\ &\quad + (1-t)\varphi_k(w) - \varphi_k(z_t) \\ &\leq t\left[\Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)\right] + (1-t)\langle z_t - w, B_k z_t \rangle \\ &= t\left[\Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)\right] + (1-t)t\langle y - w, B_k z_t \rangle, \end{aligned} \tag{114}$$

and hence

$$0 \le \Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t) + (1-t)\langle y - w, B_k z_t\rangle. \tag{115}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \le \Theta_k(w, y) + \varphi_k(y) - \varphi_k(w) + \langle y - w, B_k w\rangle. \tag{116}$$

This implies that $w \in \mathrm{GMEP}(\Theta_k, \varphi_k, B_k)$ and hence $w \in \cap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, B_k)$.

Further, we prove that $w \in \mathrm{I}(B, R)$. In fact, since B is β-inverse strongly monotone, B is monotone and Lipschitzian. It follows from Lemma 9 that $R + B$ is maximal monotone. Let $(v, g) \in G(R + B)$; that is, $g - Bv \in Rv$. Again, since $y_{n_i} = J_{R,\lambda}(v_{n_i} - \lambda B v_{n_i})$, we have $v_{n_i} - \lambda B v_{n_i} \in (I + \lambda R) y_{n_i}$; that is, $(1/\lambda)(v_{n_i} - y_{n_i} - \lambda B v_{n_i}) \in R y_{n_i}$. By virtue of the monotonicity of R, we have

$$\langle v - y_{n_i}, g - Bv - \frac{1}{\lambda}(v_{n_i} - y_{n_i} - \lambda B v_{n_i})\rangle \ge 0, \tag{117}$$

and so

$$\begin{aligned}
&\langle v - y_{n_i}, g\rangle \\
&\quad \ge \langle v - y_{n_i}, Bv + \frac{1}{\lambda}(v_{n_i} - y_{n_i} - \lambda B v_{n_i})\rangle \\
&\quad = \langle v - y_{n_i}, Bv - B y_{n_i} + B y_{n_i} - B v_{n_i} + \frac{1}{\lambda}(v_{n_i} - y_{n_i})\rangle \\
&\quad \le \langle v - y_{n_i}, B y_{n_i} - B v_{n_i}\rangle + \langle v - y_{n_i}, \frac{1}{\lambda}(v_{n_i} - y_{n_i})\rangle.
\end{aligned} \tag{118}$$

Since $\|v_n - y_n\| \to 0$, $\|Bv_n - By_n\| \to 0$, and $y_{n_i} \rightharpoonup w$, we have

$$\lim_{n_i \to \infty} \langle v - y_{n_i}, g\rangle = \langle v - w, g\rangle \ge 0. \tag{119}$$

It follows from the maximal monotonicity of $B + R$ that $0 \in (R + B)w$; that is, $w \in \mathrm{I}(B, R)$. Therefore, $w \in \Omega$. This shows that $\omega_w(\{x_n\}) \subset \Omega$. Consequently, it follows from (103) that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle (\gamma f - \mu F) x^*, x_n - x^*\rangle \\
&\quad = \lim_{j\to\infty} \langle (\gamma f - \mu F) x^*, x_{n_j} - x^*\rangle \\
&\quad = \langle (\gamma f - \mu F) x^*, w - x^*\rangle \le 0.
\end{aligned} \tag{120}$$

Step 6. Let us show that $x_n \to x^*$ as $n \to \infty$.

Indeed, from (49) and Lemma 20 it follows that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - \mu F x^*) + \beta_n(x_n - x^*) \\
&\quad + ((1-\beta_n)I - \alpha_n\mu F)W_n y_n - ((1-\beta_n)I - \alpha_n\mu F)W_n x^*\|^2 \\
&\le \|\beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n\mu F)W_n y_n \\
&\quad - ((1-\beta_n)I - \alpha_n\mu F)W_n x^*\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - \mu F x^*, x_{n+1} - x^*\rangle \\
&\le [\|((1-\beta_n)I - \alpha_n\mu F)W_n y_n - ((1-\beta_n)I - \alpha_n\mu F)W_n x^*\| \\
&\quad + \beta_n\|x_n - x^*\|]^2 + 2\alpha_n\gamma\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle \\
&\quad + 2\alpha_n\langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle \\
&= \left[(1-\beta_n)\left\|\left(I - \frac{\alpha_n}{1-\beta_n}\mu F\right)W_n y_n - \left(I - \frac{\alpha_n}{1-\beta_n}\mu F\right)x^*\right\|\right. \\
&\quad \left. + \beta_n\|x_n - x^*\|\right]^2 + 2\alpha_n\gamma\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle \\
&\quad + 2\alpha_n\langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\le \left[(1-\beta_n)\left(1 - \frac{\alpha_n\tau}{1-\beta_n}\right)\|y_n - x^*\| + \beta_n\|x_n - x^*\|\right]^2 \\
&\quad + 2\alpha_n\gamma\rho\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n\langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\le [(1-\beta_n - \alpha_n\tau)\|x_n - x^*\| + \beta_n\|x_n - x^*\|]^2 \\
&\quad + 2\alpha_n\gamma\rho\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n\langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n\tau)^2\|x_n - x^*\|^2 + \alpha_n\gamma\rho[\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2] \\
&\quad + 2\alpha_n\langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle,
\end{aligned} \tag{121}$$

which immediately yields

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \frac{1 - 2\alpha_n\tau + \alpha_n^2\tau^2 + \alpha_n\gamma\rho}{1 - \alpha_n\gamma\rho}\|x_n - x^*\|^2 \\
&\qquad + \frac{2\alpha_n}{1 - \alpha_n\gamma\rho}\langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad = \left[1 - \frac{2(\tau - \gamma\rho)\alpha_n}{1 - \alpha_n\gamma\rho}\right]\|x_n - x^*\|^2 + \frac{(\alpha_n\tau)^2}{1 - \alpha_n\gamma\rho}\|x_n - x^*\|^2 \\
&\qquad + \frac{2\alpha_n}{1 - \alpha_n\gamma\rho}\langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle
\end{aligned}$$

$$\leq \left[1 - \frac{2(\tau - \gamma\rho)\alpha_n}{1 - \alpha_n \gamma\rho}\right] \|x_n - x^*\|^2 + \frac{2(\tau - \gamma\rho)\alpha_n}{1 - \alpha_n \gamma\rho} \times \left\{ \frac{\alpha_n \tau^2 \widetilde{M}_2}{2(\tau - \gamma\rho)} + \frac{1}{\tau - \gamma\rho} \langle (\gamma f - \mu F) x^*, x_{n+1} - x^* \rangle \right\} = (1 - \delta_n) \|x_n - x^*\|^2 + \delta_n \sigma_n, \tag{122}$$

where

$$\widetilde{M}_2 = \sup\left\{\|x_n - x^*\|^2 : n \geq 1\right\}, \qquad \delta_n = \frac{2(\tau - \gamma\rho)\alpha_n}{1 - \alpha_n \gamma\rho}, \quad \sigma_n = \frac{\alpha_n \tau^2 \widetilde{M}_2}{2(\tau - \gamma\rho)} + \frac{1}{\tau - \gamma\rho} \langle (\gamma f - \mu F) x^*, x_{n+1} - x^* \rangle. \tag{123}$$

It is easy to see that $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \sigma_n \leq 0$. Hence, by Lemma 17 we conclude that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

From Theorem 22, we can readily derive the following.

Corollary 23. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $\mathscr{B} : H \to H$ and $A_i : C \to H$ be ζ-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $i = 1, 2$. Let $F : C \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$ and let $f : H \to H$ be a ρ-Lipschitzian mapping with constant $\rho \geq 0$. Let $R : C \to 2^H$ be a maximal monotone mapping and let the mapping $B : C \to H$ be β-inverse strongly monotone. Let $0 < \lambda < 2\beta$, $0 < \mu < 2\eta/\kappa^2$, and $0 \leq \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$ and let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that $\Omega := \mathrm{GMEP}(\Theta, \varphi, \mathscr{B}) \cap \mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2) \cap \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{I}(B, R) \neq \emptyset$. For arbitrarily given $x_1 \in H$, let the sequence $\{x_n\}$ be generated by*

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle \mathscr{B} x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$
$$v_n = P_C(I - \lambda_{2,n} A_2) P_C(I - \lambda_{1,n} A_1) u_n, \tag{124}$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n) I - \alpha_n \mu F] \times W_n J_{R,\lambda}(v_n - \lambda B v_n), \quad \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0, 1]$ and W_n is the W-mapping defined by (14). Assume that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ *and* $\lim_{n\to\infty} |\lambda_{i,n+1} - \lambda_{i,n}| = 0$ *for* $i = 1, 2$;

(iv) $\{r_n\} \subset [e, f] \subset (0, 2\zeta)$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Assume that either (B1) or (B2) holds. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_{\Omega}(I - \mu F + \gamma f) x^*$ is a unique solution of the VIP:

$$\langle (\mu F - \gamma f) x^*, x^* - y \rangle \leq 0, \quad \forall y \in \Omega. \tag{125}$$

Corollary 24. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $\mathscr{B} : H \to H$ and $\mathscr{A} : C \to H$ be ζ-inverse strongly monotone and ξ-inverse strongly monotone, respectively. Let $F : C \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$ and let $f : H \to H$ be a ρ-Lipschitzian mapping with constant $\rho \geq 0$. Let $R : C \to 2^H$ be a maximal monotone mapping and let the mapping $B : C \to H$ be β-inverse strongly monotone. Let $0 < \lambda < 2\beta$, $0 < \mu < 2\eta/\kappa^2$, and $0 \leq \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$ and let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that $\Omega := \mathrm{GMEP}(\Theta, \varphi, \mathscr{B}) \cap \mathrm{VI}(C, \mathscr{A}) \cap \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{I}(B, R) \neq \emptyset$. For arbitrarily given $x_1 \in H$, let the sequence $\{x_n\}$ be generated by*

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle \mathscr{B} x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$
$$v_n = P_C(I - \rho_n \mathscr{A}) u_n, \tag{126}$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n) I - \alpha_n \mu F] \times W_n J_{R,\lambda}(v_n - \lambda B v_n), \quad \forall n \geq 1,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0, 1]$ and W_n is the W-mapping defined by (14). Assume that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\{\rho_n\} \subset [a, b] \subset (0, 2\xi)$ *and* $\lim_{n\to\infty} |\rho_{n+1} - \rho_n| = 0$;

(iv) $\{r_n\} \subset [e, f] \subset (0, 2\zeta)$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Assume that either (B1) or (B2) holds. Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_{\Omega}(I - \mu F + \gamma f) x^*$ is a unique solution of the VIP:

$$\langle (\mu F - \gamma f) x^*, x^* - y \rangle \leq 0, \quad \forall y \in \Omega. \tag{127}$$

Corollary 25. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $\mathscr{A} : C \to H$ be ξ-inverse strongly monotone and $f : H \to H$ a ρ-contractive* *mapping with constant* $\rho \in [0,1)$. *Let* $R : C \to 2^H$ *be a maximal monotone mapping and let the mapping* $B : C \to H$ *be* β*-inverse strongly monotone. Let* $0 < \lambda < 2\beta$. *Let* $\{\lambda_n\}_{n=1}^{\infty}$ *be a sequence of positive numbers in* $(0,b]$ *for some* $b \in (0,1)$ *and let* $\{T_n\}_{n=1}^{\infty}$ *be an infinite family of nonexpansive self-mappings on* C *such that* $\Omega := \mathrm{MEP}(\Theta,\varphi) \cap \mathrm{VI}(C,\mathcal{A}) \cap \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{I}(B,R) \neq \emptyset$. *For arbitrarily given* $x_1 \in H$, *let the sequence* $\{x_n\}$ *be generated by*

$$\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, \quad \forall y \in C,$$
$$v_n = P_C(I-\rho_n\mathcal{A})u_n,$$
$$x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+(1-\alpha_n-\beta_n)W_nJ_{R,\lambda}(v_n-\lambda Bv_n), \quad \forall n \geq 1, \tag{128}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are two real sequences in* $[0,1]$ *and* W_n *is the* W*-mapping defined by* (14). *Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\{\rho_n\} \subset [a,b] \subset (0,2\xi)$ *and* $\lim_{n\to\infty}|\rho_{n+1}-\rho_n| = 0$;

(iv) $\{r_n\} \subset [e,f] \subset (0,\infty)$ *and* $\lim_{n\to\infty}|r_{n+1}-r_n| = 0$.

Assume that either (B1) or (B2) holds. Then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega f(x^*)$ *is a unique solution of the* VIP:

$$\langle (I-f)x^*, x^*-y\rangle \leq 0, \quad \forall y \in \Omega. \tag{129}$$

Proof. In Corollary 24, put $\mathcal{B} = 0$, $F = (1/2)I$, $\mu = 2$, and $\gamma = 1$. Then from Remark 16 (ii), we get $\tau = 1$. Moreover, for $\{r_n\} \subset [e,f] \subset (0,\infty)$, we can choose a positive constant $\zeta > 0$ such that $\{r_n\} \subset [e,f] \subset (0,2\zeta)$. It is easy to see that $\mathcal{B}$ is ζ-inverse strongly monotone. In addition, for the contraction $f : H \to H$, we have $0 \leq \gamma\rho < \tau$. Hence, all the conditions of Corollary 24 are satisfied. Thus, in terms of Corollary 24, we obtain the desired result. □

Corollary 26. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* Θ *be a bifunction from* $C \times C$ *to* $\mathbf{R}$ *satisfying (A1)–(A4) and let* $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* $\mathcal{A} : C \to H$ *be* ξ*-inverse strongly monotone. Let* $F : C \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with positive constants* $\kappa, \eta > 0$ *and let* $f : H \to H$ *be a* ρ*-Lipschitzian mapping with constant* $\rho \geq 0$. *Let* $R : C \to 2^H$ *be a maximal monotone mapping and let the mapping* $B : C \to H$ *be* β*-inverse strongly monotone. Let* $0 < \lambda < 2\beta$, $0 < \mu < 2\eta/\kappa^2$, *and* $0 \leq \gamma\rho < \tau$, *where* $\tau = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}$. *Let* $\{\lambda_n\}_{n=1}^{\infty}$ *be a sequence of positive numbers in* $(0,b]$ *for some* $b \in (0,1)$ *and let* $\{T_n\}_{n=1}^{\infty}$ *be an infinite family of nonexpansive self-mappings on* C *such that* $\Omega := \mathrm{MEP}(\Theta,\varphi) \cap \mathrm{VI}(C,\mathcal{A}) \cap \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{I}(B,R) \neq \emptyset$. *For arbitrarily given* $x_1 \in H$, *let the sequence* $\{x_n\}$ *be generated by*

$$\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, \quad \forall y \in C,$$
$$v_n = P_C(I-\rho_n\mathcal{A})u_n,$$
$$x_{n+1}=\alpha_n\gamma f(x_n)+\beta_n x_n+[(1-\beta_n)I-\alpha_n\mu F] \times W_nJ_{R,\lambda}(v_n-\lambda Bv_n), \quad \forall n \geq 1, \tag{130}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are two real sequences in* $[0,1]$ *and* W_n *is the* W*-mapping defined by* (14). *Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\{\rho_n\} \subset [a,b] \subset (0,2\xi)$ *and* $\lim_{n\to\infty}|\rho_{n+1}-\rho_n| = 0$;

(iv) $\{r_n\} \subset [e,f] \subset (0,\infty)$ *and* $\lim_{n\to\infty}|r_{n+1}-r_n| = 0$.

Assume that either (B1) or (B2) holds. Then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega(I-\mu F+\gamma f)x^*$ *is a unique solution of the* VIP:

$$\langle (\mu F-\gamma f)x^*, x^*-y\rangle \leq 0, \quad \forall y \in \Omega. \tag{131}$$

Corollary 27. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* Θ *be a bifunction from* $C \times C$ *to* $\mathbf{R}$ *satisfying (A1)–(A4) and let* $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* $\mathcal{B} : H \to H$ *and* $\mathcal{A} : C \to H$ *be* ζ*-inverse strongly monotone and* ξ*-inverse strongly monotone, respectively. Let* $F : C \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with positive constants* $\kappa, \eta > 0$ *and let* $f : H \to H$ *be a* ρ*-Lipschitzian mapping with constant* $\rho \geq 0$. *Let* $R : C \to 2^H$ *be a maximal monotone mapping and let the mapping* $B : C \to H$ *be* β*-inverse strongly monotone. Let* $0 < \lambda < 2\beta$, $0 < \mu < 2\eta/\kappa^2$, *and* $0 \leq \gamma\rho < \tau$, *where* $\tau = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}$. *Assume that* $\Omega := \mathrm{GMEP}(\Theta,\varphi,\mathcal{B}) \cap \mathrm{VI}(C,\mathcal{A}) \cap \mathrm{I}(B,R) \neq \emptyset$. *For arbitrarily given* $x_1 \in H$, *let the sequence* $\{x_n\}$ *be generated by*

$$\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\langle \mathcal{B}x_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, \quad \forall y \in C,$$
$$v_n = P_C(I-\rho_n\mathcal{A})u_n, \tag{132}$$
$$x_{n+1}=\alpha_n\gamma f(x_n)+\beta_n x_n+[(1-\beta_n)I-\alpha_n\mu F] \times J_{R,\lambda}(v_n-\lambda Bv_n), \quad \forall n \geq 1,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are two real sequences in* $[0,1]$. *Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\{\rho_n\} \subset [a,b] \subset (0,2\xi)$ *and* $\lim_{n\to\infty}|\rho_{n+1}-\rho_n| = 0$;

(iv) $\{r_n\} \subset [e,f] \subset (0,2\zeta)$ *and* $\lim_{n\to\infty}|r_{n+1}-r_n| = 0$.

Assume that either (B1) or (B2) holds. Then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega(I-\mu F+\gamma f)x^*$ *is a unique solution of the* VIP:

$$\langle (\mu F - \gamma f)\, x^*, x^* - y\rangle \le 0, \quad \forall y \in \Omega. \tag{133}$$

Corollary 28. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* Θ *be a bifunction from* $C\times C$ *to* $\mathbf{R}$ *satisfying (A1)–(A4). Let* $\mathscr{B}: H \to H$ *and* $\mathscr{A}: C \to H$ *be* ζ*-inverse strongly monotone and* ξ*-inverse strongly monotone, respectively. Let* $F: C \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with positive constants* $\kappa,\eta > 0$ *and let* $f: H \to H$ *be a* ρ*-Lipschitzian mapping with constant* $\rho \ge 0$. *Let* $R: C \to 2^H$ *be a maximal monotone mapping and let the mapping* $B: C \to H$ *be* β*-inverse strongly monotone. Let* $0 < \lambda < 2\beta$, $0 < \mu < 2\eta/\kappa^2$, *and* $0 \le \gamma\rho < \tau$, *where* $\tau = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}$. *Let* $\{\lambda_n\}_{n=1}^\infty$ *be a sequence of positive numbers in* $(0,b]$ *for some* $b \in (0,1)$ *and let* $\{T_n\}_{n=1}^\infty$ *be an infinite family of nonexpansive self-mappings on* C *such that* $\Omega :=$ GEP$(\Theta,\mathscr{B})\cap$ VI$(C,\mathscr{A})\cap\bigcap_{n=1}^\infty$ Fix$(T_n)\cap$ I$(B,R)\ne\emptyset$. *For arbitrarily given* $x_1 \in H$, *let the sequence* $\{x_n\}$ *be generated by*

$$\begin{gathered}\Theta(u_n,y)+\langle \mathscr{B}x_n, y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \ge 0, \\ \forall y\in C,\\ v_n = P_C(I-\rho_n\mathscr{A})u_n,\\ x_{n+1} = \alpha_n\gamma f(x_n)+\beta_n x_n+[(1-\beta_n)I-\alpha_n\mu F] \\ \times W_n J_{R,\lambda}(v_n-\lambda B v_n), \quad \forall n\ge 1,\end{gathered} \tag{134}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are two real sequences in* $[0,1]$ *and* W_n *is the* W*-mapping defined by* (14). *Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\{\rho_n\} \subset [a,b] \subset (0,2\xi)$ *and* $\lim_{n\to\infty}|\rho_{n+1}-\rho_n| = 0$;

(iv) $\{r_n\} \subset [e,f] \subset (0,2\zeta)$ *and* $\lim_{n\to\infty}|r_{n+1}-r_n| = 0$.

Assume that either (B1) or (B2) holds. Then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega(I-\mu F+\gamma f)x^*$ *is a unique solution of the* VIP:

$$\langle (\mu F - \gamma f)\, x^*, x^* - y\rangle \le 0, \quad \forall y \in \Omega. \tag{135}$$

Corollary 29. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* Θ *be a bifunction from* $C\times C$ *to* $\mathbf{R}$ *satisfying (A1)–(A4). Let* $\mathscr{B}: H \to H$ *and* $\mathscr{A}: C \to H$ *be* ζ*-inverse strongly monotone and* ξ*-inverse strongly monotone, respectively. Let* $F: C \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with positive constants* $\kappa,\eta > 0$ *and let* $f: H \to H$ *be a* ρ*-Lipschitzian mapping with constant* $\rho \ge 0$. *Let* $R: C \to 2^H$ *be a maximal monotone mapping and let the mapping* $B: C \to H$ *be* β*-inverse strongly monotone. Let* $0 < \lambda < 2\beta$, $0 < \mu < 2\eta/\kappa^2$, *and* $0 \le \gamma\rho < \tau$, *where* $\tau = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}$. *Assume that* $\Omega :=$ GEP$(\Theta,\mathscr{B})\cap$ VI$(C,\mathscr{A})\cap$ I$(B,R)\ne\emptyset$. *For arbitrarily given* $x_1 \in H$, *let the sequence* $\{x_n\}$ *be generated by*

$$\begin{gathered}\Theta(u_n,y)+\langle \mathscr{B}x_n, y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \ge 0, \\ \forall y\in C,\\ v_n = P_C(I-\rho_n\mathscr{A})u_n,\\ x_{n+1} = \alpha_n\gamma f(x_n)+\beta_n x_n+[(1-\beta_n)I-\alpha_n\mu F] \\ \times J_{R,\lambda}(v_n-\lambda B v_n), \quad \forall n\ge 1,\end{gathered} \tag{136}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are two real sequences in* $[0,1]$. *Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\{\rho_n\} \subset [a,b] \subset (0,2\xi)$ *and* $\lim_{n\to\infty}|\rho_{n+1}-\rho_n| = 0$;

(iv) $\{r_n\} \subset [e,f] \subset (0,2\zeta)$ *and* $\lim_{n\to\infty}|r_{n+1}-r_n| = 0$.

Assume that either (B1) or (B2) holds. Then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega(I-\mu F+\gamma f)x^*$ *is a unique solution of the* VIP:

$$\langle (\mu F - \gamma f)\, x^*, x^* - y\rangle \le 0, \quad \forall y \in \Omega. \tag{137}$$

Corollary 30. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* Θ *be a bifunction from* $C\times C$ *to* $\mathbf{R}$ *satisfying (A1)–(A4). Let* $\mathscr{B}: H \to H$ *and* $\mathscr{A}: C \to H$ *be* ζ*-inverse strongly monotone and* ξ*-inverse strongly monotone, respectively. Let* $f: H \to H$ *be a* ρ*-contractive mapping with constant* $\rho \in [0,1)$. *Let* $R: C \to 2^H$ *be a maximal monotone mapping and let the mapping* $B: C \to H$ *be* β*-inverse strongly monotone. Let* $0 < \lambda < 2\beta$. *Assume that* $\Omega :=$ GEP$(\Theta,\mathscr{B})\cap$ VI$(C,\mathscr{A})\cap$ I$(B,R)\ne\emptyset$. *For arbitrarily given* $x_1 \in H$, *let the sequence* $\{x_n\}$ *be generated by*

$$\begin{gathered}\Theta(u_n,y)+\langle \mathscr{B}x_n, y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \ge 0, \\ \forall y\in C,\\ v_n = P_C(I-\rho_n\mathscr{A})u_n,\\ x_{n+1} = \alpha_n f(x_n)+\beta_n x_n+(1-\alpha_n-\beta_n) J_{R,\lambda}(v_n-\lambda B v_n), \\ \forall n\ge 1,\end{gathered} \tag{138}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are two real sequences in* $[0,1]$. *Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\{\rho_n\} \subset [a,b] \subset (0,2\xi)$ *and* $\lim_{n\to\infty}|\rho_{n+1}-\rho_n| = 0$;

(iv) $\{r_n\} \subset [e,f] \subset (0,2\zeta)$ *and* $\lim_{n\to\infty}|r_{n+1}-r_n| = 0$.

Assume that either (B1) or (B2) holds. Then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega f(x^*)$ *is a unique solution of the* VIP:

$$\langle (I-f)x^*, x^*-y\rangle \le 0, \quad \forall y \in \Omega. \tag{139}$$

Remark 31. Theorem 22 extends, improves, and supplements [4, Theorem 3.2] in the following aspects.

(i) The problem of finding a point $p \in \cap_{k=1}^{M}\mathrm{GMEP}(\Theta_k,\varphi_k,B_k) \cap \cap_{i=1}^{N}\mathrm{VI}(C,A_i) \cap \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{I}(B,R)$ in Theorem 22 is very different from the problem of finding a point $p \in \mathrm{GMEP}(\Theta,\varphi,A) \cap \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{I}(B,R)$ in [4, Theorem 3.2] (i.e., Theorem 3 in this paper). There is no doubt that the problem of finding a point $p \in \cap_{k=1}^{M}\mathrm{GMEP}(\Theta_k,\varphi_k,B_k) \cap \cap_{i=1}^{N}\mathrm{VI}(C,A_i) \cap \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{I}(B,R)$ is more general and more subtle than the problem of finding a point $p \in \mathrm{GMEP}(\Theta,\varphi,A) \cap \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{I}(B,R)$ in [4, Theorem 3.2].

(ii) If, in Corollary 24, $C = H$, $\mathscr{A} = 0$, $r_n = r > 0$ $(\forall n \ge 1)$, $\mu F = V$ is a strongly positive bounded linear operator, and f is a contraction, then Corollary 24 reduces essentially to [4, Theorem 3.2]. This shows that Theorem 22 includes [4, Theorem 3.2] as a special case.

(iii) The iterative scheme in [4, Algorithm 3.1] is extended to develop the iterative scheme in Theorem 22 by virtue of Korpelevič's extragradient method and hybrid steepest-descent method [38]. The iterative scheme in Theorem 22 is more advantageous and more flexible than the iterative scheme in [4, Algorithm 3.1] because it involves solving four problems: a finite family of GMEPs, a finite family of VIPs, the variational inclusion (11), and the fixed point problem of an infinite family of nonexpansive self-mappings.

(iv) The iterative scheme in Theorem 22 is very different from the iterative scheme in [4, Algorithm 3.1] because the iterative scheme in Theorem 22 involves Korpelevič's extragradient method and hybrid steepest-descent method.

(v) The proof of Theorem 22 combines the proof for viscosity approximation method in [4, Theorem 3.2], the proof for Korpelevič's extragradient method in [8, Theorem 3.1], and the proof for hybrid steepest-descent method in [44, Theorem 3.1].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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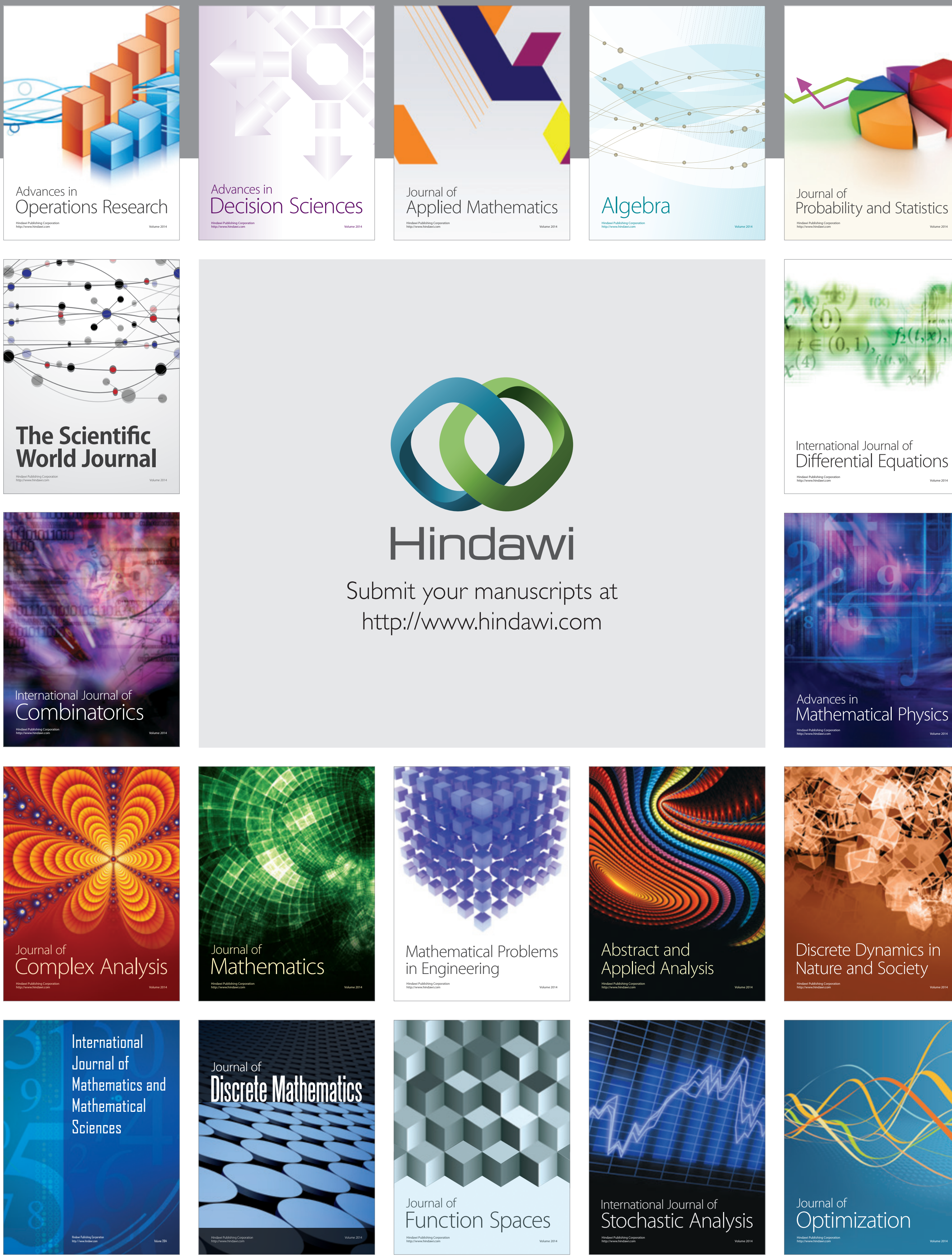
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