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Research Article
Lightlike Submanifolds of Indefinite Sasakian Manifolds

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We first prove some results on invariant lightlike submanifolds of indefinite Sasakian manifolds. Then, we introduce a general notion of contact Cauchy-Riemann (CR) lightlike submanifolds and study the geometry of leaves of their distributions. We also study a class, namely, contact screen Cauchy-Riemann (SCR) lightlike submanifolds which include invariant and screen real subcases. Finally, we prove characterization theorems on the existence of contact SCR, screen real, invariant, and contact CR minimal lightlike submanifolds.

1. Introduction

The geometry of lightlike submanifolds of indefinite Kähler manifolds was presented in a book [1, Chapter 6]. However, a general notion of lightlike submanifolds of indefinite Sasakian manifolds was not introduced yet. A significant use of the contact geometry in differential equations, optics, and phase spaces of a dynamical system (see Arnol'd [2], Maclane [3], Nazaikinskii et al. [4] and many more references therein) and only very limited specific information [5–7] on its lightlike case motivated the present authors to work on the geometry of lightlike submanifolds of indefinite Sasakian manifolds.

This paper has three objectives. First, we study invariant [8] lightlike submanifolds M of indefinite Sasakian manifolds $\overline{M}$ and prove that the geometry of a codimension-two invariant M has close relation with the nondegenerate geometry of a leaf of its integrable screen (Theorem 2.2). Also, we show that if a totally umbilical M is tangent to the characteristic vector field V, then M is totally geodesic and invariant in $\overline{M}$ (Theorem 2.5). Second, we introduce the general notion of *contact Cauchy-Riemann (CR)-lightlike* *submanifolds*, a first attempt towards the general theory of lightlike submanifolds of Sasakian manifolds, and study its properties. We study the integrability conditions of their distributions, investigate the geometry of leaves of the distributions involved in the induced contact CR-structure on M, and find geometric conditions for an irrotational [9] contact CR-submanifold M to be a product manifold. It is important to mention that contrary to the Riemannian case [8], but, similar to the Duggal-Bejancu's concept of lightlike CR-submanifolds of Kählerian manifolds [1], the contact CR-lightlike submanifolds are always nontrivial, that is, they do not include the invariant and the real subcases.

Therefore, as a third objective, we introduce a new class called *contact screen Cauchy-Riemann (SCR)-lightlike submanifolds,* which includes invariant and screen real submanifolds and study their properties. Finally, we prove characterization theorems on the existence of minimal submanifolds of all the classes studied. We follow [1] for the notations and formulas used in this paper.

A submanifold M^m immersed in a semi-Riemannian manifold $(\overline{M}^{m+k}, \overline{g})$ is called *a lightlike submanifold* if it admits a degenerate metric g induced from $\overline{g}$ whose radical distribution $\mathrm{Rad}(TM)$ is of rank r, where $1 \le r \le m$. $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$, where

$$TM^{\perp} = \bigcup_{x\in M} \{u \in T_x\overline{M} : \overline{g}(u,v) = 0,\ \forall v \in T_xM\}. \tag{1.1}$$

Let $S(TM)$ be *a screen distribution* which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is, $TM = \mathrm{Rad}(TM) \perp S(TM)$.

We consider *a screen transversal vector bundle* $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $\mathrm{Rad}(TM)$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $\mathrm{Rad}(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a *lightlike transversal vector bundle* $l\mathrm{tr}(TM)$ locally spanned by $\{N_i\}$ [1, page 144]. Let $\mathrm{tr}(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$\begin{gathered}\mathrm{tr}(TM) = l\mathrm{tr}(TM) \perp S(TM^{\perp}),\\ T\overline{M}|_M = S(TM) \perp \left[\mathrm{Rad}(TM) \oplus l\mathrm{tr}(TM)\right] \perp S(TM^{\perp}).\end{gathered} \tag{1.2}$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/\mathrm{Rad}\,TM$ [9]. The following result is important to this paper.

Proposition 1.1 [1]. *The lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM)$, $S(TM^{\perp})$, and $l\mathrm{tr}(TM)$.*

Throughout this paper, we will discuss the dependence (or otherwise) of the results on induced object(s) and refer to [1] for their transformation equations. We say that a submanifold $(M,g,S(TM),S(TM^{\perp}))$ of $\overline{M}$ is

(1) r-lightlike if $r < \min\{m,k\}$;
(2) coisotropic if $r = k < m$, $S(TM^{\perp}) = \{0\}$;
(3) isotropic if $r = m < k$, $S(TM) = \{0\}$;
(4) totally lightlike if $r = m = k$, $S(TM) = \{0\} = S(TM^{\perp})$.

The Gauss and Weingarten equations are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall X,Y \in \Gamma(TM), \tag{1.3}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall X \in \Gamma(TM),\ V \in \Gamma(\operatorname{tr}(TM)), \tag{1.4}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(\operatorname{tr}(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $\operatorname{tr}(TM)$, respectively. Moreover, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \quad \forall X,Y \in \Gamma(TM), \tag{1.5}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X,N), \quad N \in \Gamma(l\operatorname{tr}(TM)), \tag{1.6}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X,W), \quad W \in \Gamma(S(TM^{\perp})). \tag{1.7}$$

Denote the projection of TM on $S(TM)$ by $\overline{P}$. Then, by using (1.3), (1.5)–(1.7), and a metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y,D^l(X,W)) = g(A_W X,Y), \tag{1.8}$$

$$\overline{g}(D^s(X,N),W) = \overline{g}(N,A_W X). \tag{1.9}$$

From the decomposition of tangent bundle of lightlike submanifold, we have

$$\nabla_X \overline{P} Y = \nabla^*_X \overline{P} Y + h^*(X,\overline{P} Y), \tag{1.10}$$

$$\nabla_X \xi = -A^*_\xi X + {\nabla^*}^t_X \xi, \tag{1.11}$$

for $X,Y \in \Gamma(TM)$ and $\xi \in \Gamma(\operatorname{Rad} TM)$. By using the above equations, we obtain

$$\overline{g}(h^l(X,\overline{P}Y),\xi) = g(A^*_\xi X,\overline{P}Y), \tag{1.12}$$

$$\overline{g}(h^*(X,\overline{P}Y),N) = g(A_N X,\overline{P}Y), \tag{1.13}$$

$$\overline{g}(h^l(X,\xi),\xi) = 0, \qquad A^*_\xi \xi = 0. \tag{1.14}$$

In general, the induced connection ∇ on M is not a metric connection. Since $\overline{\nabla}$ is a metric connection, by using (1.5) we get

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y). \tag{1.15}$$

However, it is important to note that $\nabla^\star$ is a metric connection on $S(TM)$.

2. Invariant submanifolds

An odd-dimensional semi-Riemannian manifold $(\overline{M},\overline{g})$ is called *contact metric manifold* [10] if there are a $(1,1)$ tensor field ϕ, a vector field V, called *characteristic vector field,* and a 1-form η such that

$$\begin{gathered}\overline{g}(\phi X,\phi Y)=\overline{g}(X,Y)-\epsilon\eta(X)\eta(Y),\qquad \overline{g}(V,V)=\epsilon,\\ \phi^2(X)=-X+\eta(X)V,\qquad \overline{g}(X,V)=\epsilon\eta(X),\\ d\eta(X,Y)=\overline{g}(X,\phi Y),\quad \forall X,Y\in\Gamma(TM),\qquad \epsilon=\pm 1.\end{gathered}\tag{2.1}$$

It follows that $\phi V=0$, $\eta\circ\phi=0$, $\eta(V)=\epsilon$. Then $(\phi,V,\eta,\overline{g})$ is called *contact metric structure* of $\overline{M}$. Also, $\overline{M}$ has a *normal contact structure* if $N_\phi+d\eta\otimes V=0$, where N_ϕ is the Nijenhuis tensor field [8]. A normal contact metric $\overline{M}$ is called an *indefinite Sasakian manifold* [11, 12] for which we have

$$\overline{\nabla}_XV=\phi X,\tag{2.2}$$

$$(\overline{\nabla}_X\phi)Y=-\overline{g}(X,Y)V+\epsilon\eta(Y)X.\tag{2.3}$$

Let $(M,g,S(TM),S(TM^\perp))$ be a lightlike submanifold of $(\overline{M},\overline{g})$. For any vector field X tangent to M, we put

$$\phi X=PX+FX,\tag{2.4}$$

where PX and FX are the tangential and the transversal parts of ϕX, respectively. Moreover, P is skew-symmetric on $S(TM)$.

It is known [5] that if M is tangent to the structure vector field V, then, V belongs to $S(TM)$. Using this, we say that M is invariant in $\overline{M}$ if M is tangent to the structure vector field V and

$$\phi X=PX,\quad \text{that is, } \phi X\in\Gamma(TM),\ \forall X\in\Gamma(TM).\tag{2.5}$$

From (2.2), (2.3), (2.5), and (1.5), we get

$$h^l(X,V)=0,\qquad h^s(X,V)=0,\qquad \nabla_XV=PX,\tag{2.6}$$

$$h(X,\phi Y)=\phi h(X,Y)=h(\phi X,Y),\quad \forall X,Y\in\Gamma(TM).\tag{2.7}$$

Proposition 2.1. *Let $(M,g,S(TM),S(TM^\perp))$ be an invariant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. If the second fundamental forms h^l and h^s of M are parallel, then M is totally geodesic.*

Proof. Let us suppose that h^l is parallel, then we have

$$(\nabla^t_Xh^l)(Y,V)=\nabla_Xh^l(Y,V)-h^l(\nabla_XY,V)-h^l(Y,\nabla_XV)=0.\tag{2.8}$$

Thus, using (2.6) and (2.2), we have $h^l(Y,PX)=0$. Similarly, we have $h^s(Y,PX)=0$, which completes the proof. □

Theorem 2.2. *Let $(M,g,S(TM),S(TM^{\perp}))$ be an invariant lightlike submanifold of codimension two of an indefinite Sasakian manifold $\overline{M}$. Then,* Rad TM *defines a totally geodesic foliation on M. Moreover, $M = M_1 \times M_2$ is a lightlike product manifold if and only if $h^* = 0$, where M_1 is a leaf of the radical distribution and M_2 is a semi-Riemannian manifold.*

Proof. Since rank(Rad TM) $= 2$, for all $X,Y \in \Gamma(\text{Rad}\,TM)$ one can write ξ and $\phi\xi$ as a linear combination, that is, $X = A_1\xi + B_1\phi\xi$, $Y = A_2\xi + B_2\phi\xi$. Thus by direct calculations, using (1.5) we obtain

$$g(\nabla_X Y,\overline{P}Z) = -A_2A_1\overline{g}(\xi,h^l(\xi,\overline{P}Z)) - A_2B_1\overline{g}(h^l(\phi\xi,\overline{P}Z),\xi) - B_2A_1\overline{g}(\phi\xi,h^l(\xi,\overline{P}Z)) - B_2B_1\overline{g}(h^l(\phi\xi,\overline{P}Z),\phi\xi). \tag{2.9}$$

Now, by using (1.14) and (2.7), we derive $g(\nabla_X Y,\overline{P}Z) = 0$. This shows that Rad TM defines a totally geodesic foliation. Then, the proof of theorem follows from [1, Theorem 2.6, page 162]. □

Theorem 2.3. *Let $(M,g,S(TM),S(TM^{\perp}))$ be an invariant lightlike submanifold of codimension two of an indefinite Sasakian manifold $\overline{M}$. Suppose (M',g') is a nondegenerate submanifold of $\overline{M}$ such that M' is a leaf of integrable $S(TM)$. Then M is totally geodesic, with an induced metric connection if M' being so immersed as a submanifold of $\overline{M}$.*

Proof. Since dim(Rad TM) $=$ dim($l$tr(TM)) $= 2$, $h^l(X,Y) = A_1N + B_1\phi N$, where A_1 and B_1 are functions on M. Thus $h^l(X,\xi) = 0$ if and only if $\overline{g}(h^l(X,\xi),\xi) = 0$ and $\overline{g}(h^l(X,\xi),\phi\xi) = 0$, for all $X \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad}\,TM)$. From (1.14), we have $\overline{g}(h^l(X,\xi),\xi) = 0$. Using (2.7), we get $\overline{g}(h^l(X,\xi),\phi\xi) = -\overline{g}(h^l(\phi X,\xi),\xi) = 0$. Similarly, $h^l(X,\phi\xi) = 0$. For M', we write

$$\overline{\nabla}_X Y = \nabla'_X Y + h'(X,Y), \quad \forall X,Y \in \Gamma(TM'), \tag{2.10}$$

where ∇' is a metric connection on M' and h' is the second fundamental form of M'. Thus, $h'(X,Y) = h^*(X,Y) + h^l(X,Y)$, for all $X,Y \in \Gamma(TM')$. Also, $g(X,Y) = g'(X,Y)$, for all $X,Y \in \Gamma(TM)$, which completes the proof. □

Definition 2.4 [13]. A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\overline{M},\overline{g})$ is totally umbilical in $\overline{M}$ if there is a smooth transversal vector field $\mathbf{H} \in \Gamma(\text{tr}(TM))$ on M, called the transversal curvature vector field of M, such that for all $X,Y \in \Gamma(TM)$,

$$h(X,Y) = \mathbf{H}g(X,Y). \tag{2.11}$$

Using (1.5) and (2.11), it is easy to see that M is totally umbilical if and only if on each coordinate neighborhood $\mathcal{U}$, there exist smooth vector fields $\mathcal{H}^l \in \Gamma(l\text{tr}(TM))$ and $\mathcal{H}^s \in \Gamma(S(TM^{\perp}))$ such that

$$h^l(X,Y) = \mathcal{H}^l g(X,Y), \qquad D^l(X,W) = 0$$
$$h^s(X,Y) = \mathcal{H}^s g(X,Y), \quad \forall X,Y \in \Gamma(TM), \quad W \in \Gamma(S(TM^{\perp})). \tag{2.12}$$

THEOREM 2.5. *Let $(M,g,S(TM),S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field V, of an indefinite Sasakian manifold $(\overline{M},\overline{g})$. If M is totally umbilical, then M is totally geodesic and invariant.*

Proof. Using (2.2), (1.5), (2.4), and the transversal parts, we get

$$h^s(X,V)+h^l(X,V)=FX, \quad \forall X\in\Gamma(TM). \tag{2.13}$$

$\phi V=0$ implies that $PV=0$ and $FV=0$. Thus from (2.13), we have $h^l(V,V)=0$ and $h^s(V,V)=0$. M is totally umbilical, V is nonnull, and (2.12) implies that $h^l=0$ and $h^s=0$, so M is totally geodesic. Also, $FX=h^l(X,V)+h^s(X,V)=0$ implies that M invariant in $\overline{M}$, which completes the proof. □

Remark 2.6. As per Proposition 1.1, Definition 2.4 does not depend on $S(TM)$ and $S(TM^{\perp})$, but it depends on the transformation equations (2.60) in [1, page 165], with respect to the screen second fundamental forms h^s.

3. Contact CR-lightlike submanifolds

In this section, we follow Yano-Kon [8, page 353] definition of contact CR-submanifolds and state the following definition for a *contact CR-lightlike submanifold*.

Definition 3.1. Let $(M,g,S(TM),S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field V, immersed in an indefinite Sasakian manifold $(\overline{M},\overline{g})$. Say that M is a contact CR-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(A) $\operatorname{Rad}TM$ is a distribution on M such that $\operatorname{Rad}TM\cap\phi(\operatorname{Rad}TM)=\{0\}$;

(B) there exist vector bundles D_0 and D' over M such that

$$\begin{gathered} S(TM)=\{\phi(\operatorname{Rad}TM)\oplus D'\}\perp D_o\perp\{V\}, \\ \phi D_o=D_o, \qquad \phi(D')=L_1\perp l\operatorname{tr}(TM), \end{gathered} \tag{3.1}$$

where D_0 is nondegenerate and L_1 is a vector subbundle of $S(TM^{\perp})$.

Thus, one has the following decomposition:

$$TM=D\oplus\{V\}\oplus D', \quad D=\operatorname{Rad}TM\perp\phi(\operatorname{Rad}TM)\perp D_o. \tag{3.2}$$

A contact CR-lightlike submanifold is proper if $D_o\neq\{0\}$ and $L_1\neq\{0\}$. It follows that any contact CR-lightlike three-dimensional submanifold is 1-lightlike.

Example 3.2. Let M be a lightlike hypersurface of $\overline{M}$. For $\xi\in\Gamma(\operatorname{Rad}TM)$, we have $\overline{g}(\phi\xi,\xi)=0$. Hence, $\phi\xi\in\Gamma(TM)$. Thus, we get a rank-1 distribution $\phi(TM^{\perp})$ on M such that $\phi(TM^{\perp})\cap TM^{\perp}=\{0\}$. This enables us to choose a screen $S(TM)$ such that it contains $\phi(TM^{\perp})$ as a vector subbundle. Consider $N\in\Gamma(l\operatorname{tr}(TM))$ to obtain $\overline{g}(\phi N,\xi)=-\overline{g}(N,\phi\xi)=0$ and $\overline{g}(\phi N,N)=0$. Thus, $\phi N\in\Gamma(STM)$. Taking $D'=\phi(\operatorname{tr}(TM))$, we obtain $S(TM)=\{\phi(TM^{\perp})\oplus D'\}\perp D_o$, where D_o is a nondegenerate distribution. Moreover $\phi(D')=\operatorname{tr}(TM)$. Hence, M is a contact CR-lightlike hypersurface.

Henceforth, $(\mathbf{R}_q^{2m+1},\phi_o,V,\eta,\overline{g})$ will denote the manifold $\mathbf{R}_q^{2m+1}$ with its usual Sasakian structure given by

$$\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{m} y^i dx^i\right), \qquad V = 2\partial z,$$

$$\overline{g} = \eta\otimes\eta + \frac{1}{4}\left(-\sum_{i=1}^{q/2} dx^i\otimes dx^i + dy^i\otimes dy^i + \sum_{i=q+1}^{m} dx^i\otimes dx^i + dy^i\otimes dy^i\right), \tag{3.3}$$

$$\phi_o\left(\sum_{i=1}^{m}(X_i\partial x^i + Y_i\partial y^i) + Z\partial z\right) = \sum_{i=1}^{m}(Y_i\partial x^i - X_i\partial y^i) + \sum_{i=1}^{m} Y_i y^i\partial z,$$

where $(x^i;y^i;z)$ are the Cartesian coordinates. The above construction will help in understanding how the contact structure is recovered in the next three examples.

Example 3.3. Let $\overline{M} = (\mathbf{R}_2^9,\overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to canonical basis

$$\{\partial x_1,\partial x_2,\partial x_3,\partial x_4,\partial y_1,\partial y_2,\partial y_3,\partial y_4,\partial z\}. \tag{3.4}$$

Suppose M is a submanifold of $\mathbf{R}_2^9$ defined by

$$x^1 = y^4, \qquad x^2 = \sqrt{1-(y^2)^2}, \quad y^2 \neq \pm 1. \tag{3.5}$$

It is easy to see that a local frame of TM is given by

$$Z_1 = 2(\partial x_1 + \partial y_4 + y^1\partial z), \qquad Z_2 = 2(\partial x_4 - \partial y_1 + y^4\partial z),$$

$$Z_3 = \partial x_3 + y^3\partial z, \qquad Z_4 = \partial y_3, \qquad Z_5 = -\frac{y^2}{x^2}\partial x_2 + \partial y_2 - \frac{(y^2)^2}{x_2}\partial z, \tag{3.6}$$

$$Z_6 = \partial x_4 + \partial y_1 + y^4\partial z, \qquad Z_7 = V = 2\partial z.$$

Hence, $\operatorname{Rad} TM = \operatorname{span}\{Z_1\}$, $\phi_o \operatorname{Rad} TM = \operatorname{span}\{Z_2\}$, and $\operatorname{Rad} TM \cap \phi_o \operatorname{Rad} TM = \{0\}$. Thus (A) holds. Next, $\phi_o(Z_3) = -Z_4$ implies that $D_0 = \{Z_3,Z_4\}$ is invariant with respect to ϕ_o. By direct calculations, we get

$$S(TM^\perp) = \operatorname{span}\left\{W = \partial x_2 + \frac{y^2}{x^2}\partial y_2 + y^2\partial z\right\} \quad \text{such that } \phi_o(W) = -Z_5 \tag{3.7}$$

and $l\operatorname{tr}(TM)$ is spanned by $N = -\partial x_1 + \partial y_4 - y^1\partial z$ such that $\phi_o(N) = Z_6$. Hence, M is a contact CR-lightlike submanifold.

Proposition 3.4. *There exist no isotropic or totally lightlike contact CR-lightlike submanifolds.*

Proof. If M is isotropic or totally lightlike, then $S(TM) = \{0\}$. Hence, conditions (A) and (B) of Definition 3.1 are not satisfied. □

Proposition 3.5. *Let M be a contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then, D and $D' \oplus D$ are not integrable.*

Proof. Suppose D is integrable. Then $g([X,Y],V)=0$, for $X,Y\in\Gamma(D)$. Also from (1.7), we derive $g([X,Y],V)=\overline{g}(\overline{\nabla}_XY,V)-\overline{g}(\overline{\nabla}_YX,V)$. Then, $\overline{\nabla}$ being a metric connection and using (2.2), we have $g([X,Y],V)=-g(Y,\phi X)+g(\phi Y,X)$. Hence, $g([X,Y],V)=2g(\phi Y,X)$. Since by Proposition 3.4, M is proper and D_o is nondegenerate, we can choose nonnull vector fields $X,Y\in\Gamma(D)$ such that $g(Y,\phi X)\neq 0$, which is a contradiction, so D is not integrable. Similarly, $D'\oplus D$ is not integrable, which completes the proof. □

Denote the orthogonal complement subbundle to the vector subbundle L_1 in $S(TM^{\perp})$ by $L_1^{\perp}$. For a contact CR-lightlike submanifold M, we put

$$\phi X = fX+\omega X,\quad \forall X\in\Gamma(TM), \tag{3.8}$$

where $fX\in\Gamma(D)$ and $\omega X\in\Gamma(L_1\perp l\mathrm{tr}(TM))$. Similarly, we have

$$\phi W = BW+CW,\quad \forall W\in\Gamma(S(TM^{\perp})), \tag{3.9}$$

where $BW\in\Gamma(\phi L_1)$ and $CW\in\Gamma(L_1^{\perp})$.

Proposition 3.6. *Let M be a contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then, $D\oplus\{V\}$ is integrable if and only if*

$$h(X,\phi Y)=h(\phi X,Y),\quad \forall X,Y\in\Gamma(D\oplus\{V\}). \tag{3.10}$$

Proof. From (1.5), (3.8), (3.9), (2.3), and transversal parts, we obtain $\omega(\nabla_XY)=-Ch^s(X,Y)+h(X,\phi Y)$, for all $X,Y\in\Gamma(D\oplus\{V\})$. Consequently, $\omega[X,Y]=h(X,\phi Y)-h(\phi X,Y)$, for all $X,Y\in\Gamma(D\oplus\{V\})$, which completes the proof. □

Proposition 3.7. *Let M be a contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $D\oplus\{V\}$ is a totally geodesic foliation if and only if*

$$h^l(X,\phi Y)=0,\qquad h^s(X,Y)\ \text{has no components in}\ L_1. \tag{3.11}$$

Proof. By Definition 3.1, $D\oplus\{V\}$ defines a totally geodesic foliation if and only if $g(\nabla_XY,\phi\xi)=g(\nabla_XY,W)=0$ for $X,Y\in\Gamma(D\oplus\{V\})$ and $W\in\Gamma(\phi L_1)$. Then, from (1.5), we have $g(\nabla_XY,\phi\xi)=-\overline{g}(\phi\overline{\nabla}_XY,\xi)$. Using (2.3) and (1.5), we get

$$g(\nabla_XY,\phi\xi)=-\overline{g}(h^l(X,\phi Y),\xi). \tag{3.12}$$

In a similar way, we derive

$$g(\nabla_X\phi Y,W)=-g(h^s(X,Y),\phi W). \tag{3.13}$$

Thus, from (3.12) and (3.13), we obtain (3.11), which completes the proof. □

Proposition 3.8. *Let M be a contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then, D' is a totally geodesic foliation if and only if A_NZ has no components in $\phi L_1\perp\phi(\mathrm{Rad}\,TM)$ and $A_{\phi W}Z$ has no components in $D_o\perp\mathrm{Rad}\,TM$ for $Z,W\in\Gamma(D')$.*

Proof. We note that D' defines a totally geodesic foliation if and only if

$$\overline{g}(\nabla_Z W, N) = g(\nabla_Z W, \phi N) = g(\nabla_Z W, X) = g(\nabla_Z W, V) = 0, \tag{3.14}$$

$Z, W \in \Gamma(D')$, $N \in \Gamma(l\,\mathrm{tr}(TM))$, and $X \in \Gamma(D_o)$. From (2.2) and (1.6), we get

$$g(\nabla_Z W, V) = 0. \tag{3.15}$$

On the other hand, $\overline{\nabla}$ is a metric connection and (1.5) implies that

$$\overline{g}(\nabla_Z W, N) = g(W, A_N Z). \tag{3.16}$$

By using (2.2), (2.3), (1.8), and (3.2), we obtain

$$\overline{g}(\nabla_Z W, \phi N) = g(A_{\phi W} Z, N). \tag{3.17}$$

In a similar way, we have

$$g(\nabla_Z W, \phi X) = g(A_{\phi W} Z, X). \tag{3.18}$$

Thus the proof follows from (3.15)–(3.18). □

Recall from Kupeli [9] that a lightlike submanifold M of a semi-Riemannian manifold is said to be an *irrotational submanifold* if $\overline{\nabla}_X \xi \in \Gamma(TM)$, for all $X \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}\,TM)$. From (1.5), we conclude that M is an irrotational lightlike submanifold if and only if the following are satisfied:

$$h^s(X, \xi) = 0, \qquad h^l(X, \xi) = 0. \tag{3.19}$$

Also, we say that M is a contact CR-lightlike product if $D \oplus \{V\}$ and D' define totally geodesic foliations in M. This concept is consistent with the classical definition of product manifolds.

Theorem 3.9. *Let M be an irrotational contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then, M is contact CR-lightlike product if the following conditions are satisfied:*

(1) $\overline{\nabla}_X U \in \Gamma(S(TM^{\perp}))$, *for all* $X \in \Gamma(TM)$ *and* $U \in \Gamma(\mathrm{tr}(TM))$;

(2) $A^*_{\xi} Y$ *has no components in* $D_o \oplus \phi(l\,\mathrm{tr}(TM))$, $Y \in \Gamma(D)$.

Proof. If (1) holds, then from (1.6) and (1.7), we have $A_N X = 0$, $A_W X = 0$, and $D^l(X, W) = 0$, for $X \in \Gamma(TM)$, $W \in \Gamma(S(TM^{\perp}))$. These equations imply that D' defines a totally geodesic foliation. Moreover, from (1.8), we get $\overline{g}(h^s(X, Y), W) = -\overline{g}(Y, D^l(X, W)) = 0$. Hence, $h^s(X, Y)$ has no components in L_1. Finally, from (1.12) and M being irrotational, we have $\overline{g}(h^l(X, \phi Y), \xi) = -\overline{g}(\phi Y, A^*_{\xi} X)$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Hence, if (2) holds, then $h^l(X, \phi Y) = 0$. Thus, considering Propositions 3.7 and 3.8, we conclude that M is a contact CR-light like product, which completes the proof. □

Definition 3.10 [8, page 374]. If the second fundamental form h of a submanifold, tangent to the structure vector field V, of an indefinite Sasakian manifold $\overline{M}$ is of the form

$$h(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha + \eta(X)h(Y,V) + \eta(Y)h(X,V), \tag{3.20}$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is called totally contact umbilical and totally contact geodesic if $\alpha = 0$.

The above definition also holds for a lightlike submanifold M. For a totally contact umbilical M, we have

$$\begin{aligned} h^l(X,Y) &= [g(X,Y) - \eta(X)\eta(Y)]\alpha_L + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V), \\ h^s(X,Y) &= [g(X,Y) - \eta(X)\eta(Y)]\alpha_S + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V), \end{aligned} \tag{3.21}$$

where $\alpha_S \in \Gamma(S(TM^\perp))$ and $\alpha_L \in \Gamma(l\,\mathrm{tr}(TM))$.

Lemma 3.11. *Let M be a totally contact umbilical proper contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $\alpha_L = 0$.*

Proof. Let M be a totally contact umbilical proper contact CR-lightlike submanifold. Then, by direct calculations,using (1.5), (1.7), (2.3), and taking the tangential parts, we have

$$A_{\phi Z}Z + f\nabla_Z Z + \phi h^l(Z,Z) + Bh^s(Z,Z) = g(Z,Z)V \tag{3.22}$$

for $Z \in \Gamma(\phi L_1)$. Hence, we obtain $\overline{g}(A_{\phi Z}Z, \phi\xi) + \overline{g}(h^l(Z,Z),\xi) = 0$. Using (1.8), we get $g(h^s(Z,\phi\xi),\phi Z) + \overline{g}(h^l(Z,Z),\xi) = 0$. Thus from (3.21), we derive $-g(Z,Z)\overline{g}(\alpha_L,\xi) = 0$. Since ϕL_1 is nondegenerate, we get $\alpha_L = 0$, which completes the proof. □

Theorem 3.12. *Let M be a totally contact umbilical proper contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then either M is totally contact geodesic or $\dim(\phi L_1) = 1$.*

Proof. Assume M proper is totally contact umbilical. From (2.3), (1.5), (3.9), and (3.11), we get $\omega\nabla_X X + Ch^s(X,X) = 0$ for $X \in \Gamma(D_o)$. Hence,

$$h^s(X,X) \in \Gamma(L_1). \tag{3.23}$$

Now from (3.22) and (1.8), we have $\overline{g}(h^s(Z,Z),\phi W) = \overline{g}(h^s(Z,W),\phi Z)$ for $Z, W \in \Gamma(\phi L_1)$. Since M is totally contact umbilical, we obtain

$$g(Z,Z)\overline{g}(\alpha_S,\phi W) = g(Z,W)\overline{g}(\alpha_S,\phi Z). \tag{3.24}$$

Interchanging role Z and W and subtracting, we derive

$$\overline{g}(\alpha_S,\phi Z) = \frac{g(Z,W)^2}{g(Z,Z)g(W,W)}\overline{g}(\alpha_S,\phi Z). \tag{3.25}$$

Considering (3.23), (3.25) has solutions if either (a) $\dim(L_1) = 1$, or (b) $\alpha_S = 0$. Thus the proof follows from Lemma 3.11. □

It is known [14] that CR-submanifolds of Riemannian manifolds were designed as a generalization of both invariant and totally real submanifolds. Therefore, it is important to know whether contact CR-lightlike submanifolds admit invariant submanifolds (discussed in Section 2) and, also, are there any real submanifolds. To investigate this, we need the following definition.

Definition 3.13. Say that a lightlike submanifold M, of an indefinite Sasakian manifold $\overline{M}$, is screen real submanifold if $\mathrm{Rad}(TM)$ and $S(TM)$ are, respectively, invariant and anti-invariant with respect to ϕ.

The above definition is the lightlike version (see [15]) of the totally real submanifolds of an almost Hermitian (or contact) manifold [8].

Proposition 3.14. *Contact CR-lightlike submanifolds are nontrivial.*

Proof. Suppose M is an invariant lightlike submanifold of an indefinite Sasakian manifold. Then we can easily see that radical distribution is invariant. Thus condition (A) of the Definition 2.4 is not satisfied. Similarly, one can prove that the screen real lightlike case is not possible. □

4. Contact SCR-lightlike submanifolds

We know from Proposition 3.14 that contact CR-lightlike submanifolds exclude the invariant and the screen real subcases, and therefore, do not serve the central purpose of introducing a CR-structure. To include these two subcases, we introduce a new class, called *contact screen Cauchy-Riemann (SCR)-lightlike submanifold* as follows.

Definition 4.1. Let $(M,g,S(TM),S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field V, of an indefinite Sasakian manifold $\overline{M}$. Say that M is a contact SCR-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied.

(1) There exist real nonnull distributions D and $D^{\perp}$ such that

$$S(TM) = D \oplus D^{\perp} \perp \{V\}, \qquad \phi(D^{\perp}) \subset (S(TM^{\perp})), \qquad D \cap D^{\perp} = \{0\}, \tag{4.1}$$

where $D^{\perp}$ is orthogonally complementary to $D \perp \{V\}$ in $S(TM)$.

(2) The distributions D and $\mathrm{Rad}(TM)$ are invariant with respect to ϕ.

It follows that $l\mathrm{tr}(TM)$ is also invariant with respect to ϕ. Hence we have

$$TM = \overline{D} \oplus D^{\perp} \perp \{V\}, \qquad \overline{D} = D \perp \mathrm{Rad}(TM). \tag{4.2}$$

Denote the orthogonal complement to $\phi(D^{\perp})$ in $S(TM^{\perp})$ by μ. We say that M is a proper contact SCR-lightlike submanifold of $\overline{M}$ if $D \neq \{0\}$ and $D^{\perp} \neq \{0\}$. Note the following features of a contact SCR-lightlike submanifold:

(1) condition (2) implies that $\dim(\mathrm{Rad}\,TM) = r = 2p \geq 2$;
(2) for proper M, (2) implies that $\dim(D) = 2s \geq 2$, $\dim(D^{\perp}) \geq 1$. Thus, $\dim(M) \geq 5$, $\dim(\overline{M}) \geq 9$.

For any $X \in \Gamma(TM)$ and any $W \in \Gamma(S(TM^{\perp}))$, we put

$$\phi X = P'X + F'X, \qquad \phi W = B'W + C'W, \tag{4.3}$$

where $P'X \in \Gamma(\overline{D})$, $F'X \in \Gamma(\phi D^{\perp})$, $B'W \in \Gamma(D^{\perp})$, and $C'V \in \Gamma(\mu)$.

Example 4.2. Let M be a submanifold of $\mathbf{R}^9_2$ defined by

$$x^1 = x^2, \qquad y^1 = y^2, \qquad x^4 = \sqrt{1-(y^4)^2}, \qquad y^4 \neq \pm 1. \tag{4.4}$$

It is easy to see that a local frame of TM is given by

$$\begin{gathered} Z_1 = \partial x_1 + \partial x_2 + (y^1 + y^2)\partial z, \qquad Z_2 = \partial y_1 + \partial y_2, \\ Z_3 = \partial x_3 + y^3 \partial z, \qquad Z_4 = \partial y_3, \\ Z_5 = -y^4 \partial x_4 + x^4 \partial y_4 - (y^4)^2 \partial z, \qquad V = 2\partial z. \end{gathered} \tag{4.5}$$

Hence, $\operatorname{Rad} TM = \operatorname{Span}\{Z_1, Z_2\}$ and $\phi_o(Z_1) = -Z_2$. Thus, $\operatorname{Rad} TM$ is invariant with respect to ϕ_o. Also, $\phi_o(Z_3) = -Z_4$ implies that $D = \operatorname{Span}\{Z_3, Z_4\}$. By direct calculations, we get $S(TM^{\perp}) = \operatorname{span}\{W = x^4 \partial x_4 + y^4 \partial y_4 + x^4 y^4 \partial z\}$ such that $\phi_o(W) = -Z_5$, and lightlike transversal bundle $l\operatorname{tr}(TM)$ is spanned by

$$N_1 = 2(-\partial x_1 + \partial x_2 + (-y^1 + y^2)\partial z), \qquad N_2 = 2(-\partial y_1 + \partial y_2). \tag{4.6}$$

It follows that $\phi_o(N_2) = N_1$. Thus, $l\operatorname{tr}(TM)$ is also invariant. Hence, M is a contact SCR-lightlike submanifold.

The following results can be easily proved by direct use of Definition 4.1.

(1) A contact SCR-lightlike submanifold of $\overline{M}$ is invariant (resp., screen real) if and only if $D^{\perp} = \{0\}$ (resp., $D = \{0\}$).

(2) Any contact SCR-coisotropic, isotropic, and totally lightlike submanifold of $\overline{M}$ is an invariant lightlike submanifold. Consequently, there exist no proper contact SCR or screen real coisotropic or isotropic or totally lightlike submanifold of $\overline{M}$.

Theorem 4.3. *Let M be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the induced connection ∇ is a metric connection if and only if the following two conditions hold:*

(1) *$h^s(X,\xi)$ has no components in $\phi(D^{\perp})$,*

(2) *$A^*_{\xi}X$ has no components in D, for all $X \in \Gamma(TM)$, $\xi \in \Gamma(\operatorname{Rad} TM)$.*

Proof. Equation (2.3) implies that $\overline{\nabla}_X \phi\xi = \phi \overline{\nabla}_X \xi$ and from (1.5), (1.11), (4.2), we get

$$\nabla_X \phi\xi = B'h^s(X,\xi) + \phi \nabla^{*t}_X \xi - P'A^*_{\xi}X. \tag{4.7}$$

We know that the induced connection is a metric connection if and only if $\operatorname{Rad} TM$ is parallel with respect to ∇. Suppose that $\operatorname{Rad} TM$ is parallel. Then from (4.7), we have $B'h^s(X,\xi) = 0$ and $P'A^*_{\xi}X = 0$. Hence $h^s(X,\xi)$ has no components in $\phi(D^{\perp})$ and $A^*_{\xi}X$ has no components in D. Conversely, assume that (1) and (2) are satisfied, then from

(4.7), we get $\nabla_X \phi\xi \in \Gamma(\text{Rad}\,TM)$. Thus, Rad TM is parallel and ∇ is a metric connection, which completes the proof. □

Proposition 4.4. *There exists a Levi-Civita connection on an irrotational screen real lightlike submanifold of an indefinite Sasakian manifold.*

Proof. From (1.5), we have $\bar{g}(h^l(X,Y),\xi) = \bar{g}(\overline{\nabla}_X Y,\xi)$, for all $X,Y \in \Gamma(TM)$. By using (2.3), we get

$$\bar{g}(h^l(X,Y),\xi) = \bar{g}(\phi\overline{\nabla}_X Y,\phi\xi) = \bar{g}(-(\overline{\nabla}_X\phi)Y + \overline{\nabla}_X\phi Y,\phi\xi). \tag{4.8}$$

From (2.3), we obtain $\bar{g}(h^l(X,Y),\xi) = \bar{g}(\overline{\nabla}_X\phi Y,\phi\xi)$. Since $\overline{\nabla}$ is a metric connection, we have $\bar{g}(h^l(X,Y),\xi) = -\bar{g}(\phi Y,\overline{\nabla}_X\phi\xi)$. Using (1.5), we obtain $\bar{g}(h^l(X,Y),\xi) = -\bar{g}(\phi Y,h^s(X,\phi\xi))$. M being irrotational implies that $\bar{g}(h^l(X,Y),\xi) = 0$, that is, $h^l = 0$. Then the proof follows from (1.15). □

From (2.3), (1.5), and (4.3), we have the following:

$$(\nabla_X P')Y = A_{F'Y}X + B'h^s(X,Y) - g(X,Y)V + \eta(Y)X, \tag{4.9}$$

$$(\nabla_X F')Y = C'h^s(X,Y) - h^s(X,P'Y), \tag{4.10}$$

$$\phi h^l(X,Y) = h^l(X,P'Y) + D^l(X,F'Y), \quad \forall X,Y \in \Gamma(TM). \tag{4.11}$$

The following results are similar to those proved in Propositions 3.5 and 3.6.

Proposition 4.5. *Let M be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then*

(1) *the distribution $D^\perp$ is integrable if and only if*

$$A_{\phi X}Y = A_{\phi Y}X, \quad \forall X,Y \in \Gamma(D^\perp); \tag{4.12}$$

(2) *the distribution $\overline{D} \oplus \{V\}$ is integrable if and only if*

$$h^s(X,P'Y) = h^s(P'X,Y), \quad \forall X,Y \in \Gamma(\overline{D}); \tag{4.13}$$

(3) *the distribution $\overline{D}$ is not integrable.*

Theorem 4.6. *Let M be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $\overline{D} \oplus \{V\}$ defines a totally geodesic foliation in M if and only if $h^s(X,\phi Y)$ has no components in $\phi(D^\perp)$, for $X,Y \in \Gamma(\overline{D} \oplus \{V\})$.*

Proof. From (1.5), we have $g(\nabla_X Y,Z) = \bar{g}(\overline{\nabla}_X Y,Z)$ for $X,Y \in \Gamma(\overline{D} \oplus \{V\})$ and $Z \in \Gamma(D^\perp)$. Using (2.3), we get $g(\nabla_X Y,Z) = \bar{g}(\overline{\nabla}_X\phi Y,\phi Z)$. Hence we derive $g(\nabla_X Y,Z) = \bar{g}(h^s(X,$
$\phi Y),\phi Z)$, which proves our assertion. □

Theorem 4.7. *Let M be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the following assertions are equivalent:*

(i) *$D^\perp$ defines a totally geodesic foliation on M;*

(ii) *$A_{\phi X}Y$ has no components in $\overline{D}$;*

(iii) $B'h^s(X,\phi Z) = 0$ *and* $B'D^s(X,\phi N) = 0$, *for all* $X, Y \in \Gamma(D^\perp)$, $Z \in \Gamma(D)$, *and* $N \in \Gamma(l\mathrm{tr}(TM))$.

Proof. (i)⇒(ii). Suppose $D^\perp$ defines a totally geodesic foliation in M. Then, $\nabla_X Y \in \Gamma(D^\perp)$. From (1.5) and (2.3), we have $g(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X \phi Y, \phi Z)$ for $X, Y \in \Gamma(D^\perp)$, and $Z \in \Gamma(D)$. Using (1.7), we obtain

$$g(\nabla_X Y, Z) = -g(A_{\phi Y} X, \phi Z). \tag{4.14}$$

In a similar way, we get

$$\overline{g}(\nabla_X Y, N) = -\overline{g}(A_{\phi Y} X, \phi N), \quad \forall N \in \Gamma(l\mathrm{tr}(TM)). \tag{4.15}$$

Thus (i)⇒(ii) follows from (4.14) and (4.15).

(ii)⇒(iii) follows from (4.14), (4.15), (1.8), and (1.9).

(iii)⇒(i). By definition of contact SCR-lightlike submanifold, $D^\perp$ defines a totally geodesic foliation in M if and only if $g(\nabla_X Y, Z) = g(\nabla_X Y, V) = \overline{g}(\nabla_X Y, N) = 0$ for $X, Y \in \Gamma(D^\perp)$, $Z \in \Gamma(\overline{D})$, and $N \in \Gamma(l\mathrm{tr}(TM))$. From (1.5) and (2.2), we obtain $g(\nabla_X Y, V) = 0$. Follow a similar method to (4.14) and (4.15), we get $g(\nabla_X Y, Z) = -\overline{g}(h^s(X,\phi Z), \phi Y)$ and $g(\nabla_X Y, N) = -g(D^s(X,\phi N), \phi Y)$. By assumption, $B'h^s(X,\phi Z) = 0$ and $B'D^s(X,\phi N) = 0$. Hence we obtain $g(\nabla_X Y, Z) = 0$ and $g(\nabla_X Y, N) = 0$, which proves the assertion. □

Lemma 4.8. *Let M be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then*

$$h^l(X, V) = 0, \quad \forall X \in \Gamma(TM), \tag{4.16}$$

$$\nabla_X V = \phi X, \quad h^s(X, V) = 0, \quad \forall X \in \Gamma(\overline{D}), \tag{4.17}$$

$$\nabla_X V = 0, \quad h^s(X, V) = \phi X, \quad \forall X \in \Gamma(D^\perp). \tag{4.18}$$

Proof. Using (2.2) and (1.5), we get $\nabla_X V + h^l(X, V) + h^s(X, V) = \phi X$ for $X \in \Gamma(TM)$. Then, considering (4.2), we get (4.16)–(4.18). □

Theorem 4.9. *Any totally contact umbilical proper contact SCR-lightlike submanifold M of $\overline{M}$ admits a metric connection.*

Proof. From (4.11), we obtain $h^l(X,\phi Y) = h^l(\phi Y, X)$, for all $X, Y \in \Gamma(\overline{D})$. Using this and (3.20), we get $g(X,\phi Y)\alpha_L = g(\phi X, Y)\alpha_L$. Thus, $g(X,\phi Y)\alpha_L = 0$, since D is nondegenerate and $\alpha_L = 0$. Thus, $h^l(X, Y) = \eta(X)h^l(Y, V) + \eta(Y)h^l(X, V)$ for $X, Y \in \Gamma(TM)$. From Lemma 4.8, if $X, Y \in \Gamma(D \perp \overline{D})$, then, we obtain $h^l(X, Y) = 0$. If $X \in \Gamma(TM)$ and $Y = V$, then from (4.16), we get $h^l(X, V) = 0$. Thus $h^l = 0$ on M. Finally, our assertion follows from (1.15). □

Theorem 4.10. *Let M be a totally contact umbilical contact SCR-lightlike submanifold of $\overline{M}$. If* $\dim(D^\perp) > 1$, *then M is contact totally geodesic.*

Proof. The proof is similar to the proof of Theorem 3.12. □

A plane section p in $T_x\overline{M}$ of a Sasakian manifold $\overline{M}$ is called a ϕ-section if it is spanned by a unit vector X orthogonal to V and ϕX, where X is a nonnull vector field on $\overline{M}$. The sectional curvature $K(p)$ with respect to p determined by X is called a ϕ-sectional curvature. If $\overline{M}$ has a ϕ-sectional curvature c which does not depend on the ϕ-section at each point, then c is constant in $\overline{M}$. Then, $\overline{M}$ is called an indefinite Sasakian space form, denoted by $\overline{M}(c)$. The curvature tensor $\overline{R}$ of $\overline{M}(c)$ is given by [7]

$$\begin{aligned}\overline{R}(X,Y)Z = {} & \frac{(c+3)}{4}\{\overline{g}(Y,Z)X-\overline{g}(X,Z)Y\} \\ & +\frac{(c-1)}{4}\{\epsilon\eta(X)\eta(Z)Y-\epsilon\eta(Y)\eta(Z)X+\overline{g}(X,Z)\eta(Y)V \\ & -\overline{g}(Y,Z)\eta(X)V+\overline{g}(\phi Y,Z)\phi X+\overline{g}(\phi Z,X)\phi Y-2\overline{g}(\phi X,Y)\phi Z\}\end{aligned} \tag{4.19}$$

for any X, Y, and Z vector fields on $\overline{M}$.

THEOREM 4.11. *There exist no totally contact umbilical proper contact SCR-lightlike submanifolds in $\overline{M}(c)$ with $c \neq -3$.*

Proof. Suppose M is totally contact umbilical proper SCR-lightlike submanifold of $\overline{M}$. From Gauss equation (3.1) in [1, page 171] and (4.19), we get

$$\begin{aligned}&\frac{1}{2}(1-c)g(X,X)g(Z,Z)=\overline{g}((\nabla_X h^s)(\phi X,Z),\phi Z), \\ &-\overline{g}((\nabla_{\phi X} h^s)(X,Z),\phi Z), \quad \forall X\in\Gamma(D),\ Z\in\Gamma(D^{\perp}),\end{aligned} \tag{4.20}$$

where $(\nabla_X h^s)(\phi X,Z) = \nabla^s_X h^s(\phi X,Z) - h^s(\nabla_X\phi X,Z) - h^s(\phi X,\nabla_X Z)$. Since M is totally contact umbilical, we have $h^s(\phi X,Z)=0$, and from (3.21), we get

$$-h^s(\nabla_X\phi X,Z) = -g(\nabla_X\phi X,Z)\alpha_S - g(\nabla_X\phi X,V)\phi Z. \tag{4.21}$$

Using (1.5) and (2.2), we obtain

$$-h^s(\nabla_X\phi X,Z) = -g(\nabla_X\phi X,Z)\alpha_S + g(X,X)\phi Z. \tag{4.22}$$

In a similar way, we get

$$-h^s(\phi X,\nabla_X Z) = -g(\phi X,\nabla_X Z)\alpha_S. \tag{4.23}$$

Thus from (4.22) and (4.23), we have

$$(\nabla_X h^s)(\phi X,Z) = -g(\nabla_X\phi X,Z)\alpha_S + g(X,X)\phi Z - g(\phi X,\nabla_X Z)\alpha_S. \tag{4.24}$$

On the other hand, since $\overline{g}(\phi X,Z)=0$, taking the covariant derivative with respect to X, we obtain $g(\nabla_X\phi X,Z) = -g(\phi X,\nabla_X Z)$. Hence we get

$$(\nabla_X h^s)(\phi X,Z) = g(X,X)\phi Z. \tag{4.25}$$

In a similar way, we have

$$(\nabla_{\phi X} h^s)(X,Z) = -g(X,X)\phi Z. \tag{4.26}$$

Thus from (4.25), (4.26), and (4.20), we obtain

$$\frac{1}{2}(1-c)g(X,X)g(Z,Z) = 2g(X,X)g(Z,Z). \tag{4.27}$$

Hence, we have $(3+c)g(X,X)g(Z,Z) = 0$. Since D and $D^{\perp}$ are nondegenerate, we can choose nonnull vector fields X and Z, so $c = -3$, which proves theorem. □

5. Minimal lightlike submanifolds

Recall a general notion of minimal lightlike submanifold M, introduced by Bejan and Duggal [16], as follows.

Definition 5.1. Say that a lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is minimal if

(i) $h^s = 0$ on $\mathrm{Rad}(TM)$;

(ii) $\operatorname{trace} h = 0$, where trace is written with respect to g restricted to $S(TM)$.

In the second case, the condition (i) is trivial. Moreover, it has been shown in [16] that the above definition is independent of $S(TM)$ and $S(TM^{\perp})$, but it depends on the choice of the transversal bundle $\mathrm{tr}(TM)$.

As in the semi-Riemannian case, any lightlike totally geodesic M is minimal. Thus, from Theorem 2.5, any totally umbilical lightlike submanifold, with structure vector field tangent to submanifold, is minimal. Furthermore, from Theorems 3.12 and 4.10 of this paper, it follows that totally contact umbilical contact CR-lightlike submanifold with $(\dim(\phi L_1) > 1)$ and totally contact umbilical contact SCR-lightlike submanifolds with $(\dim(D^{\perp} > 1))$ are minimal.

Example 5.2. Let $\overline{M} = (\mathbf{R}^{11}_4, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-,-,+,+,+,-,-,+,+,+,+)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial z\}. \tag{5.1}$$

Suppose M is a submanifold of $\mathbf{R}^{11}_4$ given by

$$\begin{gathered}
x^1 = u^1, \qquad y^1 = -u^5, \\
x^2 = \cosh u^2 \cosh u^3, \qquad y^2 = \cosh u^2 \sinh u^3, \\
x^3 = \sinh u^2 \cosh u^3, \qquad y^3 = \sinh u^2 \sinh u^3, \\
x^4 = u^4, \qquad y^4 = -u^6, \\
x^5 = u^1 \cos\theta + u^5 \sin\theta, \qquad y^5 = u^1 \sin\theta - u^5 \cos\theta, \qquad z = u^7.
\end{gathered} \tag{5.2}$$

Then it is easy to see that a local frame of TM is given by

$$\begin{aligned}
Z_1 &= \partial x_1 + \cos\theta\partial x_5 + \sin\theta\partial y_5 + (y^1 + y^5\cos\theta)\partial z,\\
Z_2 &= \sin\theta\partial x_5 - \partial y_1 - \cos\theta\partial y_5 + y^5\sin\theta\partial z,\\
Z_3 &= \sinh u^2\cosh u^3\partial x_2 + \cosh u^2\cosh u^3\partial x_3 + \sinh u^2\sinh u^3\partial y_2\\
&\quad + \cosh u^2\sinh u^3\partial y_3 + (y^2\sinh u^2\cosh u^3 + y^3\cosh u^2\cosh u^3)\partial z,\\
Z_4 &= \cosh u^2\sinh u^3\partial x_2 + \sinh u^2\sinh u^3\partial x_3 + \cosh u^2\cosh u^3\partial y_2\\
&\quad + \sinh u^2\cosh u^3\partial y_3 + (y^2\sinh u^3\cosh u^2 + y^3\sinh u^2\sinh u^3)\partial z,\\
Z_5 &= \partial x_4 + y^4\partial z, \qquad Z_6 = -\partial y^4, \qquad Z_7 = 2\partial z.
\end{aligned}\tag{5.3}$$

We see that M is a 2-lightlike submanifold with $\operatorname{Rad}TM = \operatorname{span}\{Z_1,Z_2\}$ and $\phi_o Z_1 = Z_2$. Thus, $\operatorname{Rad}TM$ is invariant with respect to ϕ_o. Since $\phi_o(Z_5) = Z_6$, $D = \{Z_5,Z_6\}$ is also invariant. Moreover, since $\phi_o Z_3$ and $\phi_o Z_4$ are perpendicular to TM and they are nonnull, we can choose $S(TM^\perp) = \operatorname{Span}\{\phi_o Z_3,\phi_o Z_4\}$. Furthermore, the lightlike transversal bundle $l\operatorname{tr}(TM)$ spanned by

$$\begin{aligned}
N_1 &= 2(-\partial x_1 + \cos\theta\partial x_5 + \sin\theta\partial y_5 + (-y^1 + y^5\cos\theta)\partial z),\\
N_2 &= 2(\sin\theta\partial x_5 + \partial y_1 - \cos\theta\partial y_5 + y^5\sin\theta\partial z)
\end{aligned}\tag{5.4}$$

is also invariant. Thus we conclude that M is a contact SCR-lightlike submanifold of $\mathbf{R}^{11}_4$. Then a quasiorthonormal basis of $\overline{M}$ along M is given by

$$\begin{gathered}
\xi_1 = Z_1, \qquad \xi_2 = Z_2, \qquad e_1 = \frac{2}{\sqrt{\cosh^2 u^3 + \sinh^2 u^3}}Z_3,\\
e_2 = \frac{2}{\sqrt{\cosh^2 u^3 + \sinh^2 u^3}}Z_4, \qquad e_3 = 2Z_5, \qquad e_4 = 2Z_6, \qquad Z = Z_7,\\
W_1 = \frac{2}{\sqrt{\cosh^2 u^3 + \sinh^2 u^3}}\phi_o Z_3, \qquad W_2 = \frac{2}{\sqrt{\cosh^2 u^3 + \sinh^2 u^3}}\phi_o Z_4, N_1, N_2,
\end{gathered}\tag{5.5}$$

where $\varepsilon_1 = g(e_1,e_1) = 1$, $\varepsilon_2 = g(e_2,e_2) = -1$, and g is the degenerate metric on M. By direct calculations and using Gauss formula (1.5), we get

$$\begin{gathered}
h^s(X,\xi_1) = h^s(X,\xi_2) = h^s(X,e_3) = h^s(X,e_4) = 0, \quad h^l = 0, \quad \forall X \in \Gamma(TM),\\
h^s(e_1,e_1) = \frac{1}{\cosh^2 u^3 + \sinh^2 u^3}W_2, \qquad h^s(e_2,e_2) = \frac{1}{\cosh^2 u^3 + \sinh^2 u^3}W_2.
\end{gathered}\tag{5.6}$$

Therefore,

$$\operatorname{trace} h_{g|_{S(TM)}} = \varepsilon_1 h^s(e_1,e_1) + \varepsilon_2 h^s(e_2,e_2) = h^s(e_1,e_1) - h^s(e_2,e_2) = 0. \tag{5.7}$$

Thus, M is a minimal contact SCR-lightlike submanifold of $\mathbf{R}^{11}_4$.

Now we prove characterization results for minimal lightlike submanifolds of all the cases discussed in this paper.

THEOREM 5.3. *Let M be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then M is minimal if and only if*

$$\operatorname{trace} A_{W_j}|_{S(TM)} = 0, \quad A^*_{\xi_k} = 0 \quad \text{on } D^\perp,\ D^l(X,W) = 0 \tag{5.8}$$

for $X \in \Gamma(\operatorname{Rad} TM)$ and $W \in \Gamma(S(TM^\perp))$.

Proof. Since $\overline{\nabla}_V V = \phi V = 0$, from (1.5) we get $h^l(V,V) = h^s(V,V) = 0$. Now take an orthonormal frame $\{e_1,\dots,e_{m-r}\}$ such that $\{e_1,\dots,e_{2a}\}$ are tangent to D and $\{e_{2a+1},\dots,e_{m-2r}\}$ are tangent to $D^\perp$. First from [16], we know that $h^l = 0$ on $\operatorname{Rad}(TM)$. Now, from (4.11), for $Y,Z \in \Gamma(D)$, we have

$$h^l(\phi Y,Z) = \phi h^l(Y,Z). \tag{5.9}$$

Hence, we obtain $h^l(\phi Z,\phi Y) = -h^l(Y,Z)$. Thus $\sum_{i=1}^{2a} h^l(e_i,e_i) = 0$. Since $\operatorname{trace} h|_{S(TM)} = \sum_{i=1}^{m-2r} \varepsilon_i(h^l(e_i,e_i) + h^s(e_i,e_i))$, M is minimal if and only if

$$\sum_{i=1}^{2a} \varepsilon_i h^s(e_i,e_i) + \sum_{2a+1}^{m-2r} \varepsilon_i\big(h^l(e_i,e_i) + h^s(e_i,e_i)\big) = 0. \tag{5.10}$$

On the other hand, we have

$$\begin{aligned}\operatorname{trace} h|_{S(TM)} = {}& \sum_{i=1}^{2a} \frac{1}{n-2r} \sum_{j=1}^{n-2r} \varepsilon_i \overline{g}(h^s(e_i,e_i),W_j)W_j \\ &+ \frac{1}{n-2r} \sum_{j=1}^{n-2r} \sum_{i=2a+1}^{m-2r} \varepsilon_i \overline{g}(h^s(e_i,e_i),W_j)W_j \\ &+ \sum_{k=1}^{2r} \frac{1}{2r} \sum_{i=2a+1}^{m-2r} \varepsilon_i \overline{g}(h^l(e_i,e_i),\xi_k)N_k.\end{aligned} \tag{5.11}$$

Using (1.8) and (1.12), we get

$$\begin{aligned}\operatorname{trace} h|_{S(TM)} = {}& \sum_{i=1}^{2a} \frac{1}{n-2r} \sum_{j=1}^{n-2r} \varepsilon_i \overline{g}(A_{W_j}e_i,e_i)W_j \\ &+ \frac{1}{n-2r} \sum_{j=1}^{n-2r} \sum_{i=2a+1}^{m-2r} \varepsilon_i \overline{g}(A_{W_j}e_i,e_i)W_j \\ &+ \sum_{k=1}^{2r} \frac{1}{2r} \sum_{i=2a+1}^{m-2r} \varepsilon_i \overline{g}(A^*_{\xi_k}e_i,e_i)N_k.\end{aligned} \tag{5.12}$$

On the other hand, from (1.8) we obtain

$$\overline{g}(h^s(X,Y),W) = \overline{g}(Y,D^l(X,W)), \quad \forall X,Y \in \Gamma(\operatorname{Rad} TM),\ \forall W \in \Gamma(S(TM^\perp)). \quad (5.13)$$

Thus our assertion follows from (5.12) and (5.13). □

Theorem 5.4. *Let M be an irrotational screen real lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then M is minimal if and only if*

$$\operatorname{trace} A_{W_a} = 0 \quad \text{on } S(TM). \quad (5.14)$$

Proof. Proposition 4.4 implies that $h^l = 0$. Thus M is minimal if and only if $h^s = 0$ on $\operatorname{Rad} TM$ and $\operatorname{trace} h^s|_{S(TM)} = 0$. Then, the proof follows from Theorem 5.3. □

Theorem 5.5. *Let M be an invariant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then M is minimal in $\overline{M}$ if and only if $D^l(X,W) = 0$ for $X \in \Gamma(\operatorname{Rad} TM)$ and $W \in \Gamma(S(TM))$.*

Proof. If M is invariant, then $\phi \operatorname{Rad} TM = \operatorname{Rad} TM$ and $\phi S(TM) = S(TM)$, hence $\phi(l\operatorname{tr}(TM)) = l\operatorname{tr} TM$ and $\phi S(TM^\perp) = STM^\perp$. Then using (2.5), (1.5), and taking transversal part, we obtain

$$h(\phi X,Y) = \phi h(X,Y) \quad (5.15)$$

for $X,Y \in \Gamma(TM)$. Hence we get $h(\phi X,\phi Y) = -h(X,Y)$. Thus

$$\operatorname{trace} h|_{S(TM)} = \sum_{i=1}^{m-2r} \varepsilon_i\{h(e_i,e_i) + h(\phi e_i,\phi e_i)\} = \sum_{i=1}^{m-2r} \varepsilon_i\{h(e_i,e_i) - h(e_i,e_i)\} = 0. \quad (5.16)$$

From (1.8), we get $\overline{g}(h^s(X,Y),W) = \overline{g}(D^l(X,W),Y)$ for $X,Y \in \Gamma(\operatorname{Rad} TM)$ and $W \in \Gamma(S(TM^\perp))$. The proof follows from Definition 5.1 and $h^l = 0$ on $\operatorname{Rad} TM$. □

Theorem 5.6. *Let M be an irrotational contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then M is minimal in $\overline{M}$ if and only if*

(1) *$A^*_\xi \phi\xi$ and $A_N \phi N$ have no components in D',*

(2) *$D^s(\phi N,N)$ has no components in $L_1^\perp$,*

(3) $\operatorname{trace} A_{W_a}|_{D_0 \perp \phi L_1} = 0$, $\operatorname{trace} A^*_{\xi_k}|_{D_0 \perp \phi L_1} = 0$,

for $N \in \Gamma(l\operatorname{tr}(TM))$ and $\xi \in \Gamma(\operatorname{Rad} TM)$, where $D' = \phi(l\operatorname{tr}(TM)) \perp \phi(L_1)$.

Proof. Suppose M is irrotational. From (1.5) and (2.3), we have $\overline{g}(h^l(\phi\xi,\phi\xi),\xi_1) = -\overline{g}(\overline{\nabla}_{\phi\xi}\xi,\phi\xi_1)$. Then using (1.11) and (1.5), we obtain

$$\overline{g}(h^l(\phi\xi,\phi\xi),\xi_1) = g(A^*_\xi \phi\xi,\phi\xi_1), \quad \forall \xi,\xi_1 \in \Gamma(\operatorname{Rad} TM). \quad (5.17)$$

In a similar way, from (1.5), (2.3), (1.11), and (4.3), we get

$$\overline{g}(h^s(\phi\xi,\phi\xi),W) = g(A^*_\xi \phi\xi,BW), \quad \forall \xi \in \Gamma(\operatorname{Rad} TM),\ W \in \Gamma(S(TM^\perp)). \quad (5.18)$$

Now, using (1.7), (1.5), and (4.3), we derive

$$h(\phi N,\phi N) = -\omega A_N\phi N + CD^s(\phi N,N), \quad \forall N \in \Gamma(l\mathrm{tr}(TM)). \tag{5.19}$$

Then the proof follows from (5.17)–(5.19) and Theorem 5.3. □

6. Concluding remarks

(a) It is well known that the second fundamental forms and their shape operators of a nondegenerate submanifold are related by means of the metric tensor field. Contrary to this, we see from (1.5)–(1.9) that in case of lightlike submanifolds, there are interrelations between these geometric objects and those of their screen distributions. Thus, the geometry of lightlike submanifolds depends on the triplet $(S(TM),S(TM^{\perp}),l\mathrm{tr}(TM))$. However, it is important to highlight that as per Proposition 1.1 of this paper, our results are stable with respect to any change in the above triplet. Moreover, we have verified that the conclusions of all our results will not change with the change of any induced object on M.

(b) Note that there does not exist any inclusion relation between contact CR-lightlike and contact SCR-lightlike submanifolds. Indeed, contact CR-lightlike submanifolds are always nontrivial. Also, contrary to the case of contact CR-lightlike hypersurfaces, there do not exist any contact SCR-lightlike hypersurfaces. We, therefore, state the following problem.

Find a class of lightlike submanifolds, of an indefinite Sasakian manifold, which is an umbrella of contact CR and contact SCR-lightlike submanifolds.

The above problem is motivated from the fact that CR-submanifolds were designed as an umbrella of all types of submanifolds of a Riemannian manifold. We are working on a followup paper to address the above-stated problem.

For a similar study on all possible CR-lightlike submanifolds of Kählerian manifolds, see Duggal and Sahin [15, 17].

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References

[1] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, vol. 364 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.
[2] V. I. Arnol'd, "Contact geometry: the geometrical method of Gibbs's thermodynamics," in *Proceedings of the Gibbs Symposium (New Haven, CT, 1989)*, pp. 163–179, American Mathematical Society, Providence, RI, USA, 1990.
[3] S. Maclane, *Geometrical Mechanics II*, Lecture Notes, University of Chicago, Chicago, Ill, USA, 1968.

[4] V. E. Nazaikinskii, V. E. Shatalov, and B. Y. Sternin, *Contact Geometry and Linear Differential Equations*, vol. 6 of *De Gruyter Expositions in Mathematics*, Walter de Gruyter, Berlin, Germany, 1992.
[5] C. Călin, "Contributions to geometry of CR-submanifold," Thesis, University of Iasi, Iasi, Romania, 1998.
[6] C. Călin, "On the existence of degenerate hypersurfaces in Sasakian manifolds," *Arab Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 21–27, 1999.
[7] T. H. Kang, S. D. Jung, B. H. Kim, H. K. Pak, and J. S. Pak, "Lightlike hypersurfaces of indefinite Sasakian manifolds," *Indian Journal of Pure and Applied Mathematics*, vol. 34, no. 9, pp. 1369–1380, 2003.
[8] K. Yano and M. Kon, *Structures on Manifolds*, vol. 3 of *Series in Pure Mathematics*, World Scientific, Singapore, 1984.
[9] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, vol. 366 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.
[10] K. L. Duggal, "Space time manifolds and contact structures," *International Journal of Mathematics and Mathematical Sciences*, vol. 13, no. 3, pp. 545–553, 1990.
[11] T. Takahashi, "Sasakian manifold with pseudo-Riemannian metric," *The Tohoku Mathematical Journal*, vol. 21, pp. 271–290, 1969.
[12] S. Tanno, "Sasakian manifolds with constant ϕ-holomorphic sectional curvature," *The Tohoku Mathematical Journal*, vol. 21, pp. 501–507, 1969.
[13] K. L. Duggal and D. H. Jin, "Totally umbilical lightlike submanifolds," *Kodai Mathematical Journal*, vol. 26, no. 1, pp. 49–68, 2003.
[14] A. Bejancu, *Geometry of CR-Submanifolds*, vol. 23 of *Mathematics and Its Applications (East European Series)*, D. Reidel, Dordrecht, The Netherlands, 1986.
[15] K. L. Duggal and B. Sahin, "Screen Cauchy Riemann lightlike submanifolds," *Acta Mathematica Hungarica*, vol. 106, no. 1-2, pp. 137–165, 2005.
[16] C. L. Bejan and K. L. Duggal, "Global lightlike manifolds and harmonicity," *Kodai Mathematical Journal*, vol. 28, no. 1, pp. 131–145, 2005.
[17] K. L. Duggal and B. Sahin, "Generalized Cauchy-Riemann lightlike submanifolds of Kaehler manifolds," *Acta Mathematica Hungarica*, vol. 112, no. 1-2, pp. 107–130, 2006.

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