Zhang *Advances in Difference Equations* 2013, **2013**:243
http://www.advancesindifferenceequations.com/content/2013/1/243

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Existence and uniqueness of wave fronts in neuronal network with nonlocal post-synaptic axonal and delayed nonlocal feedback connections

Lijun Zhang*

*Correspondence:
li-jun0608@163.com
Department of Mathematics, School of Science, Zhejiang Sci-Tech University, Hangzhou, Zhejiang 310018, P.R. China

Abstract

An integral-differential model equation, arising from neuronal networks with both axonal and delayed nonlocal feedback connections, is considered in this paper. The kernel functions in the feedback channel we study here include not only pure excitations but also lateral inhibition. For the kernel functions in the synaptic coupling, pure excitations, lateral inhibition, the lateral excitations and more general synaptic couplings (*e.g.*, oscillating kernel functions) are considered. The main goal of this paper is the study of the existence and uniqueness of the traveling wave front solutions. The main method we applied is the speed index functions and principle of linear superposition.

1 Introduction

In this paper, we consider the following integral differential model equation

$$
\begin{aligned}
u_t + u = {}& \alpha \int_R K(x-y) H\left(u\left(y, t - \frac{|x-y|}{c}\right) - \theta\right) dy \\
& + \beta \int_R J(x-y) H\big(u(y, t-\tau) - \theta\big)\, dy,
\end{aligned}
\tag{1.1}
$$

which was proposed by Hutt [1] to understand the mechanism of the formation and propagation of activity patterns in neural networks. Here, $u(x,t)$ represents the effective post-synaptic potential of the neuron population at location x and time t. The first term and the second one on the right side of equation (1.1) represent the synaptic input by axonal and feedback connections, respectively. The kernel functions $K(x)$ and $J(x)$ are introduced as probability density functions of connection, which may be negative at some points to allow for inhibitory behavior in coupling. The parameters α and β represent the synaptic strength of axonal and nonlocal feedback contributions, respectively. Both the intral-areal nonlocal axonal connections with a transmission delay $\frac{|x-y|}{c}$ and nonlocal feedback connections with a constant time delay τ are incorporated in this model equation, where c is the transmission speed for both excitatory and inhibitory connections. The transfer function H is always chosen to be the Heaviside step function: $H(u-\theta) = 0$ for all $u < \theta$,

$H(\theta) = \frac{1}{2}$, and $H(u - \theta) = 1$ for all $u > \theta$. $H(u - \theta)$ denotes the output firing rate of a neuron, which means that a neuron fires at its maximum rate when the potential exceeds a threshold, and does not fire otherwise. Here, θ is assumed to be the firing threshold for all neurons functions.

The model equation (1.1) is a modification and generalization of some existing models and also relates to some model equations, which were concerned recently (see Atay and Hutt [2], Coombes *et al.* [3], Ermentrout [4], [5–8]). For example, the following equation

$$u_t + u = \alpha \int_R K(x - y)H\big(u(y,t) - \theta\big)\,dy, \tag{1.2}$$

which was derived in Amari [9] and can be used to describe the simple fields, which are one-dimensional, homogeneous, have negligible time lag and consist of only one layer. The following nonlocal nonlinear scalar integral model equation with the incorporate spatio-temporal delay was proposed to describe the dynamics of an effective post-synaptic potential $u(x,t)$ at position x and the time t

$$u_t + u = \alpha \int_R K(x - y)H\left(u\left(y, t - \frac{1}{c}|x - y|\right) - \theta\right)dy. \tag{1.3}$$

The traveling fronts of these model equations have attracted much research interest in theory partly due to experimental findings [10, 11]. It is well known that the kernel functions in these model equations reflect the underlying connectivity in neural tissue. However, the kernel functions, which were considered in previous works mostly are the following three classes:

(A) The first class consists of nonnegative kernel functions (pure excitation).

(B) The second class consists of Mexican hat kernel functions (lateral inhibition), that is, $K \geq 0$ on $(-M, N)$ and $K \leq 0$ on $(-\infty, -M) \cup (N, \infty)$ for some positive constants M and N.

(C) The third class consists of upside down Mexican hat kernel functions (lateral excitation), that is, $K \leq 0$ on $(-M, N)$ and $K \geq 0$ on $(-\infty, -M) \cup (N, \infty)$, for two positive constants M and N.

The kernel functions are always supposed to be continuous at $x = 0$, almost everywhere smooth and satisfy the following conditions

$$\begin{aligned} &\int_{-\infty}^{0} K(x)\,dx = \frac{1}{2}; \qquad \int_{-\infty}^{+\infty} K(x)\,dx = 1; \qquad \int_{-\infty}^{0} |s|K(s)\,ds > 0; \\ &\big|K(x)\big| \leq k \exp\big(-\rho|x|\big) \quad \text{in } \mathbb{R}, \end{aligned} \tag{1.4}$$

where k and ρ are positive constants.

Under the assumptions above, Zhang [12] studied the existence, uniqueness and stability of traveling wave solutions to the model equation (1.3) with three typical classes of kernel functions. The traveling wave fronts of generalized equation

$$u_t + f(u) = \alpha \int_R K(x - y)H\left(u\left(y, t - \frac{1}{c}|x - y|\right) - \theta\right)dy \tag{1.5}$$

were considered in [13] more recently. The existence and uniqueness of traveling wave fronts of equations (1.2) and (1.3) were studied for five classes of oscillatory kernel functions by Lv and Wang [14]. While for equation (1.1), only few special types of kernel functions are known. In Hutt [1], the kernel functions $K(x)$ and $J(x)$ are assumed to have the following forms

$$K(x) = \frac{a_e}{2}e^{-|x|} - \frac{a_i r}{2}e^{-r|x|}, \qquad J(x) = \frac{1}{2\sigma}e^{-|x|/\sigma}. \tag{1.6}$$

In [15], Magpantay and Zou studied the wave fronts of equation (1.1), in which the kernel function in the feedback channel is assumed to be nonnegative (pure excitation), and for the kernel function in the synaptic coupling, four types, including types (A), (B) and (C) and the pure inhibition type, were considered. Modeling of traveling phenomena in general neural systems necessitates the study for more general types of kernel functions. However, it is more complicated in practical application in neuronal network. We may meet the case that both excitation and inhibition happen at the same time. Thus, it is important to study equation (1.1) with more general kernel functions.

In the model equation (1.1), the parameters are always supposed to be positive and $0 < c \leq \infty$. To include the case $c \to \infty$, it is understood that $\frac{c-\mu}{c\mu} = \frac{1}{\mu}$ when $c = \infty$. In this paper, for the kernel functions mentioned below, we suppose that they all satisfy the condition (1.4). We say $K(x) \in (B)$ if $K(x)$ satisfies condition (B).

Motivated by their exciting pioneering works [12–17], in this paper, we aim to study the existence and uniqueness of the wave front solutions of IDE (1.1) with more general kernel functions. The main idea in this paper is employing the speed index functions (the main idea in [12, 17] and other pioneering works) and the principle of linear superposition. It is easy to see that the kernel functions that were studied before are included in our study. For example, if the kernel functions $K(x)$ satisfy (L_4) and (R_3) with $n = 1$ (see Section 4), then they are upside down Mexican hat kernel functions; actually, if the kernel functions $K(x)$ satisfy (L_3) and (R_5) with $n = 1$, then they are of case (A) in [14]. We also prove the existence and uniqueness of the traveling wave solutions of this model equation with the general classes of kernel functions under less restrictive conditions.

2 Preliminaries

The traveling wave solutions of an equation are the solutions of the form $u(x,t) = U(x+\mu t)$, where $z = x + \mu t$ is the moving coordinate and μ is a constant, which represents the speed of the traveling wave. Generally speaking, there are two kinds of traveling waves that attracted much more research concerns, because they possess some important and practical meanings in a neural network, which are the traveling wave front and the traveling pulse. We mainly focus on the traveling wave front in this paper.

To study the traveling wave solutions of the integral-differential equation (1.1), we substitute $u(x,t) = U(x + \mu t) = U(z)$ into equation (1.1), then we get

$$\begin{aligned}\mu U'(z) + U(z) &= \alpha \int_R K(x-y)H\left(U\left(y + \mu t - \frac{\mu}{c}|x-y|\right) - \theta\right) dy \\ &\quad + \beta \int_R J(x-y)H\big(U(y + \mu t - \mu\tau) - \theta\big)\, dy.\end{aligned} \tag{2.1}$$

By transformation of the variable x to $z = x + \mu t$ and the variable of integration y to $y + \mu t$, the integral IDE (2.1) can be written as

$$\mu U' + U = \alpha \int_R K(z-y) H\left(U\left(y - \frac{\mu}{c}|z-y|\right) - \theta\right) dy + \beta \int_R J(z - \mu\tau - y) H\big(U(y) - \theta\big)\, dy. \tag{2.2}$$

Let $0 < \mu < c$ and $t = y - \frac{\mu}{c}|z-y|$ in the first term of the right side of equation above, then we have

$$\mu U' + U = \alpha \left(\frac{1}{1 + \operatorname{sgn}(z-t)\frac{\mu}{c}}\right) \int_R K\left(\frac{z-t}{1 + \operatorname{sgn}(z-t)\frac{\mu}{c}}\right) H\big(U(t) - \theta\big)\, dt + \beta \int_R J(z - \mu\tau - y) H\big(U(y) - \theta\big)\, dy, \tag{2.3}$$

where $\operatorname{sgn}(x)$ is the sign-function, *i.e.*, $\operatorname{sgn}(x) = 1$ when $x > 0$; $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(x) = -1$ when $x < 0$. Clearly, the nonlinear terms are on the right side of equation (2.3). Because of the special property of Heaviside step function, we know that equation (2.3) can be simplified if some properties of the function $U(x)$ are known. Let $R[U,\theta] = \{y | U(y) > \theta\}$, then equation (2.3) can be reduced to

$$\mu U' + U = \alpha \left(\frac{1}{1 + \operatorname{sgn}(z-t)\frac{\mu}{c}}\right) \int_{R[U,\theta]} K\left(\frac{z-y}{1 + \operatorname{sgn}(z-y)\frac{\mu}{c}}\right) dy + \beta \int_{R[U,\theta]} J(z - \mu\tau - y)\, dy.$$

According to the property of the traveling wave front, we know that if $U(z)$ is a traveling wave front of equation (1.1), then $U_+ = \lim_{z\to+\infty} U(z)$ and $U_- = \lim_{z\to-\infty} U(z)$ exist, $U_+ \neq U_-$ and U_+, U_- should be two different constant solutions of equation (1.1) or equation (2.3). However, it is easy to see that the constant solutions of equation (1.1) only could be 0 or $\alpha + \beta$, since $R[U_{+(-)}, \theta] = \emptyset$ or $R[U_{+(-)}, \theta] = (-\infty, \infty)$.

Suppose that $U(z)$ is a traveling wave front of equation (1.1) satisfying $U(0) = \theta$, $U(z) < \theta$ when $z < 0$, $U(z) > \theta$ when $z > 0$, then $U(z)$ satisfies the following equation

$$\mu U' + U = \alpha \left(\frac{1}{1 + \operatorname{sgn}(z-t)\frac{\mu}{c}}\right) \int_0^{+\infty} K\left(\frac{z-t}{1 + \operatorname{sgn}(z-t)\frac{\mu}{c}}\right) dt + \beta \int_0^{+\infty} J(z - \mu\tau - y)\, dy, \tag{2.4}$$

which can be further rewritten as

$$\mu U' + U = \alpha \int_{-\infty}^{\frac{cz}{c + \operatorname{sgn}(z)\mu}} K(t)\, dt + \beta \int_{-\infty}^{z - \mu\tau} J(y)\, dy. \tag{2.5}$$

Obviously, equation (2.5) is a linear ordinary differential equation having two equilibria as 0 and $\alpha + \beta$, which means the wave front of equation (1.1) satisfying $U(0) = \theta$, $U(z) < \theta$ when $z < 0$, and $U(z) > \theta$ when $z > 0$ must have the limits $\lim_{z\to+\infty} U(z) = \alpha + \beta$ and

$\lim_{z\to-\infty} U(z) = 0$. Solving equation (2.5), we get the solution

$$U(z) = \alpha \int_{-\infty}^{\frac{cz}{c+\operatorname{sgn}(z)\mu}} K(s)\,ds - \alpha \int_{-\infty}^{z} \exp\left(\frac{s-z}{\mu}\right) K(s) \frac{c}{c+\operatorname{sgn}(s)\mu}\,ds + \beta \int_{-\infty}^{z-\mu\tau} J(s)\,ds - \beta \int_{-\infty}^{z} \exp\left(\frac{s-z}{\mu}\right) J(s-\mu\tau)\,ds \tag{2.6}$$

and

$$U' = \frac{\alpha}{\mu} \int_{-\infty}^{z} \exp\left(\frac{s-z}{\mu}\right) K(s) \frac{c}{c+\operatorname{sgn}(s)\mu}\,ds + \frac{\beta}{\mu} \int_{-\infty}^{z} \exp\left(\frac{s-z}{\mu}\right) J(s-\mu\tau)\,ds. \tag{2.7}$$

Notice that function (2.6) we obtained is just a solution of equation (2.5), but not necessarily a solution of equation (2.1), that is to say that it is not necessarily a traveling wave front of equation (1.1). Function (2.6) could be a traveling wave front of equation (1.1) only if it satisfies $U(0) = \theta$, $U(z) < \theta$ when $z < 0$, and $U(z) > \theta$ when $z > 0$. Based on the discussion above, we study the existence and uniqueness of the wave front solution of IDE (1.1) in the following two sections under some conditions by proving the existence of a unique wave speed μ such that function (2.6) with this unique μ satisfies $U(0) = \theta$, $U(z) < \theta$ when $z < 0$, and $U(z) > \theta$ when $z > 0$.

3 Existence and uniqueness of the wave solution of IDE (1.1) with $\alpha = 0$

In this section, we study the IDE (1.1) with $\alpha = 0$, *i.e.*,

$$u_t + u = \beta \int_R J(x-y) H\big(u(y,t-\tau) - \theta\big)\,dy. \tag{3.1}$$

According to the discussion in Section 2, we know that the traveling wave equation of (3.1) is

$$\mu U'(z) + U(z) = \beta \int_R J(z-y) H\big(U(y-\mu\tau) - \theta\big) J(y)\,dy,$$

where $z = x + \mu t$. If we suppose that $U(0) = \theta$ (because of the invariant property of the traveling wave solution), $U(z) > \theta$ when $z > 0$ and $U(z) < \theta$ when $z < 0$, then the equation above can be reduced to

$$\mu U'(z) + U(z) = \beta \int_{-\infty}^{z-\mu\tau} J(y)\,dy. \tag{3.2}$$

The solution of equation (3.2) is

$$U(z) = \beta \int_{-\infty}^{z} \left[1 - \exp\left(\frac{s-z}{\mu}\right)\right] J(s-\mu\tau)\,ds \tag{3.3}$$

and

$$U'(z) = \frac{\beta}{\mu} \int_{-\infty}^{z} \exp\left(\frac{s-z}{\mu}\right) J(s-\mu\tau)\,ds. \tag{3.4}$$

Similar to the analysis given in Section 2, we know that function (3.3) could be a solution of equation (3.1) if it satisfies the phase conditions $U(0) = \theta$; $U(z) > \theta$ when $z > 0$ and $U(z) < \theta$ when $z < 0$.

In the following, we firstly prove that function (3.3) satisfies the phase condition above when the kernel function $J(x)$ is of type (A) or type (B), *i.e.*, nonnegative function or Mexican hat function. Actually, from (3.3), we know that $U(0) = \theta$ equals to

$$\theta = \beta \int_{-\infty}^{0} \left[1 - \exp\left(\frac{s}{\mu}\right)\right] J(s - \mu\tau)\, ds. \tag{3.5}$$

We prove that there exists a unique $\mu^\star(\theta)$ if θ satisfies some conditions such that equality (3.5) holds when $\mu = \mu^\star(\theta)$. To achieve the process above, we define a function, which we call speed index function

$$\varphi(\mu) = \beta \int_{-\infty}^{0} \left[1 - \exp\left(\frac{s}{\mu}\right)\right] J(s - \mu\tau)\, ds, \tag{3.6}$$

which is a continuous function with μ.

3.1 Existence and uniqueness of the wave speed

At first, we prove the existence of the solution to $\varphi(\mu) = \theta$, and then, we prove that it is unique when the kernel function $J(z)$ is of type (A) or type (B). Note that for any real number, $c > 0$,

$$\lim_{\mu \to 0^+} \varphi(\mu) = \beta \int_{-\infty}^{0} J(s)\, ds = \frac{\beta}{2}$$

and

$$\varphi(c) = \beta \int_{-\infty}^{0} \left[1 - \exp\left(\frac{s}{c}\right)\right] J(s - c\tau)\, ds, \tag{3.7}$$

then it is easy to see that $0 \leq \varphi(c) < \frac{\beta}{2}$ for any kernel function of type (A), since β, c and τ are all positive parameters. Thus, for any θ on interval $(\varphi(c), \frac{\beta}{2})$, there exists $\mu^\star(\theta)$, $0 < \mu^\star(\theta) < c$ such that $\varphi(\mu^\star) = \theta$, *i.e.*, (3.5) holds when $\mu = \mu^\star(\theta)$, which follows directly from the intermediate value theorem for continuous function.

Next, we show that $\mu^\star(\theta)$ is the unique zero of the equation $\varphi(\mu) = \theta$ if the kernel function $J(z)$ satisfies some restrictive conditions. Computing the derivative of $\varphi(\mu)$ with respect to μ gives

$$\varphi'(\mu) = -\beta \frac{e^{\tau}}{\mu^2} \int_{-\infty}^{-\mu\tau} |s| \exp\left(\frac{s}{\mu}\right) J(s)\, ds. \tag{3.8}$$

For type (A), *i.e.*, $J(z) \in A$, it is obvious that $\varphi'(\mu) \leq 0$ and $\varphi'(\mu) = 0$ if and only if $J(s) \equiv 0$ on $(-\infty, -\mu\tau)$, since $J(z) \geq 0$ on $\mathbb{R}$. So $\varphi(\mu)$ is a nonincreasing function with μ. Then the uniqueness of the zero to the equation $\varphi(\mu) = \theta$ follows from the property of $\varphi'(\mu)$. If the uniqueness of the zero to the equation $\varphi(\mu) = \theta$ does not hold, then there exists an interval $[\mu_1, \mu_2] \subset (0, c)$ such that $\varphi(\mu) \equiv \theta$ when $\mu \in [\mu_1, \mu_2]$; $\varphi(\mu) > \theta$ when $\mu \in (0, \mu_1)$ and $\varphi(\mu) < \theta$ when $\mu \in (\mu_2, c)$. Then, we know that $\varphi'(\mu) = 0$ when $\mu \in (\mu_1, \mu_2)$, which means that $J(s) \equiv 0$ on $(-\infty, -\mu_1\tau)$, and thus, $\varphi(\mu) = 0$ when $\mu \in (\mu_1, c)$, which contradicts with the conclusion above that $\varphi(\mu) \equiv \theta$ when $\mu \in [\mu_1, \mu_2]$. Consequently, $\mu^\star(\theta)$ is the unique zero of the equation $\varphi(\mu) = \theta$ if the kernel function $J(z)$ is of type (A).

For type (B), *i.e.*, $J(z) \in B$, we have the following lemma.

Lemma 3.1 *Suppose that the kernel function $J(x)$ is of type* (B), *that is, $J(x) \leq 0$ on $(-\infty, -M) \cap (N, +\infty)$ and $J(x) \geq 0$ on $[-M, N]$, and $\int_{-\infty}^{0} |s|J(s)\,ds > 0$, then there exists a unique $M_0 \in (0, M)$ such that $\int_{-\infty}^{-M_0} |s|J(s)\,ds = 0$ and $\varphi(c) > 0$ if $c < \frac{M_0}{\tau}$.*

Proof Let $\nu(z) = \int_{-\infty}^{z} |s|J(s)\,ds$. Then it is easy to see that the function $\nu(z)$ is continuous, nonincreasing on $(-\infty, -M)$ and nondecreasing on $[-M, 0]$ since $J(x) \leq 0$ on $(-\infty, -M)$ and $J(x) \geq 0$ on $[-M, 0]$. Obviously, $\nu(0) = \int_{-\infty}^{0} |s|J(s)\,ds > 0$ and $\nu(-M) < 0$. By the intermediate value theorem, we know that there exists a unique $M_0 \in (0, M)$ such that $\int_{-\infty}^{-M_0} |s|J(s)\,ds = 0$. From (3.7), we know that if $c < \frac{M_0}{\tau}$,

$$\begin{aligned}
\varphi(c) &= \beta \int_{-\infty}^{-c\tau} \left[1 - \exp\left(\frac{s + c\tau}{c}\right)\right] J(s)\,ds \\
&> \left[1 - \exp\left(\frac{-M + c\tau}{c}\right)\right] \beta \int_{-\infty}^{-c\tau} J(s)\,ds \\
&> \left[1 - \exp\left(\frac{-M + c\tau}{c}\right)\right] \beta \int_{-\infty}^{-M_0} J(s)\,ds \\
&> \left[1 - \exp\left(\frac{-M + c\tau}{c}\right)\right] \frac{\beta}{M} \int_{-\infty}^{-M_0} |s|J(s)\,ds \\
&= 0.
\end{aligned}$$

□

However, when $J(z) \in B$ and $0 < \mu\tau < M_0$, *i.e.*, $0 < \mu < \frac{M_0}{\tau}$, it is easy to see that

$$\begin{aligned}
\varphi'(\mu) &= -\beta \frac{e^\tau}{\mu^2} \int_{-\infty}^{-\mu\tau} |s| \exp\left(\frac{s}{\mu}\right) J(s)\,ds \\
&< -\beta \frac{e^\tau}{\mu^2} \exp\left(\frac{-M}{\mu}\right) \int_{-\infty}^{-\mu\tau} |s|J(s)\,ds \\
&< -\beta \frac{e^\tau}{\mu^2} \exp\left(\frac{-M}{\mu}\right) \int_{-\infty}^{-M_0} |s|J(s)\,ds \\
&= 0.
\end{aligned}$$

Consequently, for any $c \in (0, \frac{M_0}{\tau})$, function (3.6) is decreasing with μ on interval $(0, c)$, and then $0 < \varphi(c) < \frac{\beta}{2}$, which can be applied to get the uniqueness of the zero of the equation $\varphi(\mu) = \theta$ when $\varphi(c) < \theta < \frac{\beta}{2}$. We summarize the discussion above in the following theorem.

Theorem 3.1 (Existence and uniqueness of the wave speed) *For any $\theta \in (\varphi(c), \frac{\beta}{2})$,*

(1) *if the kernel function $J(x)$ is of type* (A), *then for any $c > 0$, there exists a unique $\mu^\star(\theta) \in (0, c)$, such that $\varphi(\mu^\star) = \theta$;*

(2) *if the kernel functions $J(x)$ is of type* (B), *then for any $c \in (0, \frac{M_0}{\tau})$, there exists a unique $\mu^\star(\theta) \in (0, c)$, such that $\varphi(\mu^\star) = \theta$, where $M_0 \in (0, M)$ and is uniquely determined by $\int_{-\infty}^{-M_0} |s|J(s)\,ds = 0$.*

3.2 Existence and uniqueness of the wave

We prove that function (3.3) could satisfy $U(z) > \theta$ when $z > 0$ and $U(z) < \theta$ when $z < 0$, when $\mu = \mu^\star(\theta)$ and the kernel function $J(x)$ is of type (A) or type (B), *i.e.*, nonnegative function or Mexican hat function. In the rest of this subsection, we write μ for $\mu^\star(\theta)$ for simplicity.

It is easy to see from (3.4) that

$$U'(z) = \frac{\beta}{\mu}\int_{-\infty}^{z} \exp\left(\frac{s-z}{\mu}\right) J(s-\mu\tau)\,ds = \frac{\beta}{\mu}\exp\left(\frac{\mu\tau - z}{\mu}\right)\int_{-\infty}^{z-\mu\tau} \exp\left(\frac{s}{\mu}\right) J(s)\,ds.$$

Denote

$$h(z) = \int_{-\infty}^{z-\mu\tau} \exp\left(\frac{s}{\mu}\right) J(s)\,ds.$$

Then $U'(z)$ has the same sign with $h(z)$. Obviously, $h(z) \geq 0$ when $J(x)$ is of type (A). Thus, function $U(z)$ is a nondecreasing function, since $U'(z) \geq 0$ when $J(x)$ is of type (A). Especially, we know that $U'(0) > 0$ when $J(x)$ is of type (A). Otherwise, there should be $J(x) \equiv 0$ when $x \in (-\infty, -\mu t)$, and thus, we get $\theta = 0$ from (3.5), which is contradictory to the assumption. So $U(z) > \theta$ when $z > 0$ and $U(z) < \theta$ when $z < 0$ when the kernel function $J(x)$ is of type (A), since $U'(0) > 0$ and $U(0) = \theta$.

In the following, we show that $U(z) > \theta$ when $z > 0$ and $U(z) < \theta$ when $z < 0$ when the kernel function $J(x)$ is of type (B). For type (B), it is easy to find from $0 < \mu\tau < M_0$ that $h(z)$ has the following properties:

(1) $h(z)$ is nonincreasing on $(-\infty, \mu\tau - M) \cap (\mu\tau + N, +\infty)$ and nondecreasing on $(\mu\tau - M, \mu\tau + N)$.

(2) $h(-\infty) = 0$, $h(0) > 0$ and there exists $z_0 < 0$ such that $h(z) \leq 0$ on $(-\infty, z_0)$ and $h(z) \geq 0$ on $(z_0, \mu\tau + N)$.

(3) $h(z)$ changes its sign at most once on $(\mu\tau + N, +\infty)$, that is to say, $h(z) > 0$ on $(\mu\tau + N, +\infty)$, or there exists $z_\star \in (\mu\tau + N, +\infty)$, such that $h(z) > 0$ on $(\mu\tau + N, z_\star)$ and $h(z) \leq 0$ on $(z_\star, +\infty)$.

From the discussion above, we know that there are two cases that may happen to $U'(z)$ if the kernel function $J(z)$ is of type (B).

Case (1) $U'(z) \leq 0$ on $(-\infty, z_0)$, $U'(z) \geq 0$ on $(z_0, +\infty)$.

Case (2) $U'(z) \leq 0$ on $(-\infty, z_0)$, $U'(z) \geq 0$ on $(z_0, z_\star)$ and $U'(z) \leq 0$ on $(z_\star, +\infty)$.

By a simple computation, we get $\lim_{z\to\infty} U'(z) = 0$ and $\lim_{z\to\infty} U(z) = \beta$. If the first case holds, then $U(z)$ is nonincreasing on $(-\infty, z_0)$, nondecreasing on $(z_0, +\infty)$. Note that $z_0 < 0$, $U(0) = \theta$ and $U'(0) > 0$. Consequently, $U(z) > \theta$ when $z > 0$ and $U(z) < \theta$ when $z < 0$ if case (1) holds. However, for case (2), it is easy to see that $U(z)$ is also nonincreasing on $(-\infty, z_0)$, nondecreasing on $(z_0, z_\star)$ and nonincreasing on $(z_\star, +\infty)$. Because $z_\star > 0$ and $\lim_{z\to\infty} U(z) = \beta > \theta$, $U(z) > \theta$ when $z > 0$ and $U(z) < \Theta$ when $z < 0$ if the case (2) holds. From the discussion above, we know that $U(z) > \theta$ when $z > 0$ and $U(z) < \theta$ when $z < 0$ for any kernel function $J(z)$ of type (B). Consequently, we get the following theorem on the existence and uniqueness of wave solution to equation (3.1).

Theorem 3.2 (Existence and uniqueness of the wave front) *Suppose that the kernel functions $J(x)$ and the parameter c satisfy one of the following assumptions.*

(1) *$J(x)$ is of type* (A), $c > 0$;

(2) *$J(x)$ is of type* (B), $\int_{-\infty}^{0} |s|J(s)\,ds > 0$, $c \in (0, \frac{M_0}{\tau})$, *where* $\int_{-\infty}^{-M_0} |s|J(s)\,ds = 0$.

Then for any $\theta \in (\varphi(c), \frac{\beta}{2})$, IDE (3.1) *has a unique traveling wave solution satisfying the phase conditions*

$$U(z) < \theta \quad \text{when } z < 0; \qquad U(0) = \theta; \qquad U(z) > \theta \quad \text{when } z > 0$$

and the boundary conditions

$$\lim_{z\to-\infty} U(z) = 0; \qquad \lim_{z\to+\infty} U(z) = \beta; \qquad \lim_{z\to\pm\infty} U'(z) = 0.$$

This wave front could be expressed as

$$U(z) = \beta \int_{-\infty}^{z} \left[1 - \exp\left(\frac{s-z}{\mu}\right)\right] J(s - \mu\tau)\,ds,$$

where $z = x + \mu t$, and the wave speed $\mu = \mu^\star(\theta) \in (0, c)$ is uniquely determined by the wave speed equation (3.5).

4 Existence and uniqueness of the wave solution of IDE (1.1)

In this section, we will study the existence of the traveling wave fronts to this equation (1.1) with general types of kernel functions, which includes the cases that have been studied before. To be more precise, we will show the existence of traveling wave front to this equation when the kernel function $J(x)$ in feedback channel is of nonnegative function or Mexican hat function and the kernel function $K(x)$ in synaptic coupling is of some general types of functions. Not only three typical types of kernel functions but also oscillatory kernel functions within certain range of model parameters in synaptic coupling are considered. First, we give some assumptions for the kernel functions $K(x)$. In addition to the assumptions (1.4) for kernel functions, we also assume that the kernel function $K(x)$ in $(-\infty, 0)$ satisfies one of the following conditions.

(L_1) $K(x) \geq 0$ in $x \in (-\infty, 0)$.

(L_2) $K(x) \geq 0$ when $x \in (-M_1, 0) \cup (-M_3, -M_2) \cup (-M_5, -M_4) \cup \cdots \cup (-M_{2n+1}, -M_{2n})$ and $K(x) \leq 0$ when $x \in (-M_2, -M_1) \cup (-M_4, -M_3) \cup \cdots \cup (-M_{2n}, -M_{2n-1}) \cup (-\infty, -M_{2n+1})$, where $0 < M_1 < M_2 < \cdots < M_{2n+1} < \infty$, and $K(x)$ satisfies that

$$\int_{-\infty}^{0} |s|K(s)\,ds > 0; \qquad \int_{-M_{2i}}^{-M_{2i-2}} |s|K(s)\,ds \geq 0; \qquad \frac{\alpha}{2} - \alpha \int_{-M_{2i}}^{0} K(t)\,dt \leq \theta,$$

where $i = 1, 2, \ldots, n, n+1$ and $M_0 = 0$, $M_{2(n+1)} = \infty$.

(L_3) $K(x) \geq 0$ when $x \in (-M_1, 0) \cup (-M_3, -M_2) \cup (-M_5, -M_4) \cup \cdots \cup (-M_{2n-1}, -M_{2n-2}) \cup (-\infty, -M_{2n})$ and $K(x) \leq 0$ when $x \in (-M_2, -M_1) \cup (-M_4, -M_3) \cup \cdots \cup (-M_{2n}, -M_{2n-1})$, where $0 < M_1 < M_2 < \cdots < M_{2n} < \infty$, and $K(x)$ satisfies that

$$\int_{-\infty}^{0} |s|K(s)\,ds > 0; \qquad \int_{-M_{2i}}^{0} |s|K(s)\,ds \geq 0,$$

where $i = 1, 2, \ldots, n$.

(L_4) $K(x) \geq 0$ when $x \in (-\infty, -M)$ and $K(x) \leq 0$ when $x \in (-M, 0)$, where $0 < M < \infty$.

Assume that the kernel function $K(x)$ in $(0, \infty)$ satisfies one of the following conditions.

(R_1) $K(x) \geq 0$ for all $x \in (0, +\infty)$.

(R_2) $K(x)$ satisfies that there exist $0 < N_1 < N_2 < \cdots < N_{2n+1} < \infty$, such that $K(x) \geq 0$ when $x \in (N_1, N_2) \cup (N_3, N_4) \cup \cdots \cup (N_{2n+1}, +\infty)$; $K(x) \leq 0$ when $x \in [0, N_1] \cup [N_2, N_3] \cup \cdots \cup [N_{2n}, N_{2n+1}]$, and

$$\alpha \int_0^{N_{2i-1}} K(s)\,ds \leq \theta - \frac{\alpha}{2}, \quad i = 1, 2, \ldots, n, n+1.$$

(R_3) $K(x)$ satisfies that there exist $0 < N_1 < N_2 < \cdots < N_{2n} < \infty$ such that $K(x) \geq 0$ when $x \in (N_1, N_2) \cup (N_3, N_4) \cup \cdots \cup (N_{2n-1}, N_{2n})$; $K(x) \leq 0$ when $x \in [0, N_1] \cup [N_2, N_3] \cup \cdots \cup [N_{2n}, +\infty)$, and

$$\alpha \int_0^{N_{2i-1}} K(s)\,ds \leq \theta - \frac{\alpha}{2}, \quad i = 1, 2, \ldots, n.$$

(R_4) $K(x)$ satisfies that there exist $0 < N < \infty$ such that $K(x) \geq 0$ when $x \in (N, +\infty)$ and $K(x) \leq 0$ when $x \in (0, N]$.

(R_5) $K(x)$ satisfies that there exist $0 < N_1 < N_2 < \cdots < N_{2n+1} < \infty$ such that $K(x) \geq 0$ when $x \in [0, N_1] \cup [N_2, N_3] \cup \cdots \cup [N_{2n-2}, N_{2n-1}]$; $K(x) \leq 0$ when $x \in (N_1, N_2) \cup (N_3, N_4) \cup \cdots \cup (N_{2n-1}, +\infty)$, and

$$\alpha \int_0^{N_{2i}} K(s)\,ds \leq \theta - \frac{\alpha}{2}, \quad i = 1, 2, \ldots, n-1.$$

(R_6) $K(x)$ satisfies that there exist $0 < N_1 < N_2 < \cdots < N_{2n+1} < \infty$ such that $K(x) \geq 0$ when $x \in [0, N_1] \cup [N_2, N_3] \cup \cdots \cup [N_{2n}, +\infty)$; $K(x) \leq 0$ when $x \in (N_1, N_2) \cup (N_3, N_4) \cup \cdots \cup (N_{2n-1}, N_{2n})$, and

$$\alpha \int_0^{N_{2i}} K(s)\,ds \leq \theta - \frac{\alpha}{2}, \quad i = 1, 2, \ldots, n.$$

Obviously, the kernel functions, which satisfy one of the conditions R_i ($i = 1, 2, \ldots, 6$) in the interval $(0, +\infty)$ and one of the conditions L_j ($j = 1, 2, 3, 4$) in the interval $(-\infty, 0)$, can form a very general class of functions, which includes all the classes of the kernel functions that appeared in previous works. In [18], we proved the existence of the traveling wave front of the IDE (1.1) when $\beta = 0$ with the kernel function satisfying one of the assumptions L_i ($i = 1, 2, 3, 4$) on $(-\infty, 0)$ and one of the assumptions R_j ($j = 1, 2, \ldots, 6$) on $(0, \infty)$. We cite the main results as follows.

Theorem 4.1 *Suppose that the positive parameters* α *and* θ *satisfy the condition* $0 < 2\theta < \alpha$. $K(x)$ *satisfies the basic property of kernel function and one of the assumptions* L_i ($i = 1, 2, 3, 4$) *on* $(-\infty, 0)$ *and one of* R_j ($j = 1, 2, \ldots, 6$) *on* $(0, +\infty)$. *Then equation* (1.1) *with* $\beta = 0$ *has a unique traveling wave front solution* $U(z) = U(x + \mu t)$, *which could be expressed as*

$$U(z) = \alpha \int_{-\infty}^{\frac{cz}{c+\operatorname{sgn}(z)\mu}} K(s)\,ds - \alpha \int_{-\infty}^{z} \exp\left(\frac{s-z}{\mu}\right) K(s) \frac{c}{c + \operatorname{sgn}(s)\mu}\,ds. \tag{4.1}$$

The wave speed μ $(0 < \mu < c)$ *is uniquely determined by the following speed equation*

$$\alpha \int_{-\infty}^{0} \left[1 - \exp\left(\frac{c-\mu}{c\mu} s\right)\right] K(s)\, ds = \theta, \tag{4.2}$$

and the wave front solution $U(z)$ *also satisfies phase conditions*

$$U(z) < \theta \quad \text{when } z < 0; \qquad U(0) = \theta; \qquad U(z) > \theta \quad \text{when } z > 0$$

and the boundary conditions

$$\lim_{z \to -\infty} U(z) = 0; \qquad \lim_{z \to +\infty} U(z) = \alpha; \qquad \lim_{z \to \pm\infty} U'(z) = 0.$$

Next, we give the following lemma to prove the existence of traveling wave front to equation (1.1) when the kernel function $J(x)$ is of nonnegative function or Mexican hat function and the kernel function $K(x)$ satisfy the assumptions as in Theorem 4.1 within a certain range of model parameters.

Lemma 4.1 *Let*

$$\psi(\mu) = \alpha \int_{-\infty}^{0} \left[1 - \exp\left(\frac{c-\mu}{c\mu} s\right)\right] K(s)\, ds, \tag{4.3}$$

then the following conclusions hold.

(1) *If the kernel function* $K(x)$ *satisfies one of the assumptions* L_i $(i = 1, 2, 3)$ *on* $(-\infty, 0)$ *and one of* R_j $(j = 1, 2, \ldots, 6)$ *on* $(0, +\infty)$, *then* $\psi(\mu)$ *is monotonic decreasing on* $(0, c)$.

(2) *If the kernel function* $K(x)$ *satisfies the assumptions* L_4 *on* $(-\infty, 0)$ *and one of* R_j $(j = 1, 2, \ldots, 6)$ *on* $(0, +\infty)$ *and* $\psi(\mu_0) < \frac{\alpha}{2}$, *then* $\psi(\mu)$ *is monotonic decreasing on* (μ_0, c).

Proof From (4.3), we get $\psi'(\mu) = -\frac{\alpha}{\mu^2} \int_{-\infty}^{0} |s| \exp(\frac{c-\mu}{c\mu} s) K(s)\, ds$. We prove (1) by showing that $\psi'(\mu) < 0$ when $K(x)$ satisfies the conditions. We omit the details.

(2) If $K(x)$ satisfies the assumptions L_4 on $(-\infty, 0)$ and one of R_j $(j = 1, 2, \ldots, 6)$ on $(0, +\infty)$, then

$$\begin{aligned}
\psi'(\mu) &= -\frac{\alpha}{\mu^2} \int_{-\infty}^{0} |s| \exp\left(\frac{c-\mu}{c\mu} s\right) K(s)\, ds \\
&< -\frac{M\alpha}{\mu^2} \int_{-\infty}^{0} \exp\left(\frac{c-\mu}{c\mu} s\right) K(s)\, ds \\
&= \frac{M}{\mu^2} \left[\psi(\mu) - \frac{\alpha}{2}\right].
\end{aligned}$$

Consequently, $\psi'(\mu_0) < 0$ if $\psi(\mu_0) < \frac{\alpha}{2}$, and thus, $\psi'(\mu) < 0$ when $\mu \in (\mu_0, c)$. So, $\psi(\mu)$ is monotonic decreasing on (μ_0, c). □

Theorem 4.2 (Existence and uniqueness of the wave front) *Suppose that* $\alpha > 0$, $\beta > 0$, *the kernel functions* $K(x)$ *satisfies one of* L_i $(i = 1, 2, 3)$ *on* $(-\infty, 0)$ *and one of* R_j $(j = 1, 2, \ldots, 6)$ *on* $(0, +\infty)$. *Then, when* $J(x)$ *is of type* (A) *for any* $c > 0$, *or when* $J(x)$ *is of type* (B) *for any*

$c \in (0, \frac{M_0}{\tau})$ (*where* $\int_{-\infty}^{-M_0} |s|J(s)\,ds = 0$), *for any* $\theta \in (\varphi(c), \frac{\alpha+\beta}{2})$, *equation* (1.1) *has a unique traveling wave front solution* $U(z) = U(x + \mu t)$, *which could be expressed as*

$$U(z) = \alpha \int_{-\infty}^{\frac{cz}{c+\operatorname{sgn}(z)\mu}} K(s)\,ds - \alpha \int_{-\infty}^{z} \exp\left(\frac{s-z}{\mu}\right) K(s) \frac{c}{c+\operatorname{sgn}(s)\mu}\,ds + \beta \int_{-\infty}^{z} \left[1 - \exp\left(\frac{s-z}{\mu}\right)\right] J(s-\mu\tau)\,ds. \tag{4.4}$$

The wave speed μ $(0 < \mu < c)$ *is uniquely determined by the following speed equation*

$$\alpha \int_{-\infty}^{0} \left[1 - \exp\left(\frac{c-\mu}{c\mu} s\right)\right] K(s)\,ds + \beta \int_{-\infty}^{0} \left[1 - \exp\left(\frac{s}{\mu}\right)\right] J(s-\mu\tau)\,ds = \theta. \tag{4.5}$$

The wave front solution $U(z)$ *satisfies phase conditions*

$$U(z) < \theta \quad \text{when } z < 0; \qquad U(0) = \theta; \qquad U(z) > \theta \quad \text{when } z > 0$$

and the boundary conditions

$$\lim_{z\to-\infty} U(z) = 0; \qquad \lim_{z\to+\infty} U(z) = \alpha + \beta; \qquad \lim_{z\to\pm\infty} U'(z) = 0.$$

Proof Suppose that $U(z) = u(x + \mu t)$ is a traveling wave front of equation (1.1) satisfying $U(0) = \theta$, $U(z) < \theta$ when $z < 0$ and $U(z) > \theta$ when $z > 0$, then $U(z)$ satisfies the following equation

$$\mu U' + U = \alpha \int_{-\infty}^{\frac{cz}{c+\operatorname{sgn}(z)\mu}} K(t)\,dt + \beta \int_{-\infty}^{z-\mu\tau} J(y)\,dy, \tag{4.6}$$

which is a first-order linear ODE. Solve (4.6), and let $\lim_{z\to-\infty} U(z) = 0$, we get the solution of (4.6)

$$U(z) = \alpha \int_{-\infty}^{\frac{cz}{c+\operatorname{sgn}(z)\mu}} K(s)\,ds - \alpha \int_{-\infty}^{z} \exp\left(\frac{s-z}{\mu}\right) K(s) \frac{c}{c+\operatorname{sgn}(s)\mu}\,ds + \beta \int_{-\infty}^{z} \left[1 - \exp\left(\frac{s-z}{\mu}\right)\right] J(s-\mu\tau)\,ds,$$

which is (4.5). Note that the solution of equation (4.6) is not necessary; the traveling wave solution of the model equation (1.1), unless $U(z)$ satisfies the phase conditions that $U(0) = \theta$, $U(z) < \theta$ when $z < 0$ and $U(z) > \theta$ when $z > 0$. In the following, we prove that there exists a unique wave speed $\hat{\mu}$ such that (4.4) with $\mu = \hat{\mu}$ is the unique wave front solution of equation (1.1) satisfying the required conditions in this theorem.

Let $g(\mu) = \psi(\mu) + \varphi(\mu)$, *i.e.*,

$$g(\mu) = \alpha \int_{-\infty}^{0} \left[1 - \exp\left(\frac{c-\mu}{c\mu} s\right)\right] K(s)\,ds + \beta \int_{-\infty}^{0} \left[1 - \exp\left(\frac{s}{\mu}\right)\right] J(s-\mu\tau)\,ds.$$

It is easy to see that $\lim_{\mu\to0^+} g(\mu) = \frac{\alpha+\beta}{2}$ and $\lim_{\mu\to c^-} g(\mu) = \varphi(c)$, and $g(\mu)$ is continuous with μ. Consequently, for any $\theta \in (\varphi(c), \frac{\alpha+\beta}{2})$, there exists $\hat{\mu} \in (0, c)$, such that $g(\hat{\mu}) = \theta$.

From the proofs of Theorem 3.1 and Theorem 4.1, we know that if the kernel functions $K(x)$ and $J(x)$ satisfy the conditions of this theorem, $\psi(\mu)$ and $\varphi(\mu)$ are decreasing with μ, so the function $g(\mu)$ is also decreasing with μ. Consequently, we get the wave speed, which is uniquely determined by (4.5). Let $\theta_1 = \psi(\hat{\mu})$ and $\theta_2 = \varphi(\hat{\mu})$, then $0 < \theta_1 < \frac{\alpha}{2}$, $\varphi(c) < \theta_2 < \frac{\beta}{2}$ and $\theta_1 + \theta_2 = \theta$.

From Theorem 3.2, we know that for θ_1 and $\hat{\mu}$,

$$U_\beta(z) = \beta \int_{-\infty}^{z} \left[1 - \exp\left(\frac{s-z}{\hat{\mu}}\right)\right] J(s - \hat{\mu}\tau)\, ds$$

is a wave front solution of equation (3.2) satisfying the phase conditions

$$U_\beta(z) < \theta_1 \quad \text{when } z < 0; \qquad U_\beta(0) = \theta_1; \qquad U_\beta(z) > \theta_1 \quad \text{when } z > 0$$

and the boundary conditions

$$\lim_{z\to-\infty} U_\beta(z) = 0; \qquad \lim_{z\to+\infty} U_\beta(z) = \beta; \qquad \lim_{z\to\pm\infty} U'_\beta(z) = 0.$$

From Theorem 4.1, we know that for $\theta_2 = \theta - \theta_1$,

$$U_\alpha(z) = \alpha \int_{-\infty}^{\frac{cz}{c+\operatorname{sgn}(z)\hat{\mu}}} K(s)\, ds - \alpha \int_{-\infty}^{z} \exp\left(\frac{s-z}{\hat{\mu}}\right) K(s) \frac{c}{c + \operatorname{sgn}(s)\hat{\mu}}\, ds$$

is a wave front solution of equation

$$\mu U' + U = \alpha \int_{-\infty}^{\frac{cz}{c+\operatorname{sgn}(z)\mu}} K(t)\, dt \tag{4.7}$$

with $\mu = \hat{\mu}$ and satisfying the phase conditions

$$U_\alpha(z) < \theta_2 \quad \text{when } z < 0; \qquad U_\alpha(0) = \theta_2; \qquad U_\alpha(z) > \theta_2 \quad \text{when } z > 0$$

and the boundary conditions

$$\lim_{z\to-\infty} U_\alpha(z) = 0; \qquad \lim_{z\to+\infty} U_\beta(z) = \alpha; \qquad \lim_{z\to\pm\infty} U'_\alpha(z) = 0.$$

Denote $U(z;\mu) = U_\alpha(z;\mu) + U_\beta(z;\mu)$. Obviously, $U(z;\mu)$ is a solution of equation (4.6), which is identical to (4.4) and satisfies the phase conditions

$$U(z;\mu) < \theta_1 + \theta_2 = \theta \quad \text{when } z < 0; \qquad U(0;\mu) = \theta; \qquad U(z;\mu) > \theta \quad \text{when } z > 0$$

and the boundary conditions

$$\lim_{z\to-\infty} U(z;\mu)(z) = 0; \qquad \lim_{z\to+\infty} U(z;\mu)(z) = \alpha + \beta; \qquad \lim_{z\to\pm\infty} U'(z;\mu) = 0,$$

where $\mu = \hat{\mu}$ is uniquely determined by (4.5). □

For the case that the kernel functions $K(x)$ satisfying L_4 on $(-\infty, 0)$ and R_j ($j = 1, 2, \ldots, 6$) on $(0, +\infty)$, the parameters α and β and the kernel function $J(x)$ satisfying the same conditions as those in Theorem 4.2, we can get the existence of the wave speed by the same process. But the monotonic property of the function $\varphi(\mu)$ can not be guaranteed, neither the function $g(\mu)$, so the uniqueness of the wave speed cannot be obtained in the same way as Theorem 4.2. However, we can prove the uniqueness of the wave speed when $\theta \in (\varphi(c), \frac{\alpha}{2} + \varphi(c)]$.

Theorem 4.3 *Suppose the kernel function $K(x)$ satisfies L_4 on $(-\infty, 0)$ and one of R_j ($j = 1, 2, \ldots, 6$) on $(0, +\infty)$, the parameters α and β and the kernel function $J(x)$ satisfy the same conditions as those in Theorem* 4.2. *Then for any $\theta \in (\varphi(c), \frac{\alpha}{2} + \varphi(c)]$, equation* (1.1) *has a unique traveling wave front solution satisfying the same phase and boundary conditions as in Theorem* 4.2.

Proof By the same process as the proof of Theorem 4.2, we know that for any $\theta \in (\varphi(c), \frac{\alpha}{2} + \varphi(c)]$, there exists $\hat{\mu} \in (0, c)$, such that $g(\hat{\mu}) = \theta$. We show that there would be a contradiction if there existed $0 < \mu_1 < \mu_2 < c$ such that $g(\mu_i) = \psi(\mu_i) + \varphi(\mu_i) = \theta$, $i = 1, 2$. Actually, $\varphi(\mu_1) > \varphi(\mu_2) > \varphi(c)$, since $\varphi(\mu)$ is decreasing with μ on $(0, c)$, and thus, $0 < \psi(\mu_1) < \psi(\mu_2) < \frac{\alpha}{2}$. However, according to Lemma 4.1, we know that the function $\psi(\mu)$ should be decreasing on (μ_1, μ_2). We get a contradiction. Consequently, for any $\theta \in (\varphi(c), \frac{\alpha}{2} + \varphi(c)]$, there exists a unique $\hat{\mu} \in (0, c)$, such that $g(\hat{\mu}) = \psi(\hat{\mu}) + \varphi(\hat{\mu}) = \theta_1 + \theta_2 = \theta$, $0 < \theta_1 < \frac{\alpha}{2}$ and $\varphi(c) < \theta_2 < \frac{\beta}{2}$. Just as the proof of Theorem 4.2, we can get the conclusion. □

5 Discussion and conclusion

The existence and uniqueness of the traveling wave fronts of the general integral differential model equation (1.1) arising from neuronal networks with both axonal and delayed nonlocal connections are investigated in this paper. Besides the three class of typical kernel functions, more general kernel functions are considered. Some known results are amended. In Theorem 4.1 of [15], Magpantay and Zou showed that for $K(x)$ of type (C) and $\theta < \frac{\alpha+\beta}{2}$, if there exists a unique wave speed $\hat{\mu} \in (0, c)$, then there exists a solution to equation (1.1) satisfying the phase and boundary conditions. It is easy to see that if $K(x)$ satisfies (L_4) and (R_3) with $n = 1$, then $K(x)$ is of type (C). From the results in Theorem 4.3, we know that the wave speed is unique, and thus, the unique traveling solution to equation (1.1) exists when $\theta \in (\varphi(c), \frac{\alpha}{2} + \varphi(c)]$.

We have obtained the existence and uniqueness of the wave front solution to IDE (1.1) with a very general kernel function $K(x)$ in the first nonlinear term on the right side, but the kernel function $J(x)$ is restricted to type (A) or type (B). However, at present, if (4.1) is still a wave front solution to (1.1) when the kernel function $J(x)$ of type (C) is still open.

The speed index functions are well applied in studying the existence and uniqueness of the wave speed, in which the uniqueness is always obtained by proving the monotonicity of the speed index function. However, in Theorem 4.3, we provided a method to prove the uniqueness of the wave speed in the case that the monotonicity of the speed index function does not hold any more. The principle of linear superposition is applied skillfully to deal with the obstacles that produced by the two nonlinear terms of equation (1.1), which may provide an idea to investigate the integral differential equations as follows

$$u_t + u = \sum_{i=1}^{n} \alpha_i \int_R K(x-y) H\left(u\left(y, t - \frac{|x-y|}{c_i}\right) - \theta\right) dy$$
$$+ \sum_{i=1}^{m} \beta_i \int_R J(x-y) H\big(u(y, t-\tau_i) - \theta\big)\, dy.$$

Competing interests
The author declares that she has no competing interests.

Acknowledgements
This work is supported by the Nature Science Foundation of China (No. 11101371). This work was motivated during the period that I was visiting the Department of Mathematics of Lehigh University. I would like to acknowledge the kind invitation from the Department of Mathematics of Lehigh University. Especially, I wish to express my sincere appreciation to Professor Linghai Zhang for his patient constructive instructions and discussions. I also would like to dedicate this paper to Prof. Jibin Li for his 70th birthday. I would like to express my gratitude to anonymous reviewers, whose comments helped to improve the presentation of the paper.

Received: 28 February 2013 Accepted: 29 July 2013 Published: 14 August 2013

References
1. Hutt, A: Effects of nonlocal feedback on traveling fronts in neural fields subject to transmission delay. Phys. Rev. E **70**, 052902 (2004)
2. Atay, FM, Hutt, A: Stability and bifurcations in neural fields with finite propagation speed and general connectivity. SIAM J. Appl. Math. **65**, 644-666 (2005)
3. Coombes, S, Lord, GJ, Owen, MR: Waves and bumps in neuronal networks with axo-dendritic synaptic interactions. Physica D **178**, 219-241 (2003)
4. Ermentrout, GB: Neural networks as spatio-temporal pattern-forming systems. Rep. Prog. Phys. **61**, 353-430 (1998)
5. Atay, FM, Hutt, A: Neural fields with distributed transmission speeds and long-range feedback delays. SIAM J. Appl. Dyn. Syst. **5**, 670-698 (2006)
6. Hutt, A, Longtin, A: Effects of the anesthetic agent propofol on neural populations. Cogn. Neurodyn. **4**, 37-59 (2010)
7. Pinto, DJ, Ermentrout, GB: Spatially structured activity in synaptically coupled neuronal networks, I. Traveling fronts and pulses. SIAM J. Appl. Math. **62**, 206-225 (2001)
8. Pinto, DJ, Ermentrout, GB: Spatially structured activity in synaptically coupled neuronal networks, II. Lateral inhibition and standing pulses. SIAM J. Appl. Math. **62**, 226-243 (2001)
9. Amari, SI: Dynamics of pattern formation in lateral-inhibition type neural fields. Biol. Cybern. **27**, 77-87 (1997)
10. van Saarloos, W, van Hecke, M, Holyst, R: Front propagation into unstable and metastable states in smectic-C^* liquid crystals: linear and nonlinear marginal-stability analysis. Phys. Rev. E **52**, 1773-1777 (1995)
11. Fineberg, J, Steinberg, V: Vortex-front propagation in Rayleigh-Bénard convection. Phys. Rev. Lett. **58**, 1332-1335 (1987)
12. Zhang, LH: How do synaptic coupling and spatial temporal delay influence traveling wave solution in nonlinear nonlocal neuronal network? SIAM J. Appl. Dyn. Syst. **6**, 597-644 (2007)
13. Zhang, LH, Wu, P, Stoner, MA: Influence of sodium currents on speeds of traveling wave fronts in synaptically coupled neuronal networks. Physica D **239**, 9-32 (2010)
14. Lv, G, Wang, M: Traveling waves of some integral-differential equations arising from neuronal networks with oscillatory kernels. J. Math. Anal. Appl. **370**, 82-100 (2010)
15. Magpantay, F, Zou, X: Wave fronts in neuronal fields with nonlocal post-synaptic axonal connections and delayed nonlocal feedback connections. Math. Biosci. Eng. **7**(2), 421-442 (2010)
16. Ou, C, Wu, J: Persistence of wavefronts in delayed nonlocal reaction-diffusion equations. J. Differ. Equ. **235**, 219-261 (2007)
17. Zhang, LH: Dynamic of neuronal waves. Math. Z. **255**, 283-321 (2007)
18. Zhang, LJ, Zhang, LH, Yuan, J, Liu, HH: Existence of traveling wave solutions of an integral differential equation in nonlinear nonlocal neuronal network (2012, submitted)

doi:10.1186/1687-1847-2013-243
Cite this article as: Zhang: **Existence and uniqueness of wave fronts in neuronal network with nonlocal post-synaptic axonal and delayed nonlocal feedback connections.** *Advances in Difference Equations* 2013 **2013**:243.