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Research Article

*p*th Moment Exponential Stability of Nonlinear Hybrid Stochastic Heat Equations

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We are concerned with the exponential stability problem of a class of nonlinear hybrid stochastic heat equations (known as stochastic heat equations with Markovian switching) in an infinite state space. The fixed point theory is utilized to discuss the existence, uniqueness, and pth moment exponential stability of the mild solution. Moreover, we also acquire the Lyapunov exponents by combining the fixed point theory and the Gronwall inequality. At last, two examples are provided to verify the effectiveness of our obtained results.

1. Introduction

There have been enormous theories for both definite and stochastic partial differential equations (SPDEs) since they can describe various phenomena in reality recently. Throughout the previous literature, besides the existence and uniqueness, the stability for SPDEs is a popular topic. Some results on exponential stability of SPDEs were investigated by Caraballo and Liu [1] and Luo [2].

At present, the hybrid SDEs and hybrid SPDEs (also known as SDEs and SPDEs with Markovian switching), for example, hybrid stochastic heat equations, have been paid much attention owing to their wide applications in natural science, engineering, biology, finance, and other areas. In addition, one important reason to consider hybrid SDEs or SPDEs is that real situations may exhibit sudden changes or go into different cases in different periods resulting in parameter transition and probably changes in branch structure so that we need Markov chains to characterize such systems. For example, [3, 4] discussed some properties of the solutions to SDEs with Markovian switching. So far, the pth moment exponential stability of stochastic heat equations without Markovian switching has been studied extensively, for example, [5, 6], while hybrid stochastic heat equations remain open due to the difficulty arising from Markov chains. As a consequence, it is more challenging and exciting to explore the stabilization of the systems with Markov chains.

Up to now, there have been many methods to study the stability of SPDEs as well as hybrid stochastic heat equations like Lyapunov's function method, successive approximation approach, large deviation technique, and so on. For example, Deng et al. [7] obtained some sufficient conditions on the stability of hybrid stochastic differential equations by using the Lyapunov's function method and stochastic feedback controls. In [8], Ubøe and Zhang constructed sequence solutions to approximate the exact solution successively. Gong and Qian studied the large deviation rate function of the Markov chains more completely in [9]. In [10], Bao et al. discussed the pth moment exponential stability of linear hybrid stochastic heat equations by employing the explicit formulae and large deviation technique. However, we cannot use the method of the linear case to obtain the stability property of a class of nonlinear hybrid stochastic heat equations. Generally speaking, it is very difficult to calculate the explicit formulae of solutions to nonlinear hybrid stochastic heat equations. Furthermore, the Markov chain has no unique stationary probability distribution when it is not irreducible in an infinite state space, which makes the analysis of Lyapunov exponents more difficult.

To overcome difficulties raised by what was mentioned above, we utilize a new method—the fixed point theory—to deal with the stability of a class of nonlinear hybrid stochastic heat equations in this paper. It is well known that the fixed point theory first put forward by Burton is usually applied to study the almost sure exponential stability and the pth moment exponential stability of both SDEs and SPDEs such as second-order stochastic evolution equations, for example, [11], nonlinear neutral SDEs, for example, [12], and stochastic Volterra-Levin equations, for example, [13–15]. Different from the method in [10], we use the fixed point theory to analyze the stability of nonlinear hybrid stochastic heat equations. Firstly, we prove the existence and uniqueness of the solution, as well as the pth moment exponential stability. Then, we obtain the pth moment Lyapunov exponents by using the Gronwall inequality instead of seeking the proper rate function which was complicated to compute in [10]. Moreover, we provide two examples and some comparisons to show that our results extend and improve those given in the previous literature.

The rest of this paper is organized as follows. In Section 2, we introduce the notations and the model of nonlinear hybrid stochastic heat equations along with some necessary assumptions. In Section 3, by applying the fixed point theorem, we prove the existence, uniqueness, and pth moment exponential stability of hybrid stochastic heat equations. Besides, based on the Gronwall inequality, we further obtain the Lyapunov exponents of the solutions and make some comparisons with the previous results. In Section 4, we provide two simple examples to verify the effectiveness of the obtained results. In the last section, we conclude the paper with some general remarks.

2. Preliminaries, Model, and Assumptions

In this section, we introduce some preliminaries and common notations for a more detailed description, and then give the model that we will deal with.

Throughout this paper, the following notations will be used. Let $\{B(t), t \geq 0\}$ be a real-valued Brownian motion defined on the complete probability space $\{\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P\}$ which has a filtration satisfying the usual conditions. We denote a bounded domain $\Theta \subset \mathbb{R}^n$, equipped with $\mathscr{C}^\infty$ boundary $\partial\Theta$. In this paper, $\mathscr{L}^2(\Theta)$ is defined as usual, that is, the family of all-real valued square integrable functions with their inner product $\langle f, g\rangle := \int_\Theta f(x)g(x)dx$, $f, g \in \mathscr{L}^2(\Theta)$ and norm $\|f\| := (\int_\Theta f^2(x)dx)^{1/2}$, $f \in \mathscr{L}^2(\Theta)$. Also, we can define $H^m(\Theta)$, $m = 1, 2$, by $H^m(\Theta) := \{u \in \mathscr{L}^2(\Theta) \mid D^\alpha u \in \mathscr{L}^2(\Theta), |\alpha| \leq m\}$ and $H^m_0(\Theta)$ by $H^m_0(\Theta) := \{u \in H^m(\Theta) \mid u = 0 \text{ on } \partial\Theta\}$. Then, we can denote the Laplace operator $A := \sum_{i=1}^n (\partial^2/\partial x_i^2)$, with domain $\mathscr{D}(A) := H^1_0(\Theta) \cap H^2_0(\Theta)$, which generates a strongly continuous semigroup e^{tA}. Furthermore, let $\{r(t), t \geq 0\}$ be a right continuous Markov chain which takes values in a listed state space $S = \{1, 2, \ldots, N\}$, where N is some positive integer or arrives at ∞. Moreover, we assume that the Markov chain $\{r(t), t \geq 0\}$ is independent of the Brownian motion $\{B(t), t \geq 0\}$.

Model Definition. In this paper, we consider the following hybrid stochastic heat equation:

$$\frac{\partial u(t,x)}{\partial t} = Au(t,x) + \alpha(r(t)) f(t, u(t,x), r(t)) + \beta(r(t)) g(t, u(t,x), r(t)) \dot{B}(t), \quad x \in \Theta, \quad t > 0, \tag{1}$$
$$u(t,x) = 0, \quad x \in \partial\Theta, \quad t > 0,$$
$$u(0,x) = u_0(x) \quad x \in \Theta,$$

where α, β are mappings from $S \to \mathbb{R}$ and we take $\alpha_i := \alpha(i)$, $\beta_i := \beta(i)$ simply in this paper. Letting us fix an interval $[0,T], T > 0$, then the operators $f, g : [0,T] \times \mathscr{L}^2(\Theta) \times S : \to \mathscr{L}^2(\Theta)$ are $\mathscr{F}_t$-measurable while the initial value u_0 is a $\mathscr{D}(A)$-valued, $\mathscr{F}_0$-measurable random variable, which is independent of $r(\cdot)$ and $B(\cdot)$, and, for any $p > 0$, $\mathbf{E}\|u_0\|^p < \infty$.

Next, we will give the definitions of the mild solution to (1) and pth moment exponential stability as well as C_p inequality. For the convenience of writing, we will take $u(t) := u(t, \cdot)$ and $u := \{u(t,\cdot)\}_{t\geq 0}$ such that u is a $\mathscr{L}^2(\Theta)$-valued stochastic process since $u(t)$ is a $\mathscr{L}^2(\Theta)$-valued random variable.

Definition 1. A $\mathscr{L}^2(\Theta)$-valued stochastic process $u = \{u(t)\}_{t\in[0,T]}$ is called a mild solution to (1) if the following conditions are satisfied:

(i) $u \in \mathscr{C}([0,T]; \mathscr{L}^2(\Theta))$ and for any $t \in [0,T]$, $u(t)$ is adapted to $\mathscr{F}_t$ with

$$P\left\{\omega : \int_0^t \|u(s)\|^p ds < \infty\right\} = 1; \tag{2}$$

(ii) stochastic integral equation

$$u(t,x) = e^{At}u_0 + \int_0^t e^{A(t-s)}\alpha(r(s)) f(s, u(s,x), r(s)) ds + \int_0^t e^{A(t-s)}\beta(r(s)) g(s, u(s,x), r(s)) dB(s) \tag{3}$$

holds a.s. for any $t \in [0,T]$ and $x \in \Theta$.

Definition 2. Equation (1) is called exponentially stable in the pth moment if there exists a pair of constants $\delta > 0$ and $K > 0$ such that $\mathbf{E}\|u(t)\|^p \leq K\mathbf{E}\|u_0\|^p e^{-\delta t}$, $t \geq 0$.

Lemma 3 (C_p inequality). *Let $\{\xi_k, 1 \leq k \leq n\}$ be an arbitrary family of random variables; then one has*

$$\mathbf{E}\left(\left\|\sum_{k=1}^n \xi_k\right\|^p\right) \leq C_p \sum_{k=1}^n \mathbf{E}\left(\|\xi_k\|^p\right), \tag{4}$$

where $C_p = n^{p-1}$, $p \geq 1$; $C_p = 1$, $0 < p < 1$.

Finally, we present some necessary assumptions to close this section.

Assumptions. Consider the following:

($\mathbf{A_1}$) $\|e^{tA}\| \le Me^{-\gamma t}$ for some constants $M > 0$ and $\gamma > 0$;

($\mathbf{A_2}$) the operators f, g satisfy the following properties: for any $u, v \in L^2(\Theta)$ and $p \ge 2$, there exist positive constants L_{f_i}, L_{g_i} $(i \in S)$ such that

$$\begin{gathered} f(t,0,i) = 0, \qquad g(t,0,i) = 0, \\ \left\| f(t,u(t,x),i) - f(t,v(t,x),i) \right\| \le L_{f_i} \|u(t,x) - v(t,x)\|, \\ \left\| g(t,u(t,x),i) - g(t,v(t,x),i) \right\| \le L_{g_i} \|u(t,x) - v(t,x)\|; \end{gathered} \tag{5}$$

($\mathbf{A_3}$) $\sup_{1\le i\le N}|\alpha_i|L_{f_i} < \infty$, $\sup_{1\le i\le N}|\beta_i|L_{g_i} < \infty$.

Remark 4. It is obvious to see that (1) has a trivial solution under Assumptions $\mathbf{A_1}$–$\mathbf{A_3}$ when the initial value u_0 is equal to zero.

3. pth Moment Exponential Stability

In this section, we will use the fixed point principle to discuss the existence and uniqueness of the mild solution to (1) and prove the exponential stability result.

Theorem 5. *Let $p \ge 2$ and suppose that Assumptions $\mathbf{A_1}$–$\mathbf{A_3}$ hold. Then,* (1) *is exponentially stable in the pth moment if the following holds:*

$$\frac{2^{p-1}M^p}{\gamma^p}|\alpha^*|^p L_{f^*}^p + \frac{\sqrt{2}^{p-2}M^p J_p}{\gamma^{p/2}}|\beta^*|^p L_{g^*}^p \in (0,1), \tag{6}$$

where $J_p = \left(p^{p+1}/2(p-1)^{p-1}\right)^{p/2}$, $|\alpha^*|L_{f^*} = \sup_{1\le i\le N}|\alpha_i|L_{f_i}$, $|\beta^*|L_{g^*} = \sup_{1\le i\le N}|\beta_i|L_{g_i}$.

Proof. Let H be the Banach space of all $\mathcal{F}_t$-adapted continuous processes consisting of functions $u(t,x)$ such that $\mathbf{E}\|u(t,x)\|^p \le M^*\mathbf{E}\|u_0\|^p e^{-\eta t}$, $t \ge 0$, where $M^* > 0$, $0 < \eta < \gamma$. Below we denote the norm in H by $\|u(t,x)\|_H := \sup_{t\ge0}\mathbf{E}\|u(t,x)\|^p$.

Then, we derive an operator $\varphi : H \mapsto H$ as follows:

$$\begin{aligned} \varphi(u)(t) = e^{At}u_0 &+ \int_0^t e^{A(t-s)}\alpha(r(s)) f(s,u(s,x),r(s))\,ds \\ &+ \int_0^t e^{A(t-s)}\beta(r(s)) g(s,u(s,x),r(s))\,dB(s), \\ &\qquad t > 0, \\ \varphi(u)(0) &= u_0. \end{aligned} \tag{7}$$

It is easy to prove that the following holds by C_p inequality yields:

$$\begin{aligned} \mathbf{E}\|\varphi(u)(t)\|^p &\le 3^{p-1}\mathbf{E}\left\|e^{At}u_0\right\|^p + 3^{p-1}\mathbf{E} \\ &\quad\times\left\|\int_0^t e^{A(t-s)}\alpha(r(s)) f(s,u(s,x),r(s))\,ds\right\|^p \\ &\quad+ 3^{p-1}\mathbf{E} \\ &\quad\times\left\|\int_0^t e^{A(t-s)}\beta(r(s)) g(s,u(s,x),r(s))\,dB(s)\right\|^p \\ &:= 3^{p-1}\sum_{i=1}^{3}\mathbf{E}\|I_i(t)\|^p. \end{aligned} \tag{8}$$

Next, we will divide the proof into three steps.

Claim 1. φ is continuous in the pth moment on $[0,+\infty)$.

Proof of Claim 1. Let $u \in H$, $t_1 \ge 0$, and let $|r|$ be sufficiently small,

$$\mathbf{E}\|\varphi(t_1+r) - \varphi(t_1)\|^p \le 3^{p-1}\sum_{i=1}^{3}\mathbf{E}\|I_i(t_1+r) - I_i(t_1)\|^p. \tag{9}$$

Letting $|r| \to 0$, it is easy to see that

$$\begin{aligned} \mathbf{E}\|I_1(t_1+r) - I_1(t_1)\|^p &= \mathbf{E}\left\|e^{A(t_1+r)}u_0 - e^{At_1}u_0\right\|^p \\ &= \mathbf{E}\left\|e^{At_1}u_0\left(e^{Ar}-1\right)\right\|^p \longrightarrow 0 \quad (|r| \longrightarrow 0), \end{aligned} \tag{10}$$

$$\begin{aligned} \mathbf{E}\|I_2(t_1+r) - I_2(t_1)\|^p &= \mathbf{E}\left\|\int_0^{t_1+r} e^{A(t_1+r-s)}\alpha(r(s)) f(s,u(s,x),r(s))\,ds\right. \\ &\qquad\left. - \int_0^{t_1} e^{A(t_1-s)}\alpha(r(s)) f(s,u(s,x),r(s))\,ds\right\|^p \\ &\le 2^{p-1}\mathbf{E} \\ &\quad\times\left\|\int_0^{t_1} e^{A(t_1-s)}\alpha(r(s)) f(s,u(s,x),r(s))\left(e^{Ar}-1\right)ds\right\|^p \\ &\quad+ 2^{p-1}\mathbf{E} \\ &\quad\times\left\|\int_{t_1}^{t_1+r} e^{A(t_1+r-s)}\alpha(r(s)) f(s,u(s,x),r(s))\,ds\right\|^p \\ &\quad\longrightarrow 0 \quad (|r| \longrightarrow 0). \end{aligned} \tag{11}$$

Then, by Burkhölder-Davis-Gundy inequality, the following holds when $|r| \to 0$:

$$\begin{aligned} \mathbf{E}\|I_3(t_1+r) - I_3(t_1)\|^p &= \mathbf{E}\left\|\int_0^{t_1+r} e^{A(t_1+r-s)}\beta(r(s)) g(s,u(s,x),r(s))\,dB(s)\right. \end{aligned}$$

$$
\begin{aligned}
&-\int_0^{t_1} e^{A(t_1-s)}\beta(r(s))\,g(s,u(s,x),r(s))\,dB(s)\Big\|^p \\
&\le 2^{p-1}\mathbf{E}\Big\|\int_0^{t_1} e^{A(t_1-s)}\beta(r(s)) \\
&\qquad\times g(s,u(s,x),r(s))\left(e^{Ar}-1\right)dB(s)\Big\|^p \\
&\quad+2^{p-1}\mathbf{E}\Big\|\int_{t_1}^{t_1+r} e^{A(t_1+r-s)} \\
&\qquad\times\beta(r(s))\,g(s,u(s,x),r(s))\,dB(s)\Big\|^p \\
&\le 2^{p-1}J_p\mathbf{E} \\
&\quad\times\left(\int_0^{t_1} M^2 e^{-2\gamma(t_1-s)}\right. \\
&\qquad\left.\times\left\|\beta(r(s))\,g(s,u(s,x),r(s))\left(e^{Ar}-1\right)\right\|^2 ds\right)^{p/2} \\
&\quad+2^{p-1}J_p\mathbf{E}\left(\int_{t_1}^{t_1+r} M^2 e^{-2\gamma(t_1+r-s)}\right. \\
&\qquad\left.\times\left\|\beta(r(s))\,g(s,u(s,x),r(s))\right\|^2 ds\right)^{p/2} \\
&\longrightarrow 0\quad(|r|\longrightarrow 0).
\end{aligned}
\tag{12}
$$

Hence, from (10)–(12), we see that φ is pth continuous on $[0,+\infty)$.

Claim 2. $\varphi(H)$ is contained in H.

Proof of Claim 2. It follows from (8) that

$$
I_1(t)=\mathbf{E}\left\|e^{At}u_0\right\|^p\le M^p e^{-p\gamma t}\mathbf{E}\left\|u_0\right\|^p\le M^p e^{-\eta t}\mathbf{E}\left\|u_0\right\|^p.
\tag{13}
$$

By Assumptions $\mathbf{A_1}$–$\mathbf{A_3}$ and the Hölder inequality, we have

$$
\begin{aligned}
I_2(t)&=\mathbf{E}\left\|\int_0^t e^{A(t-s)}\alpha(r(s))\,f(s,u(s,x),r(s))\,ds\right\|^p \\
&\le M^p\mathbf{E}\left(\int_0^t e^{-\gamma(t-s)}\left|\alpha(r(s))\right|\left\|f(s,u(s,x),r(s))\right\|ds\right)^p \\
&\le M^p\mathbf{E}\left[\left(\int_0^t e^{-\gamma(t-s)}ds\right)^{p-1}\right. \\
&\qquad\times\left(\int_0^t e^{-\gamma(t-s)}\left|\alpha(r(s))\right|^p\right. \\
&\qquad\left.\left.\times\left\|f(s,u(s,x),r(s))\right\|^p ds\right)\right] \\
&\le\frac{M^p}{\gamma^{p-1}}\mathbf{E}\left(\int_0^t e^{-\gamma(t-s)}\left|\alpha(r(s))\right|^p L_f^p(r(s))\left\|u(s,x)\right\|^p ds\right) \\
&\le\frac{M^p}{\gamma^{p-1}}\mathbf{E}\left(\left|\alpha^*\right|^p L_{f^*}^p\int_0^t e^{-\gamma(t-s)}\left\|u(s,x)\right\|^p ds\right) \\
&\le\frac{M^p\left|\alpha^*\right|^p L_{f^*}^p}{\gamma^{p-1}}\int_0^t e^{-\gamma(t-s)}\mathbf{E}\left\|u(s,x)\right\|^p ds \\
&\le\frac{M^p M^*\left|\alpha^*\right|^p L_{f^*}^p}{\gamma^{p-1}}\mathbf{E}\left\|u_0\right\|^p\int_0^t e^{-\gamma(t-s)}e^{-\eta s}ds \\
&\le\frac{M^p M^*\left|\alpha^*\right|^p L_{f^*}^p}{\gamma^{p-1}(\gamma-\eta)}\mathbf{E}\left\|u_0\right\|^p e^{-\eta t}, \\
I_3(t)&=\mathbf{E}\left\|\int_0^t e^{A(t-s)}\beta(r(s))\,g(s,u(s,x),r(s))\,dB(s)\right\|^p \\
&\le M^p J_p\mathbf{E} \\
&\qquad\times\left(\int_0^t e^{-2\gamma(t-s)}\left|\beta(r(s))\right|^2\left\|g(s,u(s,x),r(s))\right\|^2 ds\right)^{p/2} \\
&\le M^p J_p\mathbf{E}\left[\left(\int_0^t e^{-2\gamma(t-s)}ds\right)^{(p/2)-1}\right. \\
&\qquad\times\left(\int_0^t e^{-2\gamma(t-s)}\left|\beta(r(s))\right|^p\right. \\
&\qquad\left.\left.\times\left\|g(s,u(s,x),r(s))\right\|^p ds\right)\right] \\
&\le M^p J_p(2\gamma)^{(2-p)/2}\left|\beta^*\right|^p L_{g^*}^p\int_0^t e^{-2\gamma(t-s)}\mathbf{E}\left\|u(s,x)\right\|^p ds \\
&\le M^p M^* J_p(2\gamma)^{(2-p)/2}\left|\beta^*\right|^p L_{g^*}^p\mathbf{E}\left\|u_0\right\|^p\int_0^t e^{-2\gamma(t-s)}e^{-\eta s}ds \\
&\le\frac{M^p M^*}{2\gamma-\eta}J_p(2\gamma)^{(2-p)/2}\left|\beta^*\right|^p L_{g^*}^p\mathbf{E}\left\|u_0\right\|^p e^{-\eta t}.
\end{aligned}
\tag{14}
$$

From (13)-(14), it is easy to see that $\mathbf{E}\|\varphi(u)(t)\|^p\le k\mathbf{E}\|\varphi(u)(0)\|^p e^{-\eta t}$, where $k=3^{p-1}M^p(1+(M^*/\gamma^{p-1}(\gamma-\eta))|\alpha^*|^p L_{f^*}^p+(M^* J_p(2\gamma)^{(2-p)/2}/(2\gamma-\eta))|\beta^*|^p L_{g^*}^p)$, which implies $\varphi(H)\subseteq H$.

Claim 3. φ is contractive for arbitrary $u,v\in H$ with $u(0,x)=u_0(x)=v(0,x)$.

Proof of Claim 3. Consider the following:

$$
\begin{aligned}
&\mathbf{E}\sup_{0\le t<\infty}\left\|\varphi(u)(t)-\varphi(v)(t)\right\|^p \\
&\quad\le 2^{p-1}\mathbf{E}\sup_{0\le t<\infty}\left\|\int_0^t e^{A(t-s)}\alpha(r(s))\,f(s,u(s,x),r(s))\,ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - \int_0^t e^{A(t-s)} \alpha (r(s)) f(s, v(s,x), r(s))\, ds \Big\|^p \\
&\quad + 2^{p-1} \mathbf{E} \\
&\quad \times \sup_{0\le t<\infty} \Big\| \int_0^t e^{A(t-s)} \beta (r(s)) g(s, u(s,x), r(s))\, ds \\
&\qquad - \int_0^t e^{A(t-s)} \beta (r(s)) g(s, v(s,x), r(s))\, ds \Big\|^p \\
&\le 2^{p-1} M^p \gamma^{1-p} |\alpha^*|^p L_{f^*}^p \mathbf{E} \\
&\quad \times \sup_{0\le t<\infty} \int_0^t e^{-\gamma(t-s)} \| u(s,x) - v(s,x) \|^p ds \\
&\quad + 2^{p-1} M^p J_p (2\gamma)^{(2-p)/2} |\beta^*|^p L_{g^*}^p \mathbf{E} \\
&\quad \times \sup_{0\le t<\infty} \int_0^t e^{-2\gamma(t-s)} \| u(s,x) - v(s,x) \|^p ds \\
&\le 2^{p-1} M^p \gamma^{1-p} |\alpha^*|^p L_{f^*}^p \mathbf{E} \sup_{0\le t<\infty} \| u(t,x) - v(t,x) \|^p \\
&\quad \times \int_0^t e^{-\gamma(t-s)} ds \\
&\quad + 2^{p-1} M^p J_p (2\gamma)^{(2-p)/2} |\beta^*|^p L_{g^*}^p \\
&\quad \times \mathbf{E} \sup_{0\le t<\infty} \| u(t,x) - v(t,x) \|^p \\
&\quad \times \int_0^t e^{-2\gamma(t-s)} ds \\
&\le 2^{p-1} M^p \gamma^{-p} |\alpha^*|^p L_{f^*}^p \mathbf{E} \sup_{0\le t<\infty} \| u(t,x) - v(t,x) \|^p \\
&\quad + 2^{p-1} M^p J_p (2\gamma)^{-p/2} |\beta^*|^p L_{g^*}^p \\
&\quad \times \mathbf{E} \sup_{0\le t<\infty} \| u(t,x) - v(t,x) \|^p \\
&= \tilde{k} \mathbf{E} \sup_{0\le t<\infty} \| u(t,x) - v(t,x) \|^p,
\end{aligned} \tag{15}
$$

where $\tilde{k} = (2^{p-1} M^p / \gamma^p) |\alpha^*|^p L_{f^*}^p + (\sqrt{2}^{p-2} M^p J_p / \gamma^{p/2}) |\beta^*|^p L_{g^*}^p$.

Recalling condition (6) and noting that $\tilde{k} \in (0,1)$, we see that φ is a contraction mapping. By the fixed point theory, we derive that φ has a unique fixed point $u(t,x)$ in H, which is also exponentially stable in the pth moment from the proof of three claims. Therefore, the desired assertion in Theorem 5 is completed. □

Remark 6. If $p = 2$, then it is obvious that (1) is mean square exponentially stable.

Remark 7. In Theorem 5, we apply the fixed point theory to obtain the existence and uniqueness of the solution for a class of nonlinear hybrid stochastic heat equations. Obviously, our results extend and improve those given in [3, 5, 10].

Remark 8. According to the proof of Theorem 5 and based on the Gronwall inequality, we can easily obtain the Lyapunov exponents of the solution as follows:

$$
\limsup_{t\to\infty} \frac{1}{t} \log \left(\mathbf{E} \left(\| u(t) \|^p \right) \right) \le C - \gamma, \tag{16}
$$

where $C = 3^{p-1} M^p \gamma^{1-p} |\alpha^*|^p L_{f^*}^p + 3^{p-1} M^p J_p (2\gamma)^{(2-p)/2} |\beta^*|^p L_{g^*}^p$.

In particular, we derive that $C < \gamma$ when $\tilde{k} < 1$ in Theorem 5 such that the solution of (1) is exponentially stable in the pth moment.

Remark 9. In [10], Bao et al. discussed the Lyapunov exponents of linear hybrid stochastic heat equations by employing the explicit formula of solution. However, we consider a class of nonlinear hybrid stochastic heat equations in this paper. Obviously, the method in [10] fails in our result since the explicit formula of solution cannot be obtained in the nonlinear case. So we apply succesfully the fixed point principle to study the pth moment exponential stability of a class of nonlinear hybrid stochastic heat equations, which obviously generalizes the linear case.

4. Two Examples

In this section, we provide two examples of heat equations with Markovian switching as the applications of our main results and give some remarks compared to the previous articles.

Example 1. Consider the following stochastic heat equation with Markovian switching:

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= A u(x,t) + \alpha (r(t)) \sin \frac{u(t,x)}{3^{r(t)}} \\
&\quad + \beta (r(t)) u(t,x) \dot{B}(t) \quad x \in (0,\pi), \ t > 0, \\
u(t,0) &= u(t,\pi) = 0, \quad t > 0, \\
u(0,x) &= \sqrt{\frac{2}{\pi}} \cos x \quad x \in (0,\pi),
\end{aligned} \tag{17}
$$

where $\{r(t), t \ge 0\}$ is a right continuous Markov chain which takes values in an infinite state space $S = \{1, 2, \ldots\}$. Take $\alpha_i = i$, $\beta_i = J_p^{-1/p}/i$, $i \in S$. Below we recall that A is an infinitesimal generator with a strongly continuous semigroup e^{tA}, $t \ge 0$ so that the eigenfunctions of $-A$ are $e_n(x) = \sqrt{2/\pi} \sin nx \in \mathscr{D}(A)$, $n = 1, 2, \ldots$, and the relevant eigenvalues of A are $\lambda_n = n^2$. Observing that $e^{tA} u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, e_n \rangle_H e_n$, $u \in H$, and $\| e^{tA} \| \le e^{-\pi^2 t}$, $t \ge 0$, we have $\gamma = \pi^2$, $M = 1$.

Since

$$\|f(t,u(t,x),i)\| = \left\|\sin\frac{u(t,x)}{3^i}\right\| \le \frac{1}{3^i}\|u(t,x)\|, \quad i \in S, \tag{18}$$

we can choose $L_{f_i} = 1/3^i$ and similarly $L_{g_i} = 1$. It is easy to calculate that $\widetilde{k} = (1/3\pi^{2p}) + (\sqrt{2}^{p-2}/\pi^p) \in (0,1)$ with $\alpha^* L_{f^*} = 1/3$, $\beta^* L_{g^*} = J_p^{-1/p}$. Therefore, by Theorem 5 we see that (17) is pth moment exponentially stable.

Remark 10. The state space S in (17) is infinite in which Markov chain $\{r(t), t \ge 0\}$ takes values while generally we assume the state space to be finite to gain the pth moment exponential stability of the solution, so our results generalize those in the previous literature (e.g., see [4, 10]).

Remark 11. If $p = 2$, then (17) is mean square exponentially stable.

Example 2. We compare the following linear stochastic heat equation with that in [10]:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= Au(x,t) + \alpha(r(t))u(t,x) \\ &\quad + \beta(r(t))u(t,x)\dot{B}(t) \quad x \in (0,\pi),\ t > 0, \\ u(t,0) &= u(t,\pi) = 0, \quad t > 0, \\ u(0,x) &= \sqrt{\frac{2}{\pi}}\sin x \quad x \in (0,\pi), \end{aligned} \tag{19}$$

where $\{r(t), t \ge 0\}$ takes values in $S = \{1,2\}$ with the generator $\Gamma = (\gamma_{ij})_{2\times 2}$:

$$-\gamma_{11} = \gamma_{12} = 1, \qquad -\gamma_{22} = \gamma_{21} = q > 0, \tag{20}$$

which implies that the Markov chain $\{r(t), t \ge 0\}$ has a unique invariant measure $\pi = (\pi_1, \pi_2) = (q/(q+1), 1/(q+1))$. Take $\alpha(1) = a$, $\alpha(2) = b$, $\beta(1) = c$, $\beta(2) = d, a, b, c, d \in \mathbb{R}$. Then, it is easy to compute $\widetilde{k} = (2^{p-1}/\pi^{2p})|a \vee b|^p + (\sqrt{2}^{p-2}J_p/\pi^p)|c \vee d|^p$. So we can conclude that the system of (19) is pth moment exponentially stable if $\widetilde{k} < 1$.

Remark 12. Example 2 is also considered by Bao et al. [10] based on the large deviation technique, in which a proper q should be found to calculate the rate function $I(\mu)$ to testify whether the solution of (19) is pth moment exponentially stable or not. However, it is very difficult and complicated in comparison with our conditions, especially in the nonlinear hybrid stochastic heat equations. So our results generalize and imporove those given in [10].

5. Conclusion

In this paper, we have studied the stability problem of a class of nonlinear hybrid stochastic heat equations. Based on the fixed point theory and Burkhölder-Davis-Gundy inequality, we not only establish the existence and uniqueness of the equation in an infinite state space, but also prove the pth moment exponential stability of the system. Moreover, we give two simple examples to verify all our conditions at the end of this paper. In the future work, we will focus on the stability of more complicity models such as neutral stochastic differential equations with Markov chains and variable time delay.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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