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*Research Article*

# Positive Solutions for a Class of Third-Order Three-Point Boundary Value Problem

**Xiaojie Lin and Zhengmin Fu**

*School of Mathematical Sciences, Jiangsu Normal University, Xuzhou, Jiangsu 221116, China*

Correspondence should be addressed to Xiaojie Lin, linxiaojie1973@163.com

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We investigate the problem of existence of positive solutions for the nonlinear third-order three-point boundary value problem $u'''(t) + \lambda a(t) f(u(t)) = 0$, $0 < t < 1$, $u(0) = u'(0) = 0$, $u''(1) = \alpha u''(\eta)$, where $\lambda$ is a positive parameter, $\alpha \in (0,1)$, $\eta \in (0,1)$, $f : (0,\infty) \to (0,\infty)$, $a : (0,1) \to (0,\infty)$ are continuous. Using a specially constructed cone, the fixed point index theorems and Leray-Schauder degree, this work shows the existence and multiplicities of positive solutions for the nonlinear third-order boundary value problem. Some examples are given to demonstrate the main results.

## 1. Introduction

This paper deals with the following third-order nonlinear boundary value problem:

$$\begin{gathered} u'''(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad u''(1) = \alpha u''(\eta). \end{gathered} \tag{1.1}$$

Third-order boundary value problems arise in a variety of different areas of applied mathematics and physics. In the few years, there has been increasing interest in studying certain third-order boundary value problems for nonlinear differential equation and have received much attention. To identify a few, we refer the reader to [1–6].

Recently, El-Shahed [1] discussed the following third-order two-point boundary value problem:

$$\begin{gathered} u'''(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad \alpha u'(1) + \beta u''(1) = 0. \end{gathered} \tag{1.2}$$

The methods employed in [1] are Kransnoselskii's fixed-point theorem of cone.

In later work, by placing restrictions on the nonlinear term $f$, Sun [2] studied the following boundary value problems and obtained the three solution via leggett-williams fixed point theorem:

$$\begin{gathered} u'''(t) = a(t)f\left(t, u(t), u'(t), u''(t)\right), \quad 0 < t < 1, \\ u(0) = \delta u(\eta) = 0, \qquad u'(\eta) = 0, \qquad u''(1) = 0. \end{gathered} \tag{1.3}$$

The upper and lower solution is a powerful tool for proving existence for boundary value problems, Ma [7] studied the multiplicity of positive solutions of three-point boundary value problem of second-order ordinary differential equations. Du et al. [5] investigated a class of third-order nonlinear problem.

Motivated by the work of the above papers, the purpose of this article is to study the existence of solution for boundary value problem (1.1) using a new technique (different from the proof of [1, 2, 7]) and we get a new existence result. The tools are based on the fixed point index theorems and Leray-Schauder degree.

The paper is organized as follows: Section 2 states some definitions and some lemmas which are important to obtain our main result. Section 3 is devoted to the existence result of BVP (1.1). Section 4 gives some examples to illustrate our main results.

## 2. Preliminary

*Definition 2.1.* Let $E$ be a real Banach space. A nonempty closed convex set $K \subset E$ is called a cone of $E$ if it satisfies the following two conditions:

(1) $x \in K$, $\lambda \geq 0$ implies $\lambda x \in K$;

(2) $x \in K$, $-x \in K$ implies $x = 0$.

*Definition 2.2.* An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

**Lemma 2.3.** *Let $y \in C[0,1]$, then the following boundary value problem:*

$$u'''(t) + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u'(0) = 0, \qquad u''(1) = \alpha u''(\eta), \tag{2.2}$$

*has the unique solution*

$$u(t) = \int_0^1 G(t,s)y(s)ds, \tag{2.3}$$

*where*

$$G(t,s) = \begin{cases} -\dfrac{1}{2}(t-s)^2 + \dfrac{t^2}{2}, & s \leq \eta, \ s \leq t, \\ \dfrac{t^2}{2}, & t \leq s \leq \eta, \\ -\dfrac{1}{2}(t-s)^2 + \dfrac{t^2}{2(1-\alpha)}, & \eta \leq s \leq t, \\ \dfrac{t^2}{2(1-\alpha)}, & \eta \leq s, \ t \leq s. \end{cases} \tag{2.4}$$

*Proof.* From (2.1), we have

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + At^2 + Bt + C. \tag{2.5}$$

In particular,

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + At^2 + Bt + C, \\ u'(t) &= -t\int_0^t y(s)ds + \int_0^t sy(s)ds + 2At + B, \\ u''(t) &= -\int_0^t y(s)ds + 2A. \end{aligned} \tag{2.6}$$

Combining this with boundary conditions (2.2), we conclude that

$$\begin{gathered} A = \frac{\int_0^1 y(s)ds}{2(1-\alpha)} - \frac{\alpha\int_0^\eta y(s)ds}{2(1-\alpha)}, \\ B = 0, \\ C = 0. \end{gathered} \tag{2.7}$$

Therefore, BVP (2.1)-(2.2) has a unique solution:

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds - \frac{\alpha t^2\int_0^\eta y(s)ds}{2(1-\alpha)} + \frac{t^2\int_0^1 y(s)ds}{2(1-\alpha)} \\ &= \begin{cases} \displaystyle\int_0^t \left[-\frac{1}{2}(t-s)^2 + \frac{t^2}{2}\right] y(s)ds + \int_t^\eta \frac{t^2}{2} y(s)ds + \int_\eta^1 \frac{t^2}{2(1-\alpha)} y(s)ds, & t \le \eta, \\ \displaystyle\int_0^\eta \left[-\frac{1}{2}(t-s)^2 + \frac{t^2}{2}\right] y(s)ds + \int_\eta^t \left[-\frac{1}{2}(t-s)^2 + \frac{t^2}{2(1-\alpha)}\right] y(s)ds & \\ \displaystyle\quad + \int_t^1 \frac{t^2}{2(1-\alpha)} y(s)ds, & t \ge \eta, \end{cases} \\ &= \int_0^1 G(t,s)y(s)ds. \end{aligned} \tag{2.8}$$

The proof is completed. □

**Lemma 2.4.** *For all* $(t,s) \in [0,1] \times [0,1]$, *one has* $G(t,s) \ge 0$.

**Lemma 2.5.** *for all* $(t,s) \in [\tau,1] \times [0,1]$, *one has*

$$\gamma G(1,s) \le G(t,s) \le G(1,s), \tag{2.9}$$

*where* $\gamma = \alpha\tau^2/2$, *and* $\tau$ *statisfies* $\int_\tau^1 G(t,s)a(s)ds > 0$.

*Proof.* For $s \leq t,\ s \leq \eta$,

$$\begin{gathered} G(t,s) = -\frac{1}{2}(t-s)^2 + \frac{t^2}{2} = \frac{s(2t-s)}{2} \leq G(1,s), \\ \frac{G(t,s)}{G(1,s)} = \frac{2t-s}{2-s} = \frac{t+t-s}{2-s} \geq \frac{t}{2}. \end{gathered} \tag{2.10}$$

For $t \leq s \leq \eta$,

$$\begin{gathered} G(t,s) = \frac{t^2}{2} \leq G(1,s), \\ \frac{G(t,s)}{G(1,s)} = \frac{t^2/2}{1/2} = t^2. \end{gathered} \tag{2.11}$$

For $\eta \leq s \leq t$,

$$\begin{gathered} G(t,s) = -\frac{1}{2}(t-s)^2 + \frac{t^2}{2(1-\alpha)} = \frac{\alpha t^2 + 2ts(1-\alpha) + s^2(1-\alpha)}{2(1-\alpha)} \leq G(1,s), \\ \frac{G(t,s)}{G(1,s)} = \frac{\alpha t^2 + 2ts(1-\alpha) + s^2(1-\alpha)}{\alpha + 2s(1-\alpha) + s^2(1-\alpha)} \geq \alpha t^2. \end{gathered} \tag{2.12}$$

For $\eta \leq s,\ t \leq s$,

$$\begin{gathered} G(t,s) = \frac{t^2}{2(1-\alpha)} \leq G(1,s), \\ \frac{G(t,s)}{G(1,s)} = t^2. \end{gathered} \tag{2.13}$$

Thus,

$$\frac{\alpha t^2}{2} G(1,s) \leq G(t,s) \leq G(1,s), \quad \text{for } (t,s) \in [0,1] \times [0,1]. \tag{2.14}$$

Therefore,

$$\gamma G(1,s) \leq G(t,s) \leq G(1,s), \quad \forall (t,s) \in [\tau,1] \times [0,1]. \tag{2.15}$$

The proof is completed. □

**Lemma 2.6.** *If $y \in C[0,1]$ and $y \geq 0$, then the unique solution $u(t)$ of the BVP* (2.1)-(2.2) *is nonnegative and satisfies*

$$\min_{t \in [\tau,1]} u(t) \geq \gamma \|u\|. \tag{2.16}$$

*Proof.* Let $y \in C^{+}[0,1]$, it is obvious that it is nonnegative. For any $t \in [0,1]$, by (2.3) and Lemma 2.5, it follows that

$$u(t) = \int_0^1 G(t,s) y(s) ds \leq \int_0^1 G(1,s) y(s) ds, \tag{2.17}$$

and thus,

$$\|u\| \le \int_0^1 G(1,s)y(s)ds. \tag{2.18}$$

On the other hand, (2.3) and Lemma 2.5 imply, for any $t \in [\tau, 1]$,

$$u(t) = \int_0^1 G(t,s)y(s)ds \ge \gamma \int_0^1 G(1,s)y(s)ds. \tag{2.19}$$

Therefore,

$$\min_{t\in[\tau,1]} u(t) \ge \gamma\|u\|. \tag{2.20}$$

This completes the proof. □

Let $E = C[0,1]$ with the usual normal $\|u\| = \max_{t\in[0,1]}|u(t)|$.
Define the cone $K$ by

$$K = \left\{u \in C^+[0,1] : \min_{t\in[\tau,1]} u(t) \ge \gamma\|u\|\right\}. \tag{2.21}$$

Define an operator $T$ by

$$Tu(t) = \lambda\int_0^1 G(t,s)a(s)f(u(s))ds. \tag{2.22}$$

By Lemma 2.3, BVP (1.1) has a positive solution $u = u(t)$ if and only if $u$ is a fixed point of $T$.

**Lemma 2.7.** *Assume that* $0 < \lambda < \infty$. *Then,* $T : K \to K$ *is completely continuous.*

*Proof.* Firstly, it is easy to check that $T : K \to K$ is well defined. By Lemma 2.6, we know that $T(K) \subset K$.

Let $\Omega$ be any boundary subset of $K$, then there exists $r > 0$, $\|u\| \le r$, for all $u \in \Omega$. Therefore, we have

$$|Tu| = \lambda\left|\int_0^1 G(t,s)a(s)f(u(s))ds\right| \le \lambda\left|\int_0^1 G(1,s)a(s)f(u(s))ds\right|. \tag{2.23}$$

So $T\Omega$ is boundary. Moreover, for any $t_1, t_2 \in [0,1]$, $|t_1 - t_2| \le \delta$, $\delta > 0$, we have

$$|Tu(t_1) - Tu(t_2)| \le \lambda\int_0^1 |G(t_1,s) - G(t_2,s)|a(s)f(u(s))ds. \tag{2.24}$$

By the continuity of $f$ and $a$, we have $a(t)$ and $f(u(t))$ are boundary on $u \in \Omega, t \in [0,1]$, which means that there exists a constant $M_a^f > 0$, depending only on $\Omega$ such that

$$\left|a(t)f(u(t))\right| < M_a^f, \tag{2.25}$$

and thus for any $\varepsilon > 0$,

$$\begin{gathered}|G(t_1,s) - G(t_2,s)| \le \frac{\varepsilon}{\lambda M_a^f},\\ |Tu(t_1) - Tu(t_2)| < \varepsilon.\end{gathered} \tag{2.26}$$

Therefore, we can get $T\Omega$ is equicontinuity. Thirdly, we prove that $T$ is continuous. Let $u_n \to u$ as $n \to \infty$, $u_n \in K$. Then, the continuity of $f$, we can get

$$\begin{aligned}|Tu_n(t) - Tu(t)| &= \left|\lambda\int_0^1 G(t,s)a(s)f(u_n(s))ds - \lambda\int_0^1 G(t,s)a(s)f(u(s))ds\right|\\ &= \left|\lambda\int_0^1 G(t,s)a(s)\big(f(u_n(s)) - f(u(s))\big)ds\right|\\ &\le \left|\lambda\int_0^1 G(1,s)a(s)\big(f(u_n(s)) - f(u(s))\big)ds\right| \longrightarrow 0, \quad n \longrightarrow \infty.\end{aligned} \tag{2.27}$$

Then, $Tu_n(t) \to Tu(t)$. Therefore, $T$ is continuous.The operator $T$ is completely continuous by an application of the Ascoli-Arzela theorem. This completes the proof. □

**Lemma 2.8** (see [7, 8]). *Let $E$ be a real Banach space and let $K$ be a cone in $E$. For $r \ge 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume $T : \overline{K}_r \to K$ is a completely continuous operator such that $Tx \ne x$ for $x \in \partial K_r = \{x \in K : \|x\| = r\}$.*

(1) *If $\|Tx\| \ge \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0. \tag{2.28}$$

(2) *If $\|Tx\| \le \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1. \tag{2.29}$$

## 3. Main Results

**Theorem 3.1.** *Assume that*

(A1) *$\lambda$ is a positive parameter, $\eta \in (0,1)$ and $\alpha \in (0,1)$;*

(A2) *$a : [0,1] \to (0,\infty)$ is continuous;*

(A3) *$f : [0,\infty) \to (0,\infty)$ is continuous;*

(A4) *$f_\infty := \lim_{u\to\infty}(f(u)/u) = \infty$.*

*When $\lambda$ is sufficiently small,* (1.1) *has at least one positive solution, whereas for $\lambda$ is sufficiently large,* (1.1) *has no positive solution.*

*Proof.* If $q > 0$, then

$$\beta(q) = \max_{u\in K,\|u\|=q}\left[\int_0^1 G(t,s)a(s)f(u(s))ds\right] > 0. \tag{3.1}$$

For any number $0 < r_1$, let $\delta_1 = r_1/\beta(r_1)$, and set

$$K_{r_1} = \{u \in K : \|u\| < r_1\}. \tag{3.2}$$

Then, for $\lambda \in (0,\delta_1)$ any $u \in \partial K_{r_1}$, we have

$$Tu(t) < \delta_1\left[\int_0^1 G(t,s)f(u(s))ds\right] \le \delta_1\beta(r_1) = r_1. \tag{3.3}$$

Thus, Lemma 2.8 implies

$$i(T,K_{r_1},K) = 1. \tag{3.4}$$

Since $f_\infty = \infty$, there is $M > 0$, such that $f(u) \ge \mu u$, for $u > M$, where $\mu$ is chosen so that

$$\lambda\mu\gamma\int_\tau^1 G(1,s)a(s)ds > 1. \tag{3.5}$$

Let $r_2 > M/\gamma$, and set

$$K_{r_2} = \{u \in K : \|u\| < r_2\}. \tag{3.6}$$

If $u \in \partial K_{r_2}$, then

$$\min_{t\in[\tau,1]} u(t) \ge \gamma\|u\| \ge M. \tag{3.7}$$

Therefore,

$$\begin{aligned}Tu(1) &= \lambda\int_0^1 G(1,s)a(s)f(u(s))ds\\ &\ge \lambda\int_\tau^1 G(1,s)a(s)f(u(s))ds\\ &\ge \lambda\int_\tau^1 G(1,s)a(s)\mu u(s)ds\end{aligned}$$

$$
\begin{aligned}
&\geq \lambda\mu \int_{\tau}^{1} G(1,s)a(s)ds\gamma\|u\| \\
&\geq \lambda\mu\gamma \int_{\tau}^{1} G(1,s)a(s)ds\|u\| \\
&> \|u\|,
\end{aligned}
\tag{3.8}
$$

which implies that

$$\|Tu\| \geq \|u\|, \tag{3.9}$$

for $u \in \partial K_{r_2}$. An application of Lemma 2.8 again shows that

$$i(T, K_{r_2}, K) = 0. \tag{3.10}$$

Since we can adjust $r_1, r_2$ so that $r_1 < r_2$, it follows the additivity of the fixed-point index that

$$i\left(T, K_{r_2} \setminus \overline{K}_{r_1}, K\right) = -1. \tag{3.11}$$

Thus, $T$ has a fixed point in $K_{r_2} \setminus \overline{K}_{r_1}$ which is the desired positive solution of (1.1).

We verify that BVP of (1.1) has no positive solution for $\lambda$ large enough.

Otherwise, there exist $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, with $\lim_{n\to\infty}\lambda_n = +\infty$, such that for any positive integer $n$, the BVP,

$$
\begin{aligned}
&u'''(t) + \lambda_n a(t) f(u(t)) = 0, \quad 0 < t < 1, \\
&u(0) = u'(0) = 0, \qquad u''(1) = \alpha u''(\eta),
\end{aligned}
\tag{3.12}
$$

has a positive solution $u_n(t)$. By (2.22), we have

$$u_n = \lambda_n \int_0^1 G(t,s)a(s)f(u_n(s)) \longrightarrow +\infty, \quad (n \longrightarrow \infty). \tag{3.13}$$

Thus,

$$u_n \longrightarrow \infty, \quad (n \longrightarrow \infty). \tag{3.14}$$

Since $f_\infty$, for $c_0 > 0$, there exists $r_3 > 0$, such that $f(u)/u > c_0$, for $u \in [r_3, \infty)$, which implies that

$$f(u) > c_0 u, \quad \text{for } u \in [r_3, \infty). \tag{3.15}$$

Let $n$ be large enough that $\|u_n\| \geq r_3$, then

$$\begin{aligned}
\|u_n\| &\geq u_n(1) \\
&= \lambda_n \int_0^1 G(1,s)a(s)f(u_n(s))ds \\
&\geq \lambda_n \gamma \int_0^1 G(1,s)a(s)ds c_0 \|u_n\| \\
&> \|\mathrm{u}_n\|.
\end{aligned} \tag{3.16}$$

Choose $n$ so that $c_0 \lambda_n \gamma \int_0^1 G(1,s)a(s)ds > 1$ which is a contradiction. The proof is completed. □

**Theorem 3.2.** *Assume that*

(B1) *$\lambda$ is a positive parameter; $\eta \in (0,1)$ and $\alpha \in (0,1)$;*

(B2) *$a : [0,1] \to (0,\infty)$ is continuous and there exists $m > 0$ such that $a(t) \geq m$;*

(B3) *$f : [0,\infty) \to (0,\infty)$ is continuous;*

(B4) *$f_\infty = \lim_{u\to\infty}(f(u)/u) = 0$, $f_0 = \lim_{u\to 0}(f(u)/u) = 0$;*

(B5) *there exists $\sigma > 0$, for $u \geq \sigma$, such that $f(u) \geq \beta$, where $\beta > 0$, then there exists $\delta_2 > 0$, such that, for $\lambda > \delta_2$, BVP (1.1) has at least two positive solutions $u_\lambda^1, u_\lambda^2$ and $\max u_\lambda^1 > \sigma$.*

*Proof.* Let $\delta_2 = (M\gamma m\beta)^{-1}\sigma$, then for $\lambda > \delta_2$, Lemma 2.7 implies that $T : K \to K$ is completely continuous. Considering (B4), there exists $0 < r < \sigma$ such that $f(u) \leq u/2\Lambda\lambda$, for $0 \leq u \leq r$, where $\Lambda = \int_0^1 G(1,s)a(s)ds$.

So, for $u \in \partial K_r$, we have from (2.4)

$$\begin{aligned}
(Tu)(t) &= \lambda \left[ \int_0^1 G(t,s)a(s)f(s)ds \right] \\
&\leq \lambda \int_0^1 G(1,s)a(s)f(u(s))ds \\
&\leq \lambda \left[ \int_0^1 G(1,s)a(\mathrm{s})ds \right] \frac{\|u\|}{2\Lambda\lambda} \\
&= \frac{\|u\|}{2} < \|u\| = r.
\end{aligned} \tag{3.17}$$

Consequently, for $u \in \partial K_r$, we have $\|Tu\| < \|u\|$, by Lemma 2.8,

$$i(T, K_r, K) = 1. \tag{3.18}$$

Now considering (B4), there exists $h > 0$, for $u > h$, such that $f(u) \leq u/2\Lambda\lambda$. Letting $\rho = \max_{0\leq u\leq h} f(u)$, then

$$0 \leq f(u) \leq \frac{u}{2\Lambda\lambda} + \rho. \tag{3.19}$$

Choose

$$R > \max\{r, 2\Lambda\rho\lambda\}. \tag{3.20}$$

So for $u \in \partial K_R$, from (3.18) and (3.19), we have

$$\begin{aligned}(Tu)(t) &= \lambda\left[\int_0^1 G(t,s)a(s)f(u)ds\right]\\ &\le \lambda\left[\int_0^1 G(1,s)a(s)f(u)ds\right]\\ &\le \lambda\left[\int_0^1 G(1,s)a(s)ds\right]\left(\frac{1}{2\Lambda\lambda}\|u\| + \rho\right)\\ &< \frac{\|u\|}{2} + \frac{R}{2} = \|u\|,\end{aligned} \tag{3.21}$$

That is, by Lemma 2.8,

$$i(T, K_R, K) = 1. \tag{3.22}$$

On the other hand, for $u \in \overline{K}_\sigma^R = \{u \in K : \|u\| \le R,\ \min_{t\in J_\theta} u(t) \ge \sigma, \theta \in (0,1/2), J_\theta = [\theta, 1-\theta]\}$, (2.3) and (2.4) yield that

$$\|Tu\| \le \lambda\left[\int_0^1 G(t,s)a(s)ds\right]\left(\frac{1}{2\Lambda\lambda}\|u\| + \rho\right) < R. \tag{3.23}$$

Furthermore, for $u \in \overline{K}_\sigma^R$, from (2.3) and (2.4), we obtain

$$\begin{aligned}\min_{t\in J_\theta}(Tu)(t) &= \min_{t\in J_\theta}\lambda\left[\int_0^1 G(1,s)a(s)f(u(s))ds\right]\\ &\ge \min_{t\in J_\theta}\lambda\int_\theta^{1-\theta} G(t,s)a(s)f(u(s))ds\\ &\ge \lambda\gamma\int_\theta^{1-\theta} G(1,s)a(s)f(u(s))ds\\ &\ge \lambda M\gamma m\beta > \delta_2 \mathrm{M}\gamma m\beta = \sigma,\end{aligned} \tag{3.24}$$

where $M = \int_\theta^{1-\theta} G(1,s)ds$. Let $u_0 \equiv (\sigma + R)/2$ and $H(t,u) = (1-t)Tu + tu_0$, then $H : [0,1] \times \overline{K}_\sigma^R \to K$ is continuous, and from the analysis above, we obtain for $(t,u) \in [0,1] \times \overline{K}_\sigma^R$:

$$H(t,u) \in K_\sigma^R. \tag{3.25}$$

Therefore, for $u \in \partial K_\sigma^R$, we have $H(t,u) \neq u$. Hence, by the normality property and the homotopy invariance property of the fixed point index, we obtain

$$i\left(T, K_\sigma^R, K\right) = i\left(u_0, K_\sigma^R, K\right) = 1. \tag{3.26}$$

Consequently, by the solution property of the fixed point index, $T$ has a fixed point $u_\lambda^1$ and $u_\lambda^1 \in K_\sigma^R$. By Lemma 2.4, it follows that $u_1^\lambda$ is a solution to BVP (1.1), and

$$\max_{t\in[0,1]} u_\lambda^1 \geq \min_{t\in J\theta} u_\lambda^1 > \gamma. \tag{3.27}$$

On the other hand, from (3.18) and (3.19) together with the additivity of the fixed point index, we get

$$i\left(T, K_R \setminus \left(\overline{K}_r \cup \overline{K}_\sigma^R\right)\right) = i(T, K_R, K) - i\left(T, K_\sigma^R, K\right) - i(T, K_r, K) = 1 - 1 - 1 = -1. \tag{3.28}$$

Hence, by the solution property of the fixed point index, $T$ has a fixed point $u_\lambda^2$ and $u_\lambda^2 \in K_R \setminus (\overline{K}_r \cup \overline{K}_\sigma^R)$. By Lemma 2.3, it follows that $u_\lambda^2$ is also a solution to BVP (1.1), and $u_\lambda^1 \neq u_\lambda^2$. The proof is completed. □

## 4. Examples

*Example 4.1.* We consider the following third-order boundary value problems:

$$\begin{gathered} u'''(t) + \lambda(2t+1)e^u = 0, \\ u(0) = u'\ (0) = 0, \qquad u''\ (1) = \frac{3}{4}u''\left(\frac{1}{4}\right), \end{gathered} \tag{4.1}$$

here $\eta = 1/4, \alpha = 3/4, f(u(t)) = e^u, a(t) = 2t+1, f_\infty = \lim_{u\to\infty}(f(u)/u) = \infty, f$ is continuous, $a(t)$ is continuous. By direct calculations, we obtain that $\lambda < r_1(1-\alpha)$, for $r_1 > 0$. Therefore, by Theorem 3.1, there exists at least one solution $u(t)$ for BVP (4.1), whereas for $\lambda$ large enough, (4.1) has no solution.

*Example 4.2.* Consider the following third-order ordinary differential equation:

$$\begin{gathered} u''' + \lambda(2t+1)f(u(t)) = 0, \\ u(0) = u'(0) = 0, \qquad u''(1) = \frac{1}{4}u''\left(\frac{1}{2}\right), \end{gathered} \tag{4.2}$$

where

$$f(u(t)) = \begin{cases} u^2 e^{-u}, & \text{if } u \le a, \\ a^{3/2}\sqrt{u}e^{-a}, & \text{if } u > a, \end{cases} \tag{4.3}$$

$f$ is continuous, $a(t)$ is continuous. Here, $m = 1$, $\alpha = 1/4$, $\beta = a^2 e^{-a}$, $\sigma = a$, $a > 0$. Choose $\delta_2 = 6a/(2\theta^3 - 3\theta^2 + 3\theta - 1), \theta \in (0, 1/2)$, $\tau \in (0, 1)$, when $\lambda > \delta_2$, by Theorem 3.2, there exist at least two solutions $u_\lambda^1(t), u_\lambda^2(t)$ for BVP (4.1).

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