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Research Article

On Variational Inclusion and Common Fixed Point Problems in q-Uniformly Smooth Banach Spaces

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We introduce a general iterative algorithm for finding a common element of the common fixed-point set of an infinite family of λ_i-strict pseudocontractions and the solution set of a general system of variational inclusions for two inverse strongly accretive operators in a q-uniformly smooth Banach space. Then, we prove a strong convergence theorem for the iterative sequence generated by the proposed iterative algorithm under very mild conditions. The methods in the paper are novel and different from those in the early and recent literature. Our results can be viewed as the improvement, supplementation, development, and extension of the corresponding results in some references to a great extent.

1. Introduction

Throughout this paper, we denote by E and E^* a real Banach space and the dual space of E, respectively. Let C be a subset of E and T a mapping on C. We use $F(T)$ to denote the set of fixed points of T. Let $q > 1$ be a real number. The (generalized) duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1} \right\} \tag{1.1}$$

for all $x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. In particular, $J = J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if E is smooth, then J_q is single-valued, which is denoted by j_q.

The norm of a Banach space E is said to be Gâteaux differentiable if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.2}$$

exists for all x, y on the unit sphere $S(E) = \{x \in E : \|x\| = 1\}$. If, for each $y \in S(E)$, the limit (1.2) is uniformly attained for $x \in S(E)$, then the norm of E is said to be uniformly Gâteaux differentiable. The norm of E is said to be Fréchet differentiable if, for each $x \in S(E)$, the limit (1.2) is attained uniformly for $y \in S(E)$.

Let $\rho_E : [0,1) \to [0,1)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{ \frac{1}{2}\left(\|x + y\| + \|x - y\|\right) - 1 : x \in S(E), \|y\| \le t \right\}. \tag{1.3}$$

A Banach space E is said to be uniformly smooth if $\rho_E(t)/t \to 0$ as $t \to 0$. Let $q > 1$. A Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \le ct^q$. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \le 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable; in particular, the norm of E is Fréchet differentiable. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

A Banach space E is said to be uniformly convex if, for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any x, $y \in S(E)$, $\|x - y\| \ge \varepsilon$ implies $\|(x + y)/2\| \le 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex.

Recall that if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $Q : C \to D$ is sunny (see [1]) provided that

$$Q(x + t(x - Q(x))) = Q(x) \tag{1.4}$$

for all $x \in C$ and $t \ge 0$, whenever $Qx + t(x - Q(x)) \in C$. A mapping $Q : C \to D$ is called a retraction if $Qx = x$ for all $x \in D$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is retraction from C onto D which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. The following proposition concerns the sunny nonexpansive retraction.

Proposition 1.1 (see [1]). *Let C be a closed convex subset of a smooth Banach space E. Let D be a nonempty subset of C. Let $Q : C \to D$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*

(a) *Q is sunny and nonexpansive,*

(b) $\|Qx - Qy\|^2 \le \langle x - y, J(Qx - Qy)\rangle$, *for all x, $y \in C$,*

(c) $\langle x - Qx, J(y - Qx)\rangle \le 0$, *for all $x \in C$, $y \in D$.*

Among nonlinear mappings, the classes of nonexpansive mappings and strict pseudo-contractions are two kinds of the most important nonlinear mappings. The studies on them have a very long history (see, e.g., [1–29] and the references therein). Recall that a mapping $T : C \to E$ is said to be nonexpansive, if

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all } x, y \in C. \tag{1.5}$$

A mapping $T : C \to E$ is said to be λ-strict pseudocontractive in the terminology of Browder and Petryshyn (see [2–4]), if there exists a constant $\lambda > 0$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \le \|x - y\|^q - \lambda\|(I - T)x - (I - T)y\|^q, \tag{1.6}$$

for every x, $y \in C$ and for some $j_q(x - y) \in J_q(x - y)$. It is clear that (1.6) is equivalent to the following:

$$\langle (I - T)x - (I - T)y, j_q(x - y)\rangle \ge \lambda\|(I - T)x - (I - T)y\|^q. \tag{1.7}$$

A mapping $T : C \to E$ is said to be L-Lipschitz if for all $x, y \in C$ there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\| \quad \text{for all } x, y \in C. \tag{1.8}$$

In particular, if $0 < L < 1$, then T is called contractive and if $L = 1$, then T reduces to a nonexpansive mapping.

A mapping $T : C \to E$ is said to be accretive if for all x, $y \in C$ there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \ge 0. \tag{1.9}$$

For some $\eta > 0$, $T : C \to E$ is said to be η-strongly accretive if for all x, $y \in C$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \ge \eta\|x - y\|^q. \tag{1.10}$$

For some $\mu > 0, T : C \to E$ is said to be μ-inverse strongly accretive if for all x, $y \in C$ there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \ge \mu\|Tx - Ty\|^q. \tag{1.11}$$

A set-valued mapping $T : D(T) \subseteq E \to 2^E$ is said to be accretive if for any x, $y \in D(T)$, there exists $j(x - y) \in J(x - y)$, such that for all $u \in T(x)$ and $v \in T(y)$

$$\langle u - v, j(x - y)\rangle \ge 0. \tag{1.12}$$

A set-valued mapping $T : D(T) \subseteq E \to 2^E$ is said to be m-accretive if T is accretive and $(I + \rho T)(D(T)) = E$ for every (equivalently, for some) $\rho > 0$, where I is the identity mapping.

Let $M : D(M) \to 2^E$ be m-accretive. The mapping $J_{M,\rho} : E \to D(M)$ defined by

$$J_{M,\rho}(u) = (I + \rho M)^{-1}(u), \quad \forall u \in E \tag{1.13}$$

is called the resolvent operator associated with M, where ρ is any positive number and I is the identity mapping. It is well known that $J_{M,\rho}$ is single valued and nonexpansive (see [5]).

In order to find the common element of the solutions set of a variational inclusion and the set of fixed points of a nonexpansive mapping S, Zhang et al. [6] introduced the following new iterative scheme in a Hilbert space H. Starting with an arbitrary point $x_1 = x \in H$, define sequences $\{x_n\}$ by

$$\begin{aligned} y_n &= J_{M,\lambda}(x_n - \lambda A x_n), \\ x_{n+1} &= \alpha_n x + (1 - \alpha_n) S y_n, \end{aligned} \tag{1.14}$$

where $A : H \to H$ is an α-cocoercive mapping, $M : H \to 2^H$ is a maximal monotone mapping, $S : H \to H$ is a nonexpansive mapping, and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Under mild conditions, they obtained a strong convergence theorem.

Let C be a nonempty closed convex subset of a real reflexive, strictly convex, and q-uniformly smooth Banach space E. In this paper, we consider the general system of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \theta &\in x^* - y^* + \rho_1 (A y^* + M_1 x^*), \\ \theta &\in y^* - x^* + \rho_2 (B x^* + M_2 y^*), \end{aligned} \tag{1.15}$$

where $A, B : C \to E$, $M_1 : D(M_1) \to 2^E$ and $M_2 : D(M_2) \to 2^E$ are nonlinear mappings.

In the case where $C = E$, a uniformly convex and 2-uniformly smooth Banach space, Qin et al. [8] introduced the following scheme for finding a common element of the solution set of the variational inclusions and the fixed-point set of a λ-strict pseudocontraction. Starting with an arbitrary point $x_1 = u \in E$, define sequences $\{x_n\}$ by

$$\begin{aligned} z_n &= J_{M_2,\rho_2}(x_n - \rho_2 A_2 x_n), \\ y_n &= J_{M_1,\rho_1}(z_n - \rho_1 A_1 z_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n)\left[\mu S x_n + (1 - \mu) y_n\right], \quad n \geq 1, \end{aligned} \tag{1.16}$$

where $A_1, A_2 : E \to E$ are two inverse strongly accretive operators, $M_1, M_2 : E \to 2^E$ are two maximal monotone mappings, $T : E \to E$ is a λ-strict pseudocontraction, and $S : E \to E$ is defined as $Sx = (1-\lambda/K^2)x + (\lambda/K^2)Tx$, for all $x \in E$. Then they proved a strong convergence theorem under mild conditions.

In this paper, motivated by Zhang et al. [6], Qin et al. [8], Yao et al. [9], Hao [10], Yao and Yao [11], and Takahashi and Toyoda [12], we consider a relaxed extragradient-type method for finding a common element of the solution set of a general system of variational inclusions for inverse strongly accretive mappings and the common fixed-point set of an infinite family of λ_i-strict pseudocontractions. Furthermore, we obtain strong convergence theorems under mild conditions. The results presented by us improve and extend the corresponding results announced by many others.

2. Preliminaries

In order to prove our main results, we need the following lemmas.

Lemma 2.1 (see [16]). *Let C be a closed convex subset of a strictly convex Banach space E. Let T_1 and T_2 be two nonexpansive mappings from C into itself with $F(T_1)\bigcap F(T_2)\neq\emptyset$. Define a mapping S by*

$$Sx = \lambda T_1 x + (1-\lambda)T_2 x, \quad \forall x \in C, \tag{2.1}$$

where λ is a constant in $(0,1)$. Then S is nonexpansive and $F(S) = F(T_1)\bigcap F(T_2)$.

Lemma 2.2 (see [30]). *Let $\{\alpha_n\}$ be a sequence of nonnegative numbers satisfying the property:*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + b_n + \gamma_n c_n, \quad n \geq 0, \tag{2.2}$$

where $\{\gamma_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy the restrictions:

(i) $\lim_{n\to\infty}\gamma_n = 0$, $\sum_{n=1}^{\infty}\gamma_n = \infty$,

(ii) $b_n \geq 0$, $\sum_{n=1}^{\infty} b_n < \infty$,

(iii) $\limsup_{n\to\infty} c_n \leq 0$.

Then, $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.3 (see [31, page 63]). *Let $q > 1$. Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{q/(q-1)} \tag{2.3}$$

for arbitrary positive real numbers a, b.

Lemma 2.4 (see [17]). *Let E be a real q-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q, \quad \forall x, y \in E. \tag{2.4}$$

In particular, if E is a real 2-uniformly smooth Banach space, then there exists a best smooth constant $K > 0$ such that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2K\|y\|^2, \quad \forall x, y \in E. \tag{2.5}$$

Lemma 2.5 (see [20]). *Let C be a nonempty convex subset of a real q-uniformly smooth Banach space E and let $T : C \to C$ be a λ-strict pseudocontraction. For $\alpha \in (0,1)$, one defines $T_\alpha x = (1-\alpha)x+\alpha Tx$. Then, as $\alpha \in (0,\mu]$, $\mu = \min\{1, \{q\lambda/C_q\}^{1/(q-1)}\}$, $T_\alpha : C \to C$ is nonexpansive such that $F(T_\alpha) = F(T)$.*

Lemma 2.6 (see [21]). *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth Banach space E which admits weakly sequentially continuous generalized duality mapping j_q from E into E^*. Let $T : C \to C$ be a nonexpansive mapping. Then, for all $\{x_n\} \subset C$, if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.7 (see [21]). *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth Banach space E. Let $V : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$. Let $0 < \mu < (q\eta/C_q k^q)^{1/(q-1)}$ and $\tau = \mu(\eta - C_q\mu^{q-1}k^q/q)$. Then for each $t \in (0, \min\{1, 1/\tau\})$, the mapping $S : C \to E$ defined by $S := (I - t\mu V)$ is a contraction with a constant $1 - t\tau$.*

Lemma 2.8 (see [21]). *Let C be a nonempty, closed and convex subset of a real q-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let $V : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$, $f : C \to E$ a L-Lipschitzian mapping with constant $L \geq 0$, and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $0 < \mu < (q\eta/C_q k^q)^{1/(q-1)}$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - C_q\mu^{q-1}k^q/q)$. Then $\{x_t\}$ defined by*

$$x_t = Q_C\left[t\gamma f x_t + (I - t\mu V)Tx_t\right]. \tag{2.6}$$

Has the following properties:

(i) *$\{x_t\}$ is bounded for each $t \in (0, \min\{1, 1/\tau\})$,*

(ii) *$\lim_{t\to 0}\|x_t - Tx_t\| = 0$,*

(iii) *$\{x_t\}$ defines a continuous curve from $(0, \min\{1, 1/\tau\})$ into C.*

Lemma 2.9. *Let C be a closed convex subset of a smooth Banach space E. Let D be a nonempty subset of C. Let $Q : C \to D$ be a retraction and let j, j_q be the normalized duality mapping and generalized duality mapping on E, respectively. Then the following are equivalent:*

(a) *Q is sunny and nonexpansive,*

(b) *$\|Qx - Qy\|^2 \leq \langle x - y, j(Qx - Qy)\rangle$, for all $x, y \in C$,*

(c) *$\langle x - Qx, j(y - Qx)\rangle \leq 0$, for all $x \in C$, $y \in D$,*

(d) *$\langle x - Qx, j_q(y - Qx)\rangle \leq 0$, for all $x \in C$, $y \in D$.*

Proof. From Proposition 1.1, we have a $\Leftrightarrow$ b $\Leftrightarrow$ c. We need only to prove c $\Leftrightarrow$ d.

Indeed, if $y - Qx \neq 0$, then $\langle x - Qx, j(y - Qx)\rangle \leq 0 \Leftrightarrow \langle x - Qx, j_q(y - Qx)\rangle \leq 0$, for all $x \in C$, $y \in D$ (by the fact that $j_q(x) = \|x\|^{q-2}j(x), \forall x \neq 0$).

If $y - Qx = 0$, then $\langle x - Qx, j(y - Qx)\rangle = \langle x - Qx, j_q(y - Qx)\rangle = 0$, for all $x \in C$, $y \in D$. This completes the proof. □

Lemma 2.10. *Let C be a nonempty, closed, and convex subset of a q-uniformly smooth Banach space E which admits a weakly sequentially continuous generalized duality mapping j_q from E into E^*. Let Q_C be a sunny nonexpansive retraction from E onto C. Let $V : C \to E$ be a k-Lipschitzian and*

η-strongly accretive operator with constants $k, \eta > 0$, $f : C \to E$ a L-Lipschitzian with constant $L \geq 0$, and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $0 < \mu < (q\eta/C_q k^q)^{1/(q-1)}$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - C_q \mu^{q-1} k^q / q)$. For each $t \in (0, \min\{1, 1/\tau\})$, let $\{x_t\}$ be defined by (2.6), then $\{x_t\}$ converges strongly to $x^* \in F(T)$ as $t \to 0$, which is the unique solution of the following variational inequality:

$$\langle \gamma f x^* - \mu V x^*, j_q(p - x^*) \rangle \leq 0, \quad \forall p \in F(T). \tag{2.7}$$

Proof. We first show the uniqueness of a solution of the variational inequality (2.7). Suppose both $\hat{x} \in F(T)$ and $x^* \in F(T)$ are solutions of (2.7). It follows that

$$\begin{gathered} \langle \gamma f x^* - \mu V x^*, j_q(\hat{x} - x^*) \rangle \leq 0, \\ \langle \gamma f \hat{x} - \mu V \hat{x}, j_q(x^* - \hat{x}) \rangle \leq 0. \end{gathered} \tag{2.8}$$

Adding up (2.8), we have

$$\langle (\gamma f - \mu V)\hat{x} - (\gamma f - \mu V)x^*, j_q(x^* - \hat{x}) \rangle \leq 0. \tag{2.9}$$

On the other hand, we have that

$$\begin{aligned} \langle (\gamma f - \mu V)\hat{x} - (\gamma f - \mu V)x^*, j_q(x^* - \hat{x}) \rangle &= \mu \langle V x^* - V \hat{x}, j_q(x^* - \hat{x}) \rangle - \gamma \langle f x^* - f \hat{x}, j_q(x^* - \hat{x}) \rangle \\ &\geq \mu\eta \|x^* - \hat{x}\|^q - \gamma L \|x^* - \hat{x}\|^q \\ &\geq (\mu\eta - \gamma L) \|x^* - \hat{x}\|^q \\ &\geq (\tau - \gamma L) \|x^* - \hat{x}\|^q \\ &> 0. \end{aligned} \tag{2.10}$$

It is a contradiction. Therefore, $x^* = \hat{x}$ and the uniqueness is proved. Below we use x^* to denote the unique solution of (2.7).

Next, we prove that $x_t \to x^*$ as $t \to 0$.

Since E is reflexive and $\{x_t\}$ is bounded due to Lemma 2.8 (i), there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ and some point $\tilde{x} \in C$ such that $x_{t_n} \rightharpoonup \tilde{x}$. By Lemma 2.8(ii), we have $\lim_{t \to 0} \|x_{t_n} - T x_{t_n}\| = 0$. Taken together with Lemma 2.6, we can get that $\tilde{x} \in F(T)$. Setting $y_t = t\gamma f x_t + (I - t\mu V) T x_t$, where $t \in (0, \min\{1, 1/\tau\})$, then we can rewrite (2.6) as $x_t = Q_C y_t$.

We claim $\|x_{t_n} - \tilde{x}\| \to 0$.

From Lemma 2.9, we have

$$\langle y_t - Q_C y_t, j_q(\tilde{x} - Q_C y_t) \rangle \leq 0. \tag{2.11}$$

It follows from (2.11) and Lemma 2.7 that

$$\begin{aligned}\|x_{t_n}-\tilde{x}\|^q &= \langle Q_C y_{t_n}-y_{t_n}, j_q(x_{t_n}-\tilde{x})\rangle + \langle y_{t_n}-\tilde{x}, j_q(x_{t_n}-\tilde{x})\rangle \\ &\le \langle y_{t_n}-\tilde{x}, j_q(x_{t_n}-\tilde{x})\rangle \\ &= \langle (I-t_n\mu V)Tx_{t_n}-(I-t_n\mu V)\tilde{x}, j_q(x_{t_n}-\tilde{x})\rangle + t_n\langle \gamma f x_{t_n}-\mu V\tilde{x}, j_q(x_{t_n}-\tilde{x})\rangle \\ &\le (1-t_n\tau)\|x_{t_n}-\tilde{x}\|^q + t_n\langle \gamma f x_{t_n}-\mu V\tilde{x}, j_q(x_{t_n}-\tilde{x})\rangle.\end{aligned} \tag{2.12}$$

It follows that

$$\begin{aligned}\|x_{t_n}-\tilde{x}\|^q &\le \frac{1}{\tau}\langle \gamma f x_{t_n}-\mu V\tilde{x}, j_q(x_{t_n}-\tilde{x})\rangle \\ &= \frac{1}{\tau}\left[\langle \gamma f x_{t_n}-\gamma f\tilde{x}, j_q(x_{t_n}-\tilde{x})\rangle + \langle \gamma f\tilde{x}-\mu V\tilde{x}, j_q(x_{t_n}-\tilde{x})\rangle\right] \\ &\le \frac{1}{\tau}\left[\gamma L\|x_{t_n}-\tilde{x}\|^q + \langle \gamma f\tilde{x}-\mu V\tilde{x}, j_q(x_{t_n}-\tilde{x})\rangle\right].\end{aligned} \tag{2.13}$$

Therefore, we get

$$\|x_{t_n}-\tilde{x}\|^q \le \frac{\langle \gamma f\tilde{x}-\mu V\tilde{x}, j_q(x_{t_n}-\tilde{x})\rangle}{\tau-\gamma L}. \tag{2.14}$$

Using that the duality map j_q is weakly sequentially continuous from E to E^* and noticing (2.14), we get that

$$\lim_{n\to\infty}\|x_{t_n}-\tilde{x}\| = 0. \tag{2.15}$$

We prove that $\tilde{x}$ solves the variational inequality (2.7). Since

$$x_t = Q_C y_t = Q_C y_t - y_t + t\gamma f x_t + (I-t\mu V)Tx_t, \tag{2.16}$$

we derive that

$$(\mu V-\gamma f)x_t = \frac{1}{t}(Q_C y_t - y_t) - \frac{1}{t}(I-T)x_t + \mu(Vx_t - VTx_t). \tag{2.17}$$

For all $z \in F(T)$, note that

$$\begin{aligned}\langle (I-T)x_t-(I-T)z, j_q(x_t-z)\rangle &\ge \|x_t-z\|^q - \|Tx_t-Tz\|\|x_t-z\|^{q-1} \\ &\ge \|x_t-z\|^q - \|x_t-z\|^q \\ &= 0.\end{aligned} \tag{2.18}$$

It follows from Lemma 2.9 and (2.18) that

$$
\begin{aligned}
\langle(\mu V-\gamma f)x_t, j_q(x_t-z)\rangle &= \frac{1}{t}\langle Q_C y_t - y_t, j_q(x_t-z)\rangle - \frac{1}{t}\langle(I-T)x_t, j_q(x_t-z)\rangle \\
&\quad + \langle \mu(Vx_t - VTx_t), j_q(x_t-z)\rangle \\
&= \frac{1}{t}\langle Q_C y_t - y_t, j_q(x_t-z)\rangle - \frac{1}{t}\langle(I-T)x_t-(I-T)z, j_q(x_t-z)\rangle \\
&\quad + \langle \mu(Vx_t - VTx_t), j_q(x_t-z)\rangle \\
&\le \mu\langle Vx_t - VTx_t, j_q(x_t-z)\rangle \\
&\le \|x_t - Tx_t\| M,
\end{aligned}
\tag{2.19}
$$

where $M = \sup_{n\ge 0}\{\mu k\|x_t - z\|^{q-1}\} < \infty$.

Now replacing t in (2.19) with t_n and letting $n \to \infty$, from (2.15) and Lemma 2.8 (ii), we obtain $\langle(\mu V-\gamma f)\tilde{x}, j_q(\tilde{x}-z)\rangle \le 0$, that is, $\tilde{x} \in F(T)$ is a solution of (2.7). Hence $\tilde{x} = x^*$ by uniqueness. Therefore, $x_{t_n} \to x^*$ as $n \to \infty$. And consequently, $x_t \to x^*$ as $t \to 0$. □

Lemma 2.11. *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space E. Let the mapping $A : C \to E$ be a α-inverse-strongly accretive operator. Then the following inequality holds:*

$$
\|(I-\lambda A)x-(I-\lambda A)y\|^q \le \|x-y\|^q - \lambda\left(q\alpha - C_q\lambda^{q-1}\right)\|Ax-Ay\|^q. \tag{2.20}
$$

In particular, if $0<\lambda\le (q\alpha/C_q)^{1/(q-1)}$, then $I-\lambda A$ is nonexpansive.

Proof. Indeed, for all $x, y \in C$, it follows from Lemma 2.4 that

$$
\begin{aligned}
&\|(I-\lambda A)x-(I-\lambda A)y\|^q \\
&\quad = \|(x-y)-\lambda(Ax-Ay)\|^q \\
&\quad \le \|x-y\|^q - q\lambda\langle Ax-Ay, j_q(x-y)\rangle + C_q\lambda^q\|Ax-Ay\|^q \\
&\quad \le \|x-y\|^q - q\alpha\lambda\|Ax-Ay\|^q + C_q\lambda^q\|Ax-Ay\|^q \\
&\quad \le \|x-y\|^q - \lambda\left(q\alpha - C_q\lambda^{q-1}\right)\|Ax-Ay\|^q.
\end{aligned}
\tag{2.21}
$$

It is clear that if $0<\lambda\le (q\alpha/C_q)^{1/(q-1)}$, then $I-\lambda A$ is nonexpansive. This completes the proof. □

Lemma 2.12. *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space E. Suppose $M_1, M_2 : C \to 2^E$ are two m-accretive mappings and ρ_1, ρ_2 are two arbitrary positive* *constants. Let $A, B : C \to E$ be α-inverse strongly accretive and β-inverse strongly accretive, respectively. Let $G : C \to C$ be a mapping defined by*

$$G(x) = J_{M_1,\rho_1}\left[J_{M_2,\rho_2}(x-\rho_2 Bx) - \rho_1 A J_{M_2,\rho_2}(x-\rho_2 Bx)\right], \quad \forall x \in C. \tag{2.22}$$

If $0 < \rho_1 \leq (q\alpha/C_q)^{1/(q-1)}$ and $0 < \rho_2 \leq (q\beta/C_q)^{1/(q-1)}$, then $G : C \to C$ is nonexpansive.

Proof. For all $x, y \in C$, by Lemma 2.11, we have

$$\begin{aligned}
\|Gx - Gy\| &= \big\| J_{M_1,\rho_1}\left[J_{M_2,\rho_2}(x-\rho_2 Bx) - \rho_1 A J_{M_2,\rho_2}(x-\rho_2 Bx)\right] \\
&\qquad - J_{M_1,\rho_1}\left[J_{M_2,\rho_2}(y-\rho_2 By) - \rho_1 A J_{M_2,\rho_2}(y-\rho_2 By)\right]\big\| \\
&\leq \big\| \left[J_{M_2,\rho_2}(x-\rho_2 Bx) - \rho_1 A J_{M_2,\rho_2}(x-\rho_2 Bx)\right] \\
&\qquad - \left[J_{M_2,\rho_2}(y-\rho_2 By) - \rho_1 A J_{M_2,\rho_2}(y-\rho_2 By)\right]\big\| \\
&\leq \big\| (I-\rho_1 A) J_{M_2,\rho_2}(x-\rho_2 Bx) - (I-\rho_1 A) J_{M_2,\rho_2}(y-\rho_2 By)\big\| \\
&\leq \big\| J_{M_2,\rho_2}(x-\rho_2 Bx) - J_{M_2,\rho_2}(y-\rho_2 By)\big\| \\
&\leq \big\| (x-\rho_2 Bx) - (y-\rho_2 By)\big\| \\
&\leq \|x-y\|,
\end{aligned} \tag{2.23}$$

which implies that $G : C \to C$ is nonexpansive. This completes the proof. □

Lemma 2.13. *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space E. Suppose $A, B : C \to E$ are two inverse strongly accretive operators, M_1, $M_2 : C \to 2^E$ are two m-accretive mappings, and ρ_1, ρ_2 are two arbitrary positive constants. Then $(x^*, y^*) \in C \times C$ is a solution of general system* (1.15) *if and only if $x^* = Gx^*$, where G is defined by Lemma 2.12.*

Proof. Note that

$$\begin{gathered}
\theta \in x^* - y^* + \rho_1(Ay^* + M_1 x^*) \\
\theta \in y^* - x^* + \rho_2(Bx^* + M_2 y^*) \\
\Updownarrow \\
x^* = J_{M_1,\rho_1}(y^* - \rho_1 A y^*) \\
y^* = J_{M_2,\rho_2}(x^* - \rho_2 B x^*) \\
\Updownarrow \\
G(x^*) = J_{M_1,\rho_1}\left[J_{M_2,\rho_2}(x^*-\rho_2 Bx^*) - \rho_1 A J_{M_2,\rho_2}(x^*-\rho_2 Bx^*)\right] = x^*.
\end{gathered} \tag{2.24}$$

This completes the proof. □

Lemma 2.14 (see [18]). *Let E be a q-uniformly smooth Banach space and C a nonempty convex subset of E. Assume for each $i \geq 0$, $T_i : C \to E$ is a λ_i-strict pseudocontraction with $\lambda_i \in (0,1)$. Assume* $\inf\{\lambda_i : i \geq 1\} = \lambda > 0$ *and $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{\xi_i\}_{i=1}^{\infty}$ be a positive sequence such that $\sum_{i=1}^{\infty} \xi_i = 1$, then $\sum_{i=1}^{\infty} \xi_i T_i : C \to E$ is a λ-strict pseudocontraction and $F(\sum_{i=1}^{\infty} \xi_i T_i) = F$.*

Remark 2.15. Under the assumptions of Lemma 2.14, if for each $i \geq 1$ the mapping $T_i : C \to E$ is replaced by $T_i : C \to C$, respectively, where C is a nonempty closed convex subset of E, then noticing the fact

$$\sum_{i=1}^{\infty} \xi_i T_i x = \lim_{n \to \infty} \sum_{i=1}^{n} \xi_i T_i x = \lim_{n \to \infty} \frac{1}{\sum_{i=1}^{n} \xi_i} \sum_{i=1}^{n} \xi_i T_i x \in C, \tag{2.25}$$

by Lemma 2.14, we deduce that $\sum_{i=1}^{\infty} \xi_i T_i : C \to C$ is a λ-strict pseudocontraction with $\lambda = \inf\{\lambda_i : i \geq 1\}$ and $F(\sum_{i=1}^{\infty} \xi_i T_i) = F$.

3. Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a strictly convex, and uniformly smooth Banach space E which admits a weakly sequentially continuous generalized duality mapping $j_q : E \to E^*$. Let Q_C be a sunny nonexpansive retraction from E onto C. Assume the mappings $A, B : C \to E$ are α-inverse strongly accretive and β-inverse strongly accretive, respectively. Let M_1, $M_2 : C \to 2^E$ two m-accretive mappings and ρ_1, ρ_2 be two arbitrary positive constants. Suppose $V : C \to E$ is k-Lipschitz and η-strongly accretive with constants $k, \eta > 0$, $f : C \to E$ being L-Lipschitz with constant $L \geq 0$. Let $\{S_i : C \to C\}_{i=0}^{\infty}$ be an infinite family of λ_i-strict pseudocontractions with $\{\lambda_i\} \subset (0,1)$ and* $\inf\{\lambda_i : i \geq 0\} = \lambda > 0$. *Let $0 < \mu < (q\eta/C_q k^q)^{1/(q-1)}$, $0 < \rho_1 < (q\alpha/C_q)^{1/(q-1)}$, $0 < \rho_2 < (q\beta/C_q)^{1/(q-1)}$, $0 \leq \gamma L < \tau$, $0 < \sigma \leq d$, where $\tau = \mu(\eta - C_q \mu^{q-1} k^q / q)$ and $d =$* $\min\{1, \{q\lambda/C_q\}^{1/(q-1)}\}$. *Assume $\{\xi_i\} \subset (0,1)$ and $\sum_{i=0}^{\infty} \xi_i = 1$. Define a mapping $Tx := (1-\sigma)x + \sigma \sum_{i=0}^{\infty} \xi_i S_i x$, for all $x \in C$. For arbitrarily given $x_0 \in C$ and $\delta \in (0,1)$, let $\{x_n\}$ be the sequence generated iteratively by*

$$\begin{aligned} x_{n+1} &= Q_C\left[\alpha_n \gamma f x_n + \gamma_n x_n + \left((1-\gamma_n)I - \alpha_n \mu V\right)\left(\delta T x_n + (1-\delta) y_n\right)\right], \\ y_n &= (1-\beta_n)x_n + \beta_n k_n, \\ k_n &= J_{M_1,\rho_1}(z_n - \rho_1 A z_n), \\ z_n &= J_{M_2,\rho_2}(x_n - \rho_2 B x_n), n \geq 0. \end{aligned} \tag{3.1}$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$ satisfying the following conditions:

(i) $\sum_{i=0}^{\infty} \alpha_n = \infty, \lim_{n \to \infty} \alpha_n = 0, \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(ii) $0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1, \sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,

(iii) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \lim_{n \to \infty} \beta_n = \beta > 0$.

Suppose in addition that $F := \bigcap_{i=0}^{\infty} F(S_i) \bigcap F(G) \neq \emptyset$*. Then* $\{x_n\}$ *converges strongly to some point* $x^* \in F$*, which is the unique solution of the following variational inequality:*

$$\langle \gamma f x^* - \mu V x^*, j_q(p - x^*) \rangle \leq 0, \quad \forall p \in F. \tag{3.2}$$

Proof. We divide the proof into several steps.

Step 1. First, we show that sequences $\{x_n\}$ are bounded. From $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$, there exist some $a, b \in (0,1)$ such that $\{\gamma_n\} \subset [a,b]$. We may assume, without loss of generality, that $\{\alpha_n\} \subset (0, (1-b)\min\{1, 1/\tau\})$. From Lemma 2.7, we deduce that

$$\|(1-\gamma_n)I - \alpha_n \mu V\| \leq (1-\gamma_n) - \alpha_n \tau. \tag{3.3}$$

Taking $x^* \in F$, it follows from Lemma 2.13 that

$$x^* = J_{M_1,\rho_1}\left[J_{M_2,\rho_2}(x^* - \rho_2 B x^*) - \rho_1 A J_{M_2,\rho_2}(x^* - \rho_2 B x^*)\right]. \tag{3.4}$$

Putting $y^* = J_{M_2,\rho_2}(x^* - \rho_2 B x^*)$, then we can deduce that $x^* = J_{M_1,\rho_1}(y^* - \rho_1 A y^*)$. By Lemma 2.11, we obtain

$$\begin{aligned}
\|k_n - x^*\| &= \left\|J_{M_1,\rho_1}(z_n - \rho_1 A z_n) - J_{M_1,\rho_1}(y^* - \rho_1 A y^*)\right\| \\
&\leq \left\|(I - \rho_1 A)z_n - (I - \rho_1 A)y^*\right\| \\
&\leq \left\|z_n - y^*\right\| \\
&= \left\|J_{M_2,\rho_2}(x_n - \rho_2 B x_n) - J_{M_2,\rho_2}(x^* - \rho_2 B x^*)\right\| \\
&\leq \left\|(I - \rho_2 B)x_n - (I - \rho_2 B)x^*\right\| \\
&\leq \|x_n - x^*\|.
\end{aligned} \tag{3.5}$$

It follows from (3.5) that

$$\begin{aligned}
\left\|y_n - x^*\right\| &= \left\|\left[(1-\beta_n)x_n + \beta_n k_n\right] - x^*\right\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|k_n - x^*\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^*\| \\
&\leq \|x_n - x^*\|.
\end{aligned} \tag{3.6}$$

In view of Remark 2.15, let $S : C \to C$ be the mapping defined by $Sx = \sum_{i=0}^{\infty} \xi_i S_i x$ for all $x \in C$, then we can deduce that $S : C \to C$ is a λ-strict pseudocontraction and $F(S) = \bigcap_{i=0}^{\infty} F(S_i)$. By virtue of Lemma 2.5 and $0 < \sigma \leq d$, where $d = \min\{1, \{q\lambda/C_q\}^{1/(q-1)}\}$, we can get that

$T : C \to C$ is nonexpansive and $F(T) = F(S) = \bigcap_{i=0}^{\infty} F(S_i)$. Putting $l_n = \delta T x_n + (1-\delta) y_n$, it follows that

$$\begin{aligned}
\|l_n - x^*\| &= \left\|\delta T x_n + (1-\delta) y_n - x^*\right\| \\
&\leq \delta \|T x_n - x^*\| + (1-\delta)\left\|y_n - x^*\right\| \\
&\leq \delta \|T x_n - T x^*\| + (1-\delta)\left\|y_n - x^*\right\| \\
&\leq \delta \|x_n - x^*\| + (1-\delta)\|x_n - x^*\| \\
&= \|x_n - x^*\|.
\end{aligned} \tag{3.7}$$

It follows from (3.7) that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\|Q_C\left[\alpha_n \gamma f x_n + \gamma_n x_n + \left((1-\gamma_n) I - \alpha_n \mu V\right) l_n\right] - x^*\right\| \\
&\leq \left\|\alpha_n \gamma f x_n + \gamma_n x_n + \left[(1-\gamma_n) I - \alpha_n \mu V\right] l_n - x^*\right\| \\
&= \left\|\left[(1-\gamma_n) I - \alpha_n \mu V\right](l_n - x^*) + \alpha_n(\gamma f x_n - \mu V x^*) + \gamma_n(x_n - x^*)\right\| \\
&\leq (1-\gamma_n - \alpha_n \tau)\|l_n - x^*\| + \alpha_n\left\|\gamma f x_n - \mu V x^*\right\| + \gamma_n\|x_n - x^*\| \\
&\leq (1-\gamma_n - \alpha_n \tau)\|l_n - x^*\| + \alpha_n \gamma\left\|f x_n - f x^*\right\| + \alpha_n\left\|\gamma f x^* - \mu V x^*\right\| + \gamma_n\|x_n - x^*\| \\
&\leq (1-\gamma_n - \alpha_n \tau)\|x_n - x^*\| + \alpha_n \gamma L\|x_n - x^*\| + \alpha_n\left\|\gamma f x^* - \mu V x^*\right\| + \gamma_n\|x_n - x^*\| \\
&\leq \left[1 - \alpha_n(\tau - \gamma L)\right]\|x_n - x^*\| + \alpha_n\left\|\gamma f x^* - \mu V x^*\right\| \\
&\leq \max\left\{\|x_0 - x^*\|, \frac{\left\|\gamma f x^* - \mu V x^*\right\|}{\tau - \gamma L}\right\}.
\end{aligned} \tag{3.8}$$

Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{k_n\}$, $\{z_n\}$, and $\{l_n\}$.

Step 2. In this part, we will claim that $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$.

We observe that

$$\begin{aligned}
\|k_{n+1} - k_n\| &= \left\|J_{M_1,\rho_1}(z_{n+1} - \rho_1 A z_{n+1}) - J_{M_1,\rho_1}(z_n - \rho_1 A z_n)\right\| \\
&\leq \left\|(I - \rho_1 A) z_{n+1} - (I - \rho_1 A) z_n\right\| \\
&\leq \|z_{n+1} - z_n\| \\
&= \left\|J_{M_2,\rho_2}(x_{n+1} - \rho_2 B x_{n+1}) - J_{M_2,\rho_2}(x_n - \rho_2 B x_n)\right\| \\
&\leq \left\|(I - \rho_2 B) x_{n+1} - (I - \rho_2 B) x_n\right\| \\
&\leq \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.9}$$

It follows from (3.9) that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|[(1-\beta_{n+1})x_{n+1}+\beta_{n+1}k_{n+1}]-[(1-\beta_n)x_n+\beta_n k_n]\|\\ &= \|(1-\beta_{n+1})(x_{n+1}-x_n)+\beta_{n+1}(k_{n+1}-k_n)+(\beta_{n+1}-\beta_n)(k_n-x_n)\|\\ &\le (1-\beta_{n+1})\|x_{n+1}-x_n\|+\beta_{n+1}\|k_{n+1}-k_n\|+|\beta_{n+1}-\beta_n|\|k_n-x_n\|\\ &\le (1-\beta_{n+1})\|x_{n+1}-x_n\|+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|k_n-x_n\|\\ &\le \|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|k_n-x_n\|.\end{aligned} \tag{3.10}$$

Again from (3.1), we have

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\le \|[\alpha_{n+1}\gamma f x_{n+1}+\gamma_{n+1}x_{n+1}+((1-\gamma_{n+1})I-\alpha_{n+1}\mu V)l_{n+1}]\\ &\quad -[\alpha_n\gamma f x_n+\gamma_n x_n+((1-\gamma_n)I-\alpha_n\mu V)l_n]\|\\ &\le \alpha_{n+1}\gamma\|f x_{n+1}-f x_n\|+\gamma_{n+1}\|x_{n+1}-x_n\|+\|((1-\gamma_{n+1})I-\alpha_{n+1}\mu V)(l_{n+1}-l_n)\|\\ &\quad +|\alpha_{n+1}-\alpha_n|\gamma\|f x_n\|\\ &\quad +|\alpha_{n+1}-\alpha_n|\mu\|V l_n\|+|\gamma_{n+1}-\gamma_n|\|l_n-x_n\|\\ &\le \alpha_{n+1}\gamma L\|x_{n+1}-x_n\|+\gamma_{n+1}\|x_{n+1}-x_n\|\\ &\quad +[(1-\gamma_{n+1})-\alpha_{n+1}\tau]\|l_{n+1}-l_n\|+|\alpha_{n+1}-\alpha_n|\gamma\|f x_n\|\\ &\quad +|\alpha_{n+1}-\alpha_n|\mu\|V l_n\|+|\gamma_{n+1}-\gamma_n|\|l_n-x_n\|.\end{aligned} \tag{3.11}$$

It follows from (3.10) that

$$\begin{aligned}\|l_{n+1}-l_n\| &= \|\delta T x_{n+1}+(1-\delta)y_{n+1}-\delta T x_n-(1-\delta)y_n\|\\ &\le \delta\|T x_{n+1}-T x_n\|+(1-\delta)\|y_{n+1}-y_n\|\\ &\le \delta\|x_{n+1}-x_n\|+(1-\delta)\|y_{n+1}-y_n\|\\ &\le \|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|k_n-x_n\|.\end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.11), we have

$$\begin{aligned}&\|x_{n+2}-x_{n+1}\|\\ &\quad\le [(1-\gamma_{n+1})-\alpha_{n+1}\tau](\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|k_n-x_n\|)+|\alpha_{n+1}-\alpha_n|\gamma\|f x_n\|\\ &\qquad +\alpha_{n+1}\gamma L\|x_{n+1}-x_n\|+\gamma_{n+1}\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\mu\|V l_n\|+|\gamma_{n+1}-\gamma_n|\|l_n-x_n\|\\ &\quad\le [1-\alpha_{n+1}(\tau-\gamma L)]\|x_{n+1}-x_n\|+(|\alpha_{n+1}-\alpha_n|+|\gamma_{n+1}-\gamma_n|+|\beta_{n+1}-\beta_n|)M',\end{aligned} \tag{3.13}$$

where $M' = \sup_{n\geq 0}\{\mu\|Vl_n\| + \gamma\|fx_n\|, \|l_n - x_n\|, \|k_n - x_n\|\} < \infty$. From (i), (ii), (iii), (3.13), and Lemma 2.2, we deduce that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.14}$$

We observe that

$$\begin{aligned}
\|l_n - x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - l_n\| \\
&= \|x_{n+1} - x_n\| + \left\|Q_C\left[\alpha_n\gamma f x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu V)l_n\right] - l_n\right\| \\
&\leq \|x_{n+1} - x_n\| + \left\|\left[\alpha_n\gamma f x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu V)l_n\right] - l_n\right\| \\
&\leq \|x_{n+1} - x_n\| + \left\|\alpha_n(\gamma f x_n - \mu V l_n) + \gamma_n(x_n - l_n)\right\| \\
&\leq \|x_{n+1} - x_n\| + \alpha_n\left\|\gamma f x_n - \mu V l_n\right\| + \gamma_n\|x_n - l_n\|,
\end{aligned} \tag{3.15}$$

which implies that

$$\|l_n - x_n\| \leq \frac{1}{1-\gamma_n}\left(\|x_{n+1} - x_n\| + \alpha_n\left\|\gamma f x_n - \mu V l_n\right\|\right). \tag{3.16}$$

Noticing conditions (i) and (ii) and (3.14), we have

$$\lim_{n\to\infty}\|l_n - x_n\| = 0. \tag{3.17}$$

Let

$$Wx = \delta Tx + (1-\delta)\left[(1-\beta)x + \beta J_{M_1,\rho_1}(I-\rho_1 A)J_{M_2,\rho_2}(I-\rho_2 B)x\right], \quad \forall x \in C. \tag{3.18}$$

In view of Lemma 2.1, we see that $W : C \to C$ is nonexpansive such that

$$F(W) = F(T)\bigcap F\left(J_{M_1,\rho_1}(I-\rho_1 A)J_{M_2,\rho_2}(I-\rho_2 B)\right) = \bigcap_{i=0}^{\infty} S_i \bigcap F(G) = F. \tag{3.19}$$

Noticing that

$$\begin{aligned}
Wx_n - l_n = {} & \delta T x_n + (1-\delta)\left[(1-\beta)x_n + \beta J_{M_1,\rho_1}(I-\rho_1 A)J_{M_2,\rho_2}(I-\rho_2 B)x_n\right] - \delta T x_n \\
& - (1-\delta)\left[(1-\beta_n)x_n + \beta_n J_{M_1,\rho_1}(I-\rho_1 A)J_{M_2,\rho_2}(I-\rho_2 B)x_n\right] \\
= {} & (1-\delta)(\beta - \beta_n)\left(J_{M_1,\rho_1}(I-\rho_1 A)J_{M_2,\rho_2}(I-\rho_2 B)x_n - x_n\right),
\end{aligned} \tag{3.20}$$

one has

$$\begin{aligned}\|Wx_n - x_n\| &\le \|Wx_n - l_n\| + \|l_n - x_n\| \\ &\le (1-\delta)\big|\beta - \beta_n\big|\big\|J_{M_1,\rho_1}(I-\rho_1 A)J_{M_2,\rho_2}(I-\rho_2 B)x_n - x_n\big\| + \|l_n - x_n\|.\end{aligned} \tag{3.21}$$

In view of (3.17), (iii) and (3.21), we deduce that

$$\|Wx_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.22}$$

We define $x_t = Q_C[t\gamma f x_t + (I - t\mu V)Wx_t]$, then it follows from Lemma 2.10 that $\{x_t\}$ converges strongly to some point $x^* \in F(W) = F$, which is the unique solution of the variational inequality (3.2).

Step 3. We show that

$$\limsup_{n\to\infty}\langle \gamma f x^* - \mu V x^*, j_q(x_n - x^*)\rangle \le 0, \tag{3.23}$$

where x^* is the solution of the variational inequality of (3.2). To show this, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\big\langle \gamma f x^* - \mu V x^*, j_q(x_n - x^*)\big\rangle = \lim_{i\to\infty}\langle \gamma f x^* - \mu V x^*, j_q(x_{n_i} - x^*)\rangle. \tag{3.24}$$

Without loss of generality, we may further assume that $x_{n_i} \rightharpoonup z$ for some point $z \in C$ due to reflexivity of the Banach space E and boundness of $\{x_n\}$, it follows from (3.22) and Lemma 2.6 that $z \in F(W) = F$. Since the Banach space E has a weakly sequentially continuous generalized duality mapping $j_p : E \to E^*$, we obtain that

$$\begin{aligned}\limsup_{n\to\infty}\big\langle \gamma f x^* - \mu V x^*, j_q(x_n - x^*)\big\rangle &= \lim_{i\to\infty}\big\langle \gamma f x^* - \mu V x^*, j_q(x_{n_i} - x^*)\big\rangle \\ &= \langle \gamma f x^* - \mu V x^*, j_q(z - x^*)\rangle \\ &\le 0.\end{aligned} \tag{3.25}$$

Step 4. We prove that $\lim_{n\to\infty}\|x_n - x^*\|$. Setting $h_n = \alpha_n \gamma f x_n + \gamma_n x_n + [(1-\gamma_n)I - \alpha_n \mu V] l_n$, for all $n \geq 0$. It follows from (3.1) that $x_{n+1} = Q_C h_n$. In view of Lemmas 2.3, 2.7, and 2.9, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &= \langle Q_C h_n - h_n, j_q(x_{n+1} - x^*)\rangle + \langle h_n - x^*, j_q(x_{n+1} - x^*)\rangle \\
&\leq \langle h_n - x^*, j_q(x_{n+1} - x^*)\rangle \\
&= \langle [(1-\gamma_n)I - \alpha_n \mu V](l_n - x^*), j_q(x_{n+1} - x^*)\rangle + \gamma_n \langle x_n - x^*, j_q(x_{n+1} - x^*)\rangle \\
&\quad + \alpha_n \langle \gamma f x_n - \mu V x^*, j_q(x_{n+1} - x^*)\rangle \\
&\leq [(1-\gamma_n) - \alpha_n \tau] \|l_n - x^*\| \|x_{n+1} - x^*\|^{q-1} + \gamma_n \|x_n - x^*\| \|x_{n+1} - x^*\|^{q-1} \\
&\quad + \alpha_n \langle \gamma f x_n - \gamma f x^*, j_q(x_{n+1} - x^*)\rangle + \alpha_n \langle \gamma f x^* - \mu V x^*, j_q(x_{n+1} - x^*)\rangle \\
&\leq [(1-\gamma_n) - \alpha_n \tau] \|x_n - x^*\| \|x_{n+1} - x^*\|^{q-1} + \gamma_n \|x_n - x^*\| \|x_{n+1} - x^*\|^{q-1} \\
&\quad + \alpha_n \gamma L \|x_n - x^*\| \|x_{n+1} - x^*\|^{q-1} + \alpha_n \langle \gamma f x^* - \mu V x^*, j_q(x_{n+1} - x^*)\rangle \\
&\leq [1 - \alpha_n(\tau - \gamma L)] \|x_n - x^*\| \|x_{n+1} - x^*\|^{q-1} + \alpha_n \langle \gamma f x^* - \mu V x^*, j_q(x_{n+1} - x^*)\rangle \\
&\leq [1 - \alpha_n(\tau - \gamma L)] \left[\frac{1}{q}\|x_n - x^*\|^q + \frac{q-1}{q}\|x_{n+1} - x^*\|^q\right] \\
&\quad + \alpha_n \langle \gamma f x^* - \mu V x^*, j_q(x_{n+1} - x^*)\rangle
\end{aligned} \tag{3.26}$$

which implies

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &\leq \frac{1 - \alpha_n(\tau - \gamma L)}{1 + (q-1)(\tau - \gamma L)\alpha_n} \|x_n - x^*\|^q + \frac{q\alpha_n}{1 + (q-1)(\tau - \gamma L)\alpha_n} \\
&\quad \times \langle \gamma f x^* - \mu V x^*, j_q(x_{n+1} - x^*)\rangle \\
&\leq [1 - \alpha_n(\tau - \gamma L)] \|x_n - x^*\|^q \\
&\qquad + \frac{q\alpha_n}{1 + (q-1)(\tau - \gamma L)\alpha_n} \langle \gamma f x^* - \mu V x^*, j_q(x_{n+1} - x^*)\rangle.
\end{aligned} \tag{3.27}$$

Put $a_n = \alpha_n(\tau - \gamma L)$ and $c_n = q\langle \gamma f x^* - \mu V x^*, j_q(x_{n+1} - x^*)\rangle / [1 + (q-1)(\tau - \gamma L)\alpha_n](\tau - \gamma L)$. Apply Lemma 2.2 to (3.27) to obtain $x_n \to x^* \in F$ as $n \to \infty$. This completes the proof. □

Remark 3.2. Compared with the known results in the literature, our results are very different from those in the following aspects.

(i) The results in this paper improve and extend corresponding results in [6–13]. Especially, our result extends their results from 2-uniformly smooth Banach space or Hilbert space to more general q-uniformly smooth Banach space.

(ii) Our Theorem 3.1 extends one nonexpansive mapping in [6, Theorem 2.1], one λ-strict pseudocontraction in [8, Theorem 3.1], and an infinite family of nonexpansive mappings in [10, Theorem 3.1] to an infinite family of λ_i-strict pseudocontractions.

And our Theorem 3.1 gets a common element of the common fixed-point set of an infinite family of λ_i-strict pseudocontractions and the solution set of the general system of variational inclusions for two inverse strongly accretive mappings in a q-uniformly smooth Banach space.

(iii) We by $f(x_n)$ replace the u which is a fixed element in iterative scheme (1.16), where f is a L-Lipschitzian. And we also add a Lipschitz and strong accretive operator V in our scheme (3.1). In particular, whenever $C = E$, $f = u$, $V = I$, $\{T_n\}_{n=0}^{\infty} = \{T\}$ and $q = 2$, our scheme (3.1) reduces to (1.16).

(iv) It is worth noting that the Banach space E does not have to be uniformly convex in our Theorem 3.1. However, it is very necessary in Theorem 3.1 of Qin et al. [8] and many other literature.

Corollary 3.3. *Let C be a nonempty closed convex subset of a strictly convex, and 2-reflexive E which admits a weakly sequentially continuous normalized duality mapping $j : E \to E^*$. Let Q_C be a sunny nonexpansive retraction from E onto C. Assume the mappings $A, B : C \to E$ are α-inverse strongly accretive and β-inverse strongly accretive, respectively. Let $M_1, M_2 : C \to 2^E$ be two m-accretive operators and ρ_1, ρ_2 two arbitrary positive constants. Suppose $V : C \to E$ is a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$, $f : C \to E$ being a L-Lipschitzian with constant $L \geq 0$. Let $0 < \mu < \eta/K^2k^2$, $0 < \rho_1 < \alpha/K^2$, $0 < \rho_2 < \beta/K^2$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - K^2\mu k^2)$. Let $T : C \to C$ be a nonexpansive with $F = F(T) \bigcap F(G) \neq \emptyset$. For arbitrarily given $\delta \in (0,1)$ and $x_0 \in C$, let $\{x_n\}$ be the sequence generated iteratively by*

$$\begin{aligned} x_{n+1} &= Q_C\left[\alpha_n \gamma f x_n + \gamma_n x_n + \left((1-\gamma_n)I - \alpha_n \mu V\right)\left(\delta T x_n + (1-\delta) y_n\right)\right], \\ y_n &= (1-\beta_n) x_n + \beta_n k_n, \\ k_n &= J_{M_1,\rho_1}(z_n - \rho_1 A z_n), \\ z_n &= J_{M_2,\rho_2}(x_n - \rho_2 B x_n), \quad n \geq 0. \end{aligned} \tag{3.28}$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$ satisfying the following conditions:

(i) $\sum_{i=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(ii) $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$, $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,

(iii) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\lim_{n\to\infty}\beta_n = \beta > 0$.

Then $\{x_n\}$ converges strongly to $x^* \in F$, which is the unique solution of the following variational inequality:

$$\langle \gamma f x^* - \mu V x^*, j(p - x^*) \rangle \leq 0, \quad \forall p \in F. \tag{3.29}$$

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