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# Oscillation of half-linear differential equations with asymptotically almost periodic coefficients

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## Abstract

We investigate second-order half-linear differential equations with asymptotically almost periodic coefficients. For these equations, we explicitly find an oscillation constant. If the coefficients are replaced by constants, our main result (concerning the conditional oscillation) reduces to the classical one. We also mention examples and concluding remarks.

**MSC:** 34C10; 34C15

**Keywords:** half-linear equations; oscillation theory; almost periodicity; asymptotic almost periodicity; conditional oscillation; oscillation constant

## 1 Introduction

This paper is devoted to the study of the half-linear differential equation

$$[r(t)\Phi(x')] + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1} \operatorname{sgn} x, \quad (1.1)$$

where  $r, c$  are continuous functions,  $r$  is positive, and  $p > 1$ . An equation of this form appeared for the first time in [1]. Nevertheless, as the basic pioneering papers in the field of half-linear differential equations, we mention [2, 3]. During the last decades, these equations have been widely studied in the literature. A detailed description and a comprehensive literature overview concerning the topic can be found in [4] (see also [5, Chapter 3]).

The name *half-linear equations* was introduced in [6]. This term is motivated by the fact that the solution space of these equations is homogeneous (likewise in the linear case), but it is not additive. There are several differences between linear equations and half-linear equations. Especially, some tools widely used in the theory of linear equations are not available for half-linear equations (e.g., see [7] for the Wronskian identity and [8] for the Fredholm alternative). In fact, these differences are caused, more or less, by the lack of the additivity. On the other hand, many results from the theory of linear equations are extendable to their half-linear counterparts. Since the linear Sturmian theory extends verbatim to the half-linear case (for details, we refer to [4, Section 1.2]), we can classify Eq. (1.1) as oscillatory or non-oscillatory. We say that Eq. (1.1) is *oscillatory* if zeros of any solution tend to infinity. Otherwise, Eq. (1.1) is called *non-oscillatory*.

In this paper, we analyze a topic from the theory of half-linear equations and, at the same time, from the theory of (asymptotically) almost periodic functions. The importance of

the investigation of half-linear equations lies among others in the fact that various phenomena in physics and chemistry are studied through half-linear equations and through partial differential equations with  $p$ -Laplacian (half-linear equations can be regarded as equations with one-dimensional  $p$ -Laplacian). We refer to [9] and the last section of this paper (with references given therein). Similarly, many phenomena in nature have an oscillatory character and their models lead to the research of asymptotically almost periodic functions. There exist many remarkable books concerning (asymptotically) almost periodic differential equations and their applications. See, *e.g.*, [10–12].

Actually, we are interested in the conditional oscillation of half-linear differential equations with asymptotically almost periodic coefficients. We say that the equation

$$[r(t)\Phi(x')] + \gamma c(t)\Phi(x) = 0 \quad (1.2)$$

with positive coefficients is *conditionally oscillatory* if there exists a constant  $\Gamma > 0$  such that Eq. (1.2) is oscillatory for all  $\gamma > \Gamma$  and non-oscillatory for  $\gamma < \Gamma$ . The constant  $\Gamma$  is called an *oscillation constant* (more precisely, *oscillation constant of  $c$  with respect to  $r$* ). For results about the conditional oscillation of half-linear equations, we refer to [13, 14] and to [4, Section 5.4.2] (see also [15]). The conditional oscillation of linear equations is studied, *e.g.*, in [16, 17].

Our goal is to find the explicit oscillation constant for Eq. (1.2), where  $c(t) = t^{-p}s(t)$  and  $r, s$  are asymptotically almost periodic functions. We point out that the main motivation of our research comes from [14, 17, 18]. In [17, 18], the oscillation constant is found for linear equations with periodic coefficients. Using the notion of the principal solution, the main result of [17] is generalized in [14], where periodic half-linear equations are considered. For the discrete counterpart, we refer to [19], where the problem is solved for linear difference equations with almost periodic coefficients. It means that the presented result is stronger than the known one in the discrete case because we consider *asymptotically almost periodic half-linear* equations.

We consider the asymptotically almost periodic functions  $r, s$  in Eq. (1.2), because the known results do not cover two important cases - the almost periodic case, which is the basic generalization of the pure periodicity in applications, and perturbations vanishing at infinity. In this sense, our result is superior to the results of [14, 17, 18].

Besides already given references, let us mention a short literature overview connected to the above announced result. The conditional oscillation of linear equations is treated in [16]. We also refer to [20–23] which generalize [16]. An oscillation criterion for second-order half-linear equations is proved, *e.g.*, in [24]. The oscillation properties of second-order linear dynamic equations are studied in [25] and the oscillation and non-oscillation behavior of second-order half-linear dynamic equations is investigated in [26–28].

The oscillation of solutions of almost periodic linear equations is discussed in [29–31]. In [32], half-linear equations with (the Besicovitch) almost periodic coefficients are considered and an oscillation theorem is obtained. In [33], a necessary and sufficient condition for a second-order equation with the Besicovitch almost periodic coefficient to be oscillatory is given. Oscillation properties of these equations are mentioned in [5, Section 3.5] and [4, Section 5.3.3] as well. Other properties of half-linear equations with almost periodic coefficients are treated, *e.g.*, in [34].

The paper is organized as follows. The notion of asymptotic almost periodicity is recalled in the next section. Necessary background from the theory of half-linear equations

is presented in Section 3. In these two sections, we mention only basic definitions and results which we will use later. In Section 4, we prove the main result and we illustrate it by some examples. Our paper is finished by concluding remarks given in Section 5.

Henceforth, for given  $p > 1$ , the symbol  $q$  will denote the number which is conjugated with  $p$ , i.e.,  $p + q = pq$ . As usually,  $\mathbb{R}^+$  stands for all positive reals and  $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$ .

## 2 Asymptotic almost periodicity

At first, we recall the definition of almost periodicity.

**Definition 1** A continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is called *almost periodic* if for every  $\varepsilon > 0$ , there exists a number  $p(\varepsilon) > 0$  with the property that any interval of length  $p(\varepsilon)$  of the real line contains at least one point  $s$  for which

$$|f(t + s) - f(t)| < \varepsilon, \quad t \in \mathbb{R}.$$

Note that we mentioned the so-called Bohr definition of almost periodic functions and that it is possible to introduce almost periodic functions in different ways. As the basic definition of almost periodic functions in the literature, one can often find the Bochner definition which is equivalent with the Bohr one and which is embodied in the next theorem.

**Theorem 2.1** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Then  $f$  is almost periodic if and only if from any sequence of the form  $\{f(t + s_n)\}_{n \in \mathbb{N}}$ , where  $s_n$  are real numbers, one can extract a subsequence which converges uniformly with respect to  $t \in \mathbb{R}$ .

*Proof* See, e.g., [12, Theorem 1.14]. □

As a direct generalization of almost periodicity, we consider the notion of asymptotic almost periodicity.

**Definition 2** We say that a continuous function  $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}$  is *asymptotically almost periodic* if it can be represented in the form  $f(t) = f_1(t) + f_2(t)$ ,  $t \in \mathbb{R}_0^+$ , where  $f_1$  is almost periodic and  $f_2$  has the property that  $\lim_{t \rightarrow \infty} f_2(t) = 0$ .

Again, equivalent definitions of asymptotic almost periodicity (based on the Bohr and Bochner concept) are used in the literature. We formulate them in the following two theorems. For the proofs of these theorems, we refer to [12, Theorem 9.3, parts 2), 4)] (see also [10, Theorem 1.5.2]).

**Theorem 2.2** A continuous function  $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}$  is asymptotically almost periodic if and only if for every  $\varepsilon > 0$ , there exist numbers  $p(\varepsilon) > 0$  and  $P(\varepsilon) > 0$  with the property that any interval  $I \subset \mathbb{R}$  of length  $p(\varepsilon)$  contains at least one point  $s$  for which

$$|f(t + s) - f(t)| < \varepsilon, \quad t, t + s \geq P(\varepsilon).$$

**Theorem 2.3** A continuous function  $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}$  is asymptotically almost periodic if and only if from any sequence of the form  $\{f(t + s_n)\}_{n \in \mathbb{N}}$ , where  $s_n$  are positive real numbers

satisfying  $\lim_{n \rightarrow \infty} s_n = \infty$ , one can extract a subsequence which converges uniformly with respect to  $t \in \mathbb{R}_0^+$ .

Now we mention two well-known common properties of almost periodic and asymptotically almost periodic functions, which we will need later. To prove them, it suffices to consider Theorems 2.1 and 2.3 (or consider, e.g., the proof of [12, Theorem 1.9]).

**Corollary 2.1** *If  $f$  is an (asymptotically) almost periodic function and  $F : \mathbb{R} \rightarrow \mathbb{R}$  is uniformly continuous on the range of  $f$ , then the function  $F \circ f$  is (asymptotically) almost periodic as well.*

**Corollary 2.2** *The sum and the product of two (asymptotically) almost periodic functions are (asymptotically) almost periodic.*

Since asymptotically almost periodic functions are evidently bounded, from Corollary 2.1, we also obtain the following.

**Corollary 2.3** *If  $f$  is an asymptotically almost periodic function satisfying*

$$\inf\{f(t); t \in \mathbb{R}_0^+\} > 0,$$

*then the function  $g : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$ , defined by the formula  $g(t) = f^\alpha(t)$ , is asymptotically almost periodic for arbitrarily given  $\alpha \in \mathbb{R}$ .*

It remains to recall the notion of the mean value for asymptotically almost periodic functions.

**Theorem 2.4** *If  $f$  is an asymptotically almost periodic function, then the limit*

$$M(f) := \lim_{t \rightarrow \infty} \frac{1}{t} \int_a^{a+t} f(s) ds \tag{2.1}$$

*is finite and exists uniformly with respect to  $a \in \mathbb{R}_0^+$ .*

*Proof* The theorem follows, e.g., from [11, Theorem 3.4] (see also [11, Remark 3.15]).  $\square$

**Definition 3** Let  $f$  be asymptotically almost periodic. The number  $M(f)$  introduced in (2.1) is called the *mean value* of  $f$ .

**Remark 1** For any positive asymptotically almost periodic function  $f$  with the property that  $\limsup_{t \rightarrow \infty} f(t) > 0$ , we have  $M(f) > 0$ . To verify this fact, it suffices to use Theorem 2.2.

### 3 Half-linear differential equations

To prove the main result, we use the concept of the Riccati equation for half-linear equations. More precisely, applying the transformation

$$w(t) = r(t)\Phi\left(\frac{x'(t)}{x(t)}\right), \quad \Phi(x) = |x|^{p-1} \operatorname{sgn} x,$$

to Eq. (1.1), we obtain the *half-linear Riccati equation*

$$w'(t) + c(t) + (p - 1)r^{1-q}(t)|w(t)|^q = 0 \tag{3.1}$$

whenever  $x(t) \neq 0$ . We remark that the connection between Eq. (1.1) and Eq. (3.1) is described in the below given Theorem 3.1.

We also need the following two theorems. The first one is a well-known consequence of the half-linear Roundabout theorem which is proved, *e.g.*, in [4, Theorem 1.2.2] (or see directly [4, Theorem 2.2.1]). For the second one, we can refer to [4, Theorem 2.2.3].

**Theorem 3.1** *Equation (1.1) is non-oscillatory if and only if there exists a function  $w$  which solves Eq. (3.1) on some interval  $[T, \infty)$ .*

**Theorem 3.2** *Suppose that*

$$\int^\infty r^{1-q}(t) dt = \infty \tag{3.2}$$

and that Eq. (1.1) is non-oscillatory. Then the lower limit

$$\liminf_{t \rightarrow \infty} \frac{\int^t r^{1-q}(\varphi)(\int^\varphi c(\tau) d\tau) d\varphi}{\int^t r^{1-q}(\varphi) d\varphi} > -\infty \tag{3.3}$$

if and only if

$$\int^\infty r^{1-q}(t)|w(t)|^q dt < \infty \tag{3.4}$$

for every solution  $w$  of Eq. (3.1).

We focus our attention to Eq. (1.1) in the form

$$[r(t)\Phi(x')] + \gamma \frac{s(t)}{t^p} \Phi(x) = 0, \tag{3.5}$$

where  $\gamma \in \mathbb{R}^+$  and  $r, s$  are positive asymptotically almost periodic functions. The Riccati equation associated to Eq. (3.5) has the form

$$w'(t) + \gamma \frac{s(t)}{t^p} + (p - 1)r^{1-q}(t)|w(t)|^q = 0. \tag{3.6}$$

For  $\zeta(t) = -w(t)t^{p-1}$ , we express

$$\begin{aligned} \zeta'(t) &= [-w(t)t^{p-1}]' = -[w'(t)t^{p-1} + (p - 1)w(t)t^{p-2}] \\ &= \frac{1}{t} \left[ (p - 1)\zeta(t) + \gamma s(t) + (p - 1) \frac{|\zeta(t)|^q}{r^{q-1}(t)} \right], \end{aligned}$$

*i.e.*, we obtain the so-called *adapted Riccati equation*

$$\zeta'(t) = \frac{1}{t} \left[ (p - 1)\zeta(t) + \gamma s(t) + (p - 1) \frac{|\zeta(t)|^q}{r^{q-1}(t)} \right]. \tag{3.7}$$

In fact, we will use only partial cases of Theorems 3.1 and 3.2. For reader's convenience, we mention the corresponding corollaries for Eq. (3.5) and Eq. (3.7).

**Corollary 3.1** *Let Eq. (3.5) be non-oscillatory. There exists a negative solution  $\zeta$  of the adapted Riccati equation (3.7) on some interval  $[t_0, \infty)$ .*

*Proof* We apply Theorem 3.2, where (3.2) is satisfied (consider the boundedness of  $r$ ). Inequality (3.3) is valid, because  $r, s$  are positive functions and  $\gamma \in \mathbb{R}^+$ . Thus, (3.4) holds for all solutions  $w$  of Eq. (3.6). Let us consider an arbitrary solution  $w$  of Eq. (3.6) defined on  $[t_0, \infty)$  satisfying  $w(t_0) > 0$  (see Theorem 3.1). We know that  $w$  is decreasing which follows immediately from (3.6). By combination of (3.2) and (3.4), we obtain  $w(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Especially,  $w(t) > 0, t \geq t_0$ . This observation implies the existence of a negative solution  $\zeta$  of Eq. (3.7) on  $[t_0, \infty)$ .  $\square$

**Corollary 3.2** *If there exists a negative function  $\zeta$  which solves the adapted Riccati equation (3.7) on some interval  $[T, \infty)$ , then Eq. (3.5) is non-oscillatory.*

*Proof* The statement of the corollary follows directly from Theorem 3.1.  $\square$

#### 4 Oscillation constant

Now we can formulate and prove the announced result.

**Theorem 4.1** *Let  $\gamma \in \mathbb{R}^+$  be given and let  $r$  and  $s$  be arbitrary positive asymptotically almost periodic functions satisfying*

$$\inf\{r(t); t \in \mathbb{R}_0^+\} > 0, \quad \limsup_{t \rightarrow \infty} s(t) > 0. \tag{4.1}$$

Let

$$\Gamma := q^{-p} [M(r^{1-q})]^{1-p} [M(s)]^{-1}. \tag{4.2}$$

Consider the equation

$$[r(t)\Phi(x')] + \gamma \frac{s(t)}{t^p} \Phi(x) = 0, \quad \Phi(x) = |x|^{p-1} \operatorname{sgn} x. \tag{4.3}$$

If  $\gamma > \Gamma$ , then Eq. (4.3) is oscillatory. If  $\gamma < \Gamma$ , then Eq. (4.3) is non-oscillatory.

*Proof* In the both cases of the theorem, we consider  $\gamma \neq \Gamma$ . Let  $\alpha, T \in \mathbb{R}^+$  be such that

$$\frac{1}{\alpha} \int_t^{t+\alpha} s(\tau) d\tau > \frac{M(s)}{2} > 0, \quad t \geq T, \tag{4.4}$$

$$\left| \Gamma - q^{-p} \left( \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{r^{q-1}(\tau)} d\tau \right)^{-\frac{p}{q}} \left( \frac{1}{\alpha} \int_t^{t+\alpha} s(\tau) d\tau \right)^{-1} \right| < \frac{|\Gamma - \gamma|}{2}, \quad t \geq T, \tag{4.5}$$

i.e.,

$$\left| \gamma - q^{-p} \left( \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{r^{q-1}(\tau)} d\tau \right)^{-\frac{p}{q}} \left( \frac{1}{\alpha} \int_t^{t+\alpha} s(\tau) d\tau \right)^{-1} \right| > \frac{|\gamma - \Gamma|}{2}, \quad t \geq T. \tag{4.6}$$

The existence of such numbers follows from Corollary 2.3, Theorem 2.4, and Remark 1. We denote

$$\begin{aligned} r^+ &:= \sup\{r(t); t \in \mathbb{R}_0^+\}, & s^+ &:= \sup\{s(t); t \in \mathbb{R}_0^+\}, \\ r^- &:= \inf\{r(t); t \in \mathbb{R}_0^+\} > 0. \end{aligned} \tag{4.7}$$

Let  $\gamma > \Gamma$ . By contradiction, we suppose that Eq. (4.3) is non-oscillatory. Corollary 3.1 says that the adapted Riccati equation has a negative solution  $\zeta$ , which exists on some interval  $[t_0, \infty)$ . Without loss of generality, we can assume that  $t_0 = T$ . We recall (see (3.7))

$$\zeta'(t) = \frac{1}{t} \left[ (p-1)\zeta(t) + \gamma s(t) + (p-1) \frac{|\zeta(t)|^q}{r^{q-1}(t)} \right], \quad t > t_0. \tag{4.8}$$

Especially, we know that

$$\zeta'(t) > \frac{p-1}{t} \left[ \zeta(t) + \frac{|\zeta(t)|^q}{(r^+)^{q-1}} \right], \quad t > t_0. \tag{4.9}$$

Since  $q > 1$ , from (4.9) it follows the existence of  $K > 0$  for which

$$\zeta(t) \in (-K, 0), \quad t \geq t_0. \tag{4.10}$$

Indeed,  $\zeta'(t) > 0$  if we assume that  $\zeta(t) \leq -r^+$ .

In addition, considering (4.8) and (4.10), we have

$$\int_t^{t+a} |\zeta'(\tau)| d\tau \leq \frac{p-1}{t} \int_t^{t+a} |\zeta(\tau)| + \frac{|\zeta(\tau)|^q}{(r^-)^{q-1}} d\tau + \frac{a\gamma s^+}{t} \leq \frac{Ca}{t}, \quad t \geq t_0, a > 0,$$

where

$$C := (p-1) \left[ K + \frac{K^q}{(r^-)^{q-1}} \right] + \gamma s^+. \tag{4.11}$$

Therefore,

$$|\zeta(t_2) - \zeta(t_1)| \leq \int_{t_1}^{t_2} |\zeta'(\tau)| d\tau \leq \frac{C(t_2 - t_1)}{t_1}, \quad t_2 \geq t_1 \geq t_0, \tag{4.12}$$

i.e.,

$$\begin{aligned} |\zeta(t + \sigma) - \zeta(t + \tau)| &\leq \frac{C|t + \sigma - t - \tau|}{t} \\ &= \frac{C|\tau - \sigma|}{t} \leq \frac{C\alpha}{t}, \quad \tau, \sigma \in [0, \alpha], t \geq t_0. \end{aligned} \tag{4.13}$$

We denote

$$\xi(t) := \frac{1}{\alpha} \int_t^{t+\alpha} \zeta(\tau) d\tau, \quad t \geq t_0.$$

Since (4.13) holds for all  $\sigma \in [0, \alpha]$ , we obtain

$$|\xi(t) - \zeta(t + \tau)| \leq \frac{C\alpha}{t}, \quad t \geq t_0, \tau \in [0, \alpha]. \tag{4.14}$$

We also put

$$X(t) := (p-1) \left( \frac{p}{\alpha} \int_t^{t+\alpha} \frac{1}{r^{q-1}(\tau)} d\tau \right)^{-\frac{1}{q}}, \quad t \geq t_0, \tag{4.15}$$

$$Y(t) := |\xi(t)| \left( \frac{p}{\alpha} \int_t^{t+\alpha} \frac{1}{r^{q-1}(\tau)} d\tau \right)^{\frac{1}{q}}, \quad t \geq t_0. \tag{4.16}$$

Henceforth (in this part of the proof), we will consider  $t \geq t_0$ . We have

$$\begin{aligned} \xi'(t) &= \frac{1}{\alpha} \int_t^{t+\alpha} \zeta'(\tau) d\tau = \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{\tau} \left[ (p-1)\zeta(\tau) + \gamma s(\tau) + (p-1) \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)} \right] d\tau \\ &\geq \frac{1}{t+\alpha} \left[ (p-1)\xi(t) + \frac{\gamma}{\alpha} \int_t^{t+\alpha} s(\tau) d\tau + \frac{p-1}{\alpha} \int_t^{t+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)} d\tau \right] \\ &= \frac{1}{t+\alpha} \left[ (p-1)\xi(t) + \frac{X^p(t)}{p} + \frac{Y^q(t)}{q} + \frac{\gamma}{\alpha} \int_t^{t+\alpha} s(\tau) d\tau - \frac{X^p(t)}{p} \right. \\ &\quad \left. + \frac{p-1}{\alpha} \int_t^{t+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)} d\tau - \frac{Y^q(t)}{q} \right] \end{aligned} \tag{4.17}$$

if  $\xi'(t) > 0$  for an arbitrarily given  $t \geq t_0$ , or

$$\begin{aligned} \xi'(t) &\geq \frac{1}{t} \left[ (p-1)\xi(t) + \frac{X^p(t)}{p} + \frac{Y^q(t)}{q} + \frac{\gamma}{\alpha} \int_t^{t+\alpha} s(\tau) d\tau - \frac{X^p(t)}{p} \right. \\ &\quad \left. + \frac{p-1}{\alpha} \int_t^{t+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)} d\tau - \frac{Y^q(t)}{q} \right] \end{aligned}$$

if  $\xi'(t) < 0$ .

Since

$$(p-1)\xi(t) = -(p-1)|\xi(t)| = -X(t)Y(t),$$

we obtain

$$(p-1)\xi(t) + \frac{X^p(t)}{p} + \frac{Y^q(t)}{q} = \frac{X^p(t)}{p} + \frac{Y^q(t)}{q} - X(t)Y(t) \geq 0, \tag{4.18}$$

which follows from Young's inequality.

Next, applying (4.4), (4.5), and (4.6), we compute

$$\begin{aligned} &\frac{\gamma}{\alpha} \int_t^{t+\alpha} s(\tau) d\tau - \frac{X^p(t)}{p} \\ &= \frac{\gamma}{\alpha} \int_t^{t+\alpha} s(\tau) d\tau - \frac{(p-1)^p}{p} \left( \frac{p}{\alpha} \int_t^{t+\alpha} \frac{1}{r^{q-1}(\tau)} d\tau \right)^{-\frac{p}{q}} \end{aligned}$$



$$\begin{aligned}
 &= \frac{\gamma}{\alpha} \int_t^{t+\alpha} s(\tau) \, d\tau - q^{-p} \left( \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{r^{q-1}(\tau)} \, d\tau \right)^{-\frac{p}{q}} \frac{\frac{1}{\alpha} \int_t^{t+\alpha} s(\tau) \, d\tau}{\frac{1}{\alpha} \int_t^{t+\alpha} s(\tau) \, d\tau} \\
 &= \frac{1}{\alpha} \int_t^{t+\alpha} s(\tau) \, d\tau \left[ \gamma - q^{-p} \left( \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{r^{q-1}(\tau)} \, d\tau \right)^{-\frac{p}{q}} \left( \frac{1}{\alpha} \int_t^{t+\alpha} s(\tau) \, d\tau \right)^{-1} \right] \\
 &> \frac{M(s) \gamma - \Gamma}{2} > 0,
 \end{aligned}$$

i.e.,

$$\frac{\gamma}{\alpha} \int_t^{t+\alpha} s(\tau) \, d\tau - \frac{X^p(t)}{p} > M(s) \frac{\gamma - \Gamma}{4}. \tag{4.19}$$

The function  $y = |x|^q$  is continuously differentiable on  $[-K, 0]$ , and hence there exists  $L > 0$  satisfying

$$\left| |\xi|^q - |\zeta|^q \right| \leq L |\xi - \zeta|, \quad \xi, \zeta \in [-K, 0].$$

Using the fact that this function has the Lipschitz property and using (4.14), we obtain

$$\begin{aligned}
 &\frac{p-1}{\alpha} \int_t^{t+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)} \, d\tau - \frac{Y^q(t)}{q} \\
 &= \frac{p-1}{\alpha} \int_t^{t+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)} \, d\tau - \frac{|\xi(t)|^q}{q} \frac{q(p-1)}{\alpha} \int_t^{t+\alpha} \frac{1}{r^{q-1}(\tau)} \, d\tau \\
 &= -\frac{p-1}{\alpha} \int_t^{t+\alpha} \frac{|\xi(t)|^q - |\zeta(\tau)|^q}{r^{q-1}(\tau)} \, d\tau \geq -\frac{p-1}{\alpha} \frac{1}{(r^-)^{q-1}} \int_t^{t+\alpha} L |\xi(t) - \zeta(\tau)| \, d\tau \\
 &\geq -\frac{p-1}{\alpha} \frac{1}{(r^-)^{q-1}} \alpha L \frac{C\alpha}{t} = -\frac{CL\alpha(p-1)(r^-)^{1-q}}{t}.
 \end{aligned}$$

For sufficiently large  $t \geq t_0$ , it is valid

$$\frac{p-1}{\alpha} \int_t^{t+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)} \, d\tau - \frac{Y^q(t)}{q} > -M(s) \frac{\gamma - \Gamma}{8}. \tag{4.20}$$

Altogether (consider (4.17), (4.18), (4.19), (4.20)), there exists  $\hat{t} \geq t_0$  for which we have

$$\xi'(t) > \frac{1}{t+\alpha} \left[ 0 + M(s) \frac{\gamma - \Gamma}{4} - M(s) \frac{\gamma - \Gamma}{8} \right] = \frac{1}{t+\alpha} M(s) \frac{\gamma - \Gamma}{8}, \quad t \geq \hat{t},$$

which implies that

$$\xi(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty, \quad \text{i.e.,} \quad \limsup_{t \rightarrow \infty} \zeta(t) = \infty.$$

This contradiction proves that Eq. (4.3) is oscillatory for  $\gamma > \Gamma$ .

In the second part of the proof, we have to show that Eq. (4.3) is non-oscillatory for  $\gamma < \Gamma$ . We will consider arbitrarily given  $\gamma < \Gamma$  and the solution  $\zeta$  of the adapted Riccati equation determined by the Cauchy problem (4.8),  $\zeta(T) = 2\xi_0$ , where

$$\xi_0 := - \left[ \frac{p}{\alpha(p-1)} \int_T^{T+\alpha} \frac{1}{r^{q-1}(\tau)} \, d\tau \right]^{1-p}.$$

As in the first part of the proof (see (4.9)), we can estimate

$$\zeta'(t) > \frac{p-1}{t} \left[ \zeta(t) + \frac{|\zeta(t)|^q}{(r^+)^{q-1}} \right]$$

for all  $t$  from the domain of  $\zeta$ . Thus, there exists  $K > 0$  such that  $\zeta(t) > -K$  for all considered  $t$ . Analogously (see (4.12)), we can also prove

$$\begin{aligned} |\zeta(t_2) - \zeta(t_1)| &\leq \int_{t_1}^{t_2} |\zeta'(\tau)| \, d\tau \\ &\leq \frac{(p-1)[K + \frac{K^q}{(r^+)^{q-1}}] + \gamma s^+}{T} (t_2 - t_1), \quad t_2 \geq t_1 \geq T \end{aligned} \tag{4.21}$$

if  $\zeta(t_1), \zeta(t_2) < 0$ . Hence, we can assume that  $T$  is so large that it is true

$$|\zeta(t_2) - \zeta(t_1)| < -\frac{\xi_0}{2}, \quad t_1 + \alpha \geq t_2 \geq t_1 \geq T$$

if  $\zeta(t_1), \zeta(t_2) < 0$ . Especially, provided that  $\zeta(t) \leq \xi_0/2$  for some  $t \geq T$ , we have  $\zeta(\tau) < 0$  for all  $\tau \in [t, t + \alpha]$ . Thus, we can define

$$\xi(t) := \frac{1}{\alpha} \int_t^{t+\alpha} \zeta(\tau) \, d\tau$$

for all considered  $t \geq T$  such that  $\zeta(t) \leq \xi_0/2$ . From (4.21) it follows

$$|\xi(t) - \zeta(t + \tau)| \leq \frac{C\alpha}{T} \tag{4.22}$$

for all considered  $t \geq T$  and  $\tau \in [0, \alpha]$ , where  $C$  is introduced in (4.11).

We prove that  $\zeta(t) \leq -\delta$  for some  $\delta > 0$  and for all considered  $t \geq T$ . On the contrary, we will assume that such a number  $\delta$  does not exist. This assumption implies the existence of  $t_1 > T$  such that  $\xi'(t_1) > 0$  and  $\xi(t_1) = \xi_0$ . We estimate the value  $\xi'(t_1)$  by the following expression:

$$\begin{aligned} \xi'(t_1) &= \frac{1}{\alpha} \int_{t_1}^{t_1+\alpha} \frac{1}{\tau} \left[ (p-1)\zeta(\tau) + \gamma s(\tau) + (p-1) \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)} \right] \, d\tau \\ &\leq \frac{1}{t_1} \left[ (p-1)\xi(t_1) + \frac{\gamma}{\alpha} \int_{t_1}^{t_1+\alpha} s(\tau) \, d\tau + \frac{p-1}{\alpha} \int_{t_1}^{t_1+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)} \, d\tau \right] \\ &= \frac{1}{t_1} \left[ (p-1)\xi(t_1) + \frac{X^p(t_1)}{p} + \frac{Y^q(t_1)}{q} + \frac{\gamma}{\alpha} \int_{t_1}^{t_1+\alpha} s(\tau) \, d\tau - \frac{X^p(t_1)}{p} \right. \\ &\quad \left. + \frac{p-1}{\alpha} \int_{t_1}^{t_1+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)} \, d\tau - \frac{Y^q(t_1)}{q} \right], \end{aligned} \tag{4.23}$$

where  $X(t_1)$  and  $Y(t_1)$  are defined in (4.15) and (4.16), respectively.

Assume that

$$\frac{1}{\alpha} \int_T^{T+\alpha} \frac{1}{r^{q-1}(\tau)} \, d\tau = \frac{1}{\alpha} \int_{t_1}^{t_1+\alpha} \frac{1}{r^{q-1}(\tau)} \, d\tau.$$

In this case, considering  $X^p(t_1) = Y^q(t_1)$ , we have

$$\begin{aligned} \frac{X^p(t_1)}{p} + \frac{Y^q(t_1)}{q} &= X^p(t_1) \left( \frac{1}{p} + \frac{1}{q} \right) = X^{1+\frac{p}{q}}(t_1) \\ &= X(t_1) (Y^q(t_1))^{\frac{1}{q}} = X(t_1) Y(t_1) = -(p-1)\xi(t_1) \end{aligned}$$

and, consequently, we obtain

$$(p-1)\xi(t_1) + \frac{X^p(t_1)}{p} + \frac{Y^q(t_1)}{q} = 0.$$

We consider  $T, \alpha$  as sufficiently large numbers for which (4.4) and (4.5) are satisfied. Analogously, we can require that  $T$  and  $\alpha$  are so large that

$$\left| \frac{1}{\alpha} \int_T^{T+\alpha} \frac{1}{r^{q-1}(\tau)} d\tau - \frac{1}{\alpha} \int_{t_1}^{t_1+\alpha} \frac{1}{r^{q-1}(\tau)} d\tau \right| < \varepsilon$$

for arbitrarily given  $\varepsilon > 0$ . Note that  $t_1 > T$ . Thus, for sufficiently large  $T$ , we can assume that

$$(p-1)\xi(t_1) + \frac{X^p(t_1)}{p} + \frac{Y^q(t_1)}{q} < \frac{M(s)(\Gamma - \gamma)}{16}. \tag{4.24}$$

We compute

$$\begin{aligned} &\frac{\gamma}{\alpha} \int_{t_1}^{t_1+\alpha} s(\tau) d\tau - \frac{X^p(t_1)}{p} \\ &= \frac{\gamma}{\alpha} \int_{t_1}^{t_1+\alpha} s(\tau) d\tau - \frac{(p-1)^p}{p} \left( \frac{p}{\alpha} \int_{t_1}^{t_1+\alpha} \frac{1}{r^{q-1}(\tau)} d\tau \right)^{-\frac{p}{q}} \\ &= \frac{1}{\alpha} \int_{t_1}^{t_1+\alpha} s(\tau) d\tau \left[ \gamma - q^{-p} \left( \frac{1}{\alpha} \int_{t_1}^{t_1+\alpha} \frac{1}{r^{q-1}(\tau)} d\tau \right)^{-\frac{p}{q}} \left( \frac{1}{\alpha} \int_{t_1}^{t_1+\alpha} s(\tau) d\tau \right)^{-1} \right]. \end{aligned}$$

Therefore, using (4.4), (4.5), and (4.6), we have

$$\frac{\gamma}{\alpha} \int_{t_1}^{t_1+\alpha} s(\tau) d\tau - \frac{X^p(t_1)}{p} < \frac{M(s)}{2} \frac{\gamma - \Gamma}{2} < 0. \tag{4.25}$$

Similarly as in the first part of the proof (see (4.22)), we can show

$$\frac{p-1}{\alpha} \int_{t_1}^{t_1+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)} d\tau - \frac{Y^q(t_1)}{q} \leq \frac{p-1}{T} \frac{CL\alpha}{(r^-)^{q-1}}.$$

Assume that

$$T > \frac{16(p-1)}{M(s)(\Gamma - \gamma)} \frac{CL\alpha}{(r^-)^{q-1}},$$

which implies

$$\frac{p-1}{\alpha} \int_{t_1}^{t_1+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)} d\tau - \frac{Y^q(t_1)}{q} < \frac{M(s)(\Gamma - \gamma)}{16}. \tag{4.26}$$

Finally, considering (4.23), (4.24), (4.25), and (4.26), we obtain the following contradiction:

$$\xi'(t_1) < \frac{M(s)(\gamma - \Gamma)}{8t_1} < 0,$$

which means that there exists  $\delta > 0$  satisfying

$$\zeta(t) \leq -\delta, \quad t \geq T.$$

Especially, this negative solution exists for all  $t \geq T$ . From Corollary 3.2 it follows that Eq. (4.3) is non-oscillatory for  $\gamma < \Gamma$ .  $\square$

**Remark 2** Our result is 'backward compatible' with previous results about the conditional oscillation of half-linear equations. More precisely, if the functions  $r$  and  $s$  in Eq. (4.3) are  $\alpha$ -periodic ones (with the same period  $\alpha$ ), our constant  $\Gamma$  reduces to the one from [14]. If they are replaced by constants  $r, s \in \mathbb{R}^+$ , then  $\Gamma = q^{-p}r/s$ , which gives the classical result for half-linear Euler-type equations (see, e.g., [35] or [4, Section 1.4.2]).

**Example 1** Let  $a > 0$ ,  $\alpha, \beta \neq 0$ , and  $c > 1/2$ . We can use Theorem 4.1 for the equation

$$\left[ \frac{t^2}{1 + t^2(\cos[at] \sin[at] + c)} \Phi(x') \right]' + \gamma \frac{\arctan(\sin[\alpha t]) + |\sin[\alpha t]| + |\cos[\beta t]|}{\sqrt{t^3}} \Phi(x) = 0,$$

whose coefficients are asymptotically almost periodic (it suffices to consider directly Definition 2 and Corollaries 2.2, 2.3) and where  $p = 3/2$  (i.e.,  $q = 3$ ). It holds

$$M(r^{-2}) = M\left(\left(\cos[at] \sin[at] + c + \frac{1}{t^2}\right)^2\right) = \frac{8c^2 + 1}{8},$$

$$M(s) = M(\arctan(\sin[\alpha t]) + |\sin[\alpha t]| + |\cos[\beta t]|) = \frac{4}{\pi}.$$

Thus, the equation is oscillatory for  $6\gamma\sqrt{3(8c^2 + 1)} > \sqrt{2}\pi$  and, at the same time, non-oscillatory for  $6\gamma\sqrt{3(8c^2 + 1)} < \sqrt{2}\pi$ .

The statement of Theorem 4.1 is a new result also for  $p = 2$  (even for linear equations with almost periodic coefficients).

**Corollary 4.1** Consider the equation

$$\left[r(t)x'\right]' + \gamma \frac{s(t)}{t^2}x = 0, \tag{4.27}$$

where  $\gamma \in \mathbb{R}^+$  and  $r, s$  are positive almost periodic functions satisfying (4.1). Then the oscillation constant is

$$\Gamma = [4M(r^{-1})M(s)]^{-1}, \tag{4.28}$$

i.e., Eq. (4.27) is oscillatory for  $\gamma > \Gamma$  and non-oscillatory for  $\gamma < \Gamma$ .

If the functions  $r$  and  $s$  in Eq. (4.27) are periodic with the same period, then the oscillation constant given in (4.28) reduces to the known one (see [17]). For constant functions  $r, s$ , we obtain  $\Gamma = r/(4s)$ , which corresponds to the famous result of Kneser [36]. Note that the main result of [17] cannot be used for general periodic functions  $r, s$  (which do not have any common period). This situation is illustrated by the following example.

**Example 2** Let continuous functions  $f, g : [-1, 1] \rightarrow \mathbb{R}^+$  be given. Let us consider the equation

$$[f(\sin t)x']' + \gamma \frac{g[\sin(\sqrt{2}t)]}{t^2}x = 0.$$

The almost periodicity of the functions  $r(t) = f(\sin t), s(t) = g[\sin(\sqrt{2}t)]$  is obvious. It is also seen that (4.1) is satisfied and that

$$M(r^{-1}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{f(\sin t)}, \quad M(s) = \frac{\sqrt{2}}{2\pi} \int_{-\pi/\sqrt{2}}^{\pi/\sqrt{2}} g[\sin(\sqrt{2}t)] dt.$$

Hence, we obtain the oscillation constant

$$\Gamma = \left[ \frac{\sqrt{2}}{\pi^2} \int_{-\pi}^{\pi} \frac{dt}{f(\sin t)} \int_{-\pi/\sqrt{2}}^{\pi/\sqrt{2}} g[\sin(\sqrt{2}t)] dt \right]^{-1}.$$

### 5 Concluding remarks

There exist positive almost periodic functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  for which

$$\frac{1}{\alpha} \int_0^\alpha f(t) dt < M(f), \quad \frac{1}{\alpha} \int_0^\alpha g(t) dt > M(g), \quad \alpha > 0.$$

Such functions can be constructed using, e.g., [37, Theorem 3.1]. Based on this fact, we conjecture that there exist almost periodic functions  $r_1, s_1, r_2, s_2$  satisfying (4.1) such that Eq. (4.3) determined by the coefficients  $r_1, s_1$  and  $\gamma = \Gamma$  (see (4.2)) is oscillatory and Eq. (4.3) with the coefficients  $r_2, s_2$  and  $\gamma = \Gamma$  is non-oscillatory. Thus, in general, we conjecture that it is not possible to decide whether Eq. (4.3) is oscillatory or non-oscillatory for  $\gamma = \Gamma$ .

The statement of Theorem 4.1 remains valid also for  $\gamma \in \mathbb{R}$ . Indeed, if  $\gamma \leq 0$ , then the equation

$$[r^- \Phi(x')] + 0 \cdot \Phi(x) = 0$$

is a non-oscillatory majorant of Eq. (4.3) (consider (4.1), (4.7)).

Since the asymptotic almost periodicity of a function  $f$  implies the asymptotic almost periodicity of  $|f|$  (consider, e.g., Corollary 2.1), we can also apply Theorem 4.1 in the case when the function  $s$  in Eq. (4.3) changes its sign. More precisely, for Eq. (4.3) with  $\gamma \in \mathbb{R}$  and asymptotically almost periodic functions  $r$  and  $s$ , we obtain the following result by the Sturm comparison theorem. *If  $\inf\{r(t); t \in \mathbb{R}_0^+\} > 0, \lim_{t \rightarrow \infty} s(t) \neq 0$ , and  $\gamma < q^{-p}(M(r^{1-q}))^{1-p}(M(|s|))^{-1}$ , then Eq. (4.3) is non-oscillatory.*

Let us illustrate the connection between (ordinary) half-linear differential equations analyzed in this paper and partial differential equations with  $p$ -Laplacian. We refer, e.g., to [38–41]. For example, consider the equation

$$\operatorname{div}(A(x)\|\nabla u\|^{p-2}\nabla u) + (\vec{b}(x), \|\nabla u\|^{p-2}\nabla u) + c(x)\Phi(u) = 0, \quad (5.1)$$

where  $x \in \mathbb{R}^n$ ,  $\vec{b}$  is a continuous non-zero  $n$ -vector function,  $A$  is an elliptic  $n \times n$  matrix function with differentiable components,  $c$  is a Hölder continuous function, and  $\operatorname{div}$ ,  $\nabla$ ,  $\|\cdot\|$ , and  $\langle \cdot, \cdot \rangle$  stand for the divergence operator, the nabla operator, the Euclidean norm, and the usual scalar product in  $\mathbb{R}^n$ , respectively. Let  $l > 1$ ,  $S(a) = \{x \in \mathbb{R}^n, \|x\| = a\}$ ,  $d\sigma$  be the element of the surface of the sphere  $S(t)$ , and  $\lambda_{\min}(x)$  be the smallest eigenvalue of the matrix  $A(x)$ . According to [40, Theorem 1], we have the following result. *If the functions*

$$r(t) = \left(\frac{l}{l-1}\right)^{p-1} \int_{S(t)} \|A(x)\|^p \lambda_{\min}^{1-p}(x) d\sigma,$$
$$s(t) = \frac{t^p}{\gamma} \int_{S(t)} \left[ c(x) - \frac{t^{p-1}}{\lambda_{\min}^{p-1}(x)} \frac{\|\vec{b}(x)\|^p}{t^p} \right] d\sigma$$

*satisfy the requirements of Theorem 4.1 and  $\gamma > \Gamma$  (see (4.2)), then Eq. (5.1) is oscillatory.*

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

The authors declare that the research was realized in collaboration with the same responsibility and contributions. Both authors read and approved the final manuscript.

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