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Research Article

Legendre Invariance and Geometrothermodynamics Description of the 3D Charged-Dilaton Black Hole

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We first review Weinhold information geometry and Ruppeiner information geometry of 3D charged-dilaton black hole. Then, we use the Legendre invariant to introduce a 2-dimensional thermodynamic metric in the space of equilibrium states, which becomes singular at those points. According to the analysis of the heat capacities, these points are the places where phase transitions occur. This result is valid for the black hole, therefore, provides a geometrothermodynamics description of black hole phase transitions in terms of curvature singularities.

1. Introduction

Since Ferrara et al. [1] investigated the critical points of moduli space by using Weinhold metric and Ruppeiner metric; the black hole thermodynamic in geometry framework becomes a hot spot of theoretical physics.

It is well known that an equilibrium thermodynamic system poses interesting geometric features. An interesting inner product on the equilibrium thermodynamic state space in the energy representation was provided by Weinhold as the Hessian matrix of the internal energy U with respect to the extensive thermodynamic variables N^a, namely, [2] $g_{ij}^W = \partial_i\partial_j M(U, N^a)$. However, there was no physical interpretation associated with this metric structure. As a modification, Ruppeiner introduced Riemannian metric into thermodynamic system once more and defended it as the second derivative of entropy S (here, entropy is a function of internal energy U and its extensive variables N^a) [3] $g_{ij}^R = -\partial_i\partial_j S(U, N^a)$. An interesting phenomenon is that these two metrics are conformally related, that is, $g_{ij}^W = Tg_{ij}^R$, and the conformal factor is the temperature, $T = \partial M/\partial S$. It has been applied to all kinds of thermodynamic models, for example, the ideal gas, the van der Waals gas, and the two-dimensional Fermi gas et al. Studies showed that Ruppeiner geometry can overcome the covariant and self-consistent problem of general thermodynamics. Based on the Ruppeiner and Weinhold metrics, consideration of different black hole families under various assumptions has led to numerous puzzling results for both metrics [4–13]. So it is then natural to try to describe the phase transitions of black holes in terms of curvature singularities in the space of equilibrium states. Unfortunately, the obtained results, at least some, are contradictory. For instance, for Reissner-Nordström black hole, the Ruppeiner metric is flat [14], whereas the Weinhold metric presents a curvature singularity. Similarly, the 3D charged-dilaton black hole also showed similar result [15]. Nevertheless, a simple change of the thermodynamic potential [16] affects Ruppeiner's geometry in such a way that the resulting curvature singularity now corresponds to a phase transition. A dimensional reduction of Ruppeiner's curvature seems to affect its properties too [17]. However, it is well known that ordinary thermodynamics does not depend on the thermodynamic potential.

Recently, Quevedo [18] proposed a formalism of geometrothermodynamics (GTDs) as a geometric approach that incorporates Legendre invariance in a natural way and allows us to derive Legendre invariant metrics in the space of equilibrium states. Since Weinhold and Ruppeiner metrics are not Legendre invariant, one of the first results in the context of GTD was the derivation of simple Legendre invariant generalizations of these metrics and their application to black hole thermodynamics [19–26]. These results end the controversy regarding the application of geometric structures in black hole thermodynamics. The phase transition structure contained in the heat capacity of black holes becomes completely integrated in the scalar curvature of the Legendre invariant metric so that a curvature singularity corresponds to a phase transition.

In this paper, we first review Weinhold information geometry and Ruppeiner information geometry of 3D charged-dilaton black hole. Then, based on our previous work, the thermodynamics scalar curvature of the black hole is described. We explored the geometrothermodynamics of 3D charged-dilaton black hole.

The organization of this paper is outlined as follows. In Section 2, we review Weinhold information geometry and Ruppeiner information geometry of the 3D charged-dilaton black hole. Then, in Section 3, geometrothermodynamics of the charged black hole ise described. Finally, some discussions and conclusions are given in Section 4. Throughout the paper, the units ($G = c = \hbar = 1$) are used.

2. Information Geometry Description of 3D Charged-Dilaton Black Hole

Let us first review Weinhold information geometry and Ruppeiner information geometry of the 3D charged-dilaton black hole.

The starting action with the dilaton field ϕ is given by [27]

$$S = \int d^3x\sqrt{-g}\left[R + 2e^{4\phi}\Lambda + 4(\nabla\phi)^2 - e^{-4\phi}F_{\mu\nu}F^{\mu\nu}\right]. \quad (1)$$

The cosmology constant $\Lambda > 0$ for anti-de Sitter space-time. This action is conformably related to the low-energy string action black hole that is given by

$$ds^2 = -f(r)\,dt^2 + \frac{4r^2}{\gamma^4 f(r)}dr^2 + r^2 d^2\varphi,$$
$$\phi = \frac{1}{4}\ln\left[\frac{r}{\gamma^2}\right]; \qquad F_{rt} = \frac{Q}{r^2}, \quad (2)$$

where the metric function $f(r) = -2Mr + 8\Lambda r^2 + 8Q^2$ and an integration constant γ with dimension $[L]^{1/2}$ is necessary to have correct dimensions.

In our previous work, the mass and electric charge of the 3D charged-dilaton black hole have been expressed in terms of the inner and outer horizons as [15]

$$M = 4\Lambda(r_+ + r_-), \qquad Q^2 = \Lambda r_+ r_-, \quad (3)$$

and, the electric potential is given by

$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_S = \frac{8Q}{r_+}. \quad (4)$$

According to the energy conservation law,

$$dM = TdS + \Phi dQ. \quad (5)$$

The temperature is

$$T = \left(\frac{\partial M}{\partial S}\right)_Q = \frac{2\Lambda}{r_+^2}(r_+ - r_-). \quad (6)$$

By using the area law, the entropy of the black hole is given by

$$S = \frac{k_B}{4}A = k_B\pi r_+^2 = r_+^2, \quad (7)$$

with k_B Boltzmann's constant, and $k_B = 1/\pi$. In its natural coordinates, the Weinhold metric can be obtained as follows:

$$ds_W^2 = \frac{1}{r_+^3}\left(\frac{3Q^2}{r_+^2} - \Lambda\right)dS^2 - 8Q\,dS\,dQ + r_+^2 dQ^2, \quad (8)$$

where the index W denotes the Weinhold information geometry. Here, we have made the choice that the mass M corresponds to the thermodynamic potential; entropy S and charge Q correspond to the extensive variables, from which it can be shown that the Weinhold scalar curvature in the entropy representation becomes

$$\mathfrak{R}_W = -\frac{r_+}{4\Lambda(r_+ - r_-)^2}. \quad (9)$$

We see that the curvature $\mathfrak{R}_W$ naively diverges at the extreme limit of the black hole, where $r_+ = r_-$, which is of less interest physically since at the extreme limit, the Hawking temperature vanishes, and the thermodynamics description breaks down as mentioned above. We interpret this result as an indication of the limit of applicability of geometric thermodynamics as a geometric model for equilibrium thermodynamics.

By using the coordinate transformation $u = Q/r_+$ [14, 15], we obtain the diagonalized Ruppeiner metric for the 3D charged-dilaton black hole as follows:

$$ds_R^2 = \frac{1}{T}ds_W^2 = -\frac{1}{2S}dS^2 + \frac{4S}{\Lambda - u}du^2. \quad (10)$$

Let us do a new transformation as follows:

$$\tau = \sqrt{2}S, \qquad \sqrt{\Lambda}\sin\left(\frac{\sigma}{\sqrt{2}}\right) = u. \quad (11)$$

Then, the Ruppeiner metric can be written in the above Rindler coordinates as

$$ds_R^2 = -d\tau + \tau^2 d\sigma^2. \quad (12)$$

Obviously, this is a flat metric; its curvature is zero. The vanished thermodynamic curvature implies that no phase transition points exist and no thermodynamic interactions appear. This result implies that the Ruppeiner curvature cannot describe the phase transitions of the black hole either.

3. Geometrothermodynamics Description of 3D Charged-Dilaton Black Hole

In this section, we turn to use the recent geometric formulation of extended thermodynamic behavior of the 3D charged-dilaton black hole.

The formulation of GTD is based on the use of contact geometry as a framework for thermodynamics [18]. Consider the $(2n+1)$-dimensional thermodynamic phase space $\mathfrak{T}$ coordinated by the thermodynamic potential Φ, extensive variables E^a, and intensive variables I^a $(a = 1,\ldots,n)$. Consider on $\mathfrak{T}$ a nondegenerate metric $G = G(Z^A)$, with $Z^A = \{\Phi, E^a, I^a\}$, and the Gibbs1-form $\Theta = d\Phi - \delta_{ab}I^a dE^b$, with $\delta_{ab} = \text{diag}(1,1,\ldots,1)$. The set $(\mathfrak{T},\Theta,G)$ defines a contact Riemannian manifold if the condition $\Theta \wedge (d\Theta)^n \neq 0$ is satisfied. Moreover, the metric G is Legendre invariant if its functional dependence on Z^A does not change under a Legendre transformation. The Gibbs 1g-form Θ is also invariant with respect to Legendre transformations. Legendre invariance guarantees that the geometric properties of G do not depend on the thermodynamic potential used in its construction.

The thermodynamic phase space $\mathfrak{T}$ which in the case of the 3D charged-dilaton black hole can be defined as a 4-dimensional space with coordinates $Z^A = \{M,S,T,Q\}$, $A = 0,\ldots,4$. Equation (3) represents the fundamental relationship $M = (S,Q)$ from which all the thermodynamic information can be obtained; therefore, we would like to consider a 5-dimensional phase space $\mathfrak{T}$ with coordinates (M,S,T,Q,Φ), a contact one-form

$$\Theta = dM - TdS - \Phi dQ, \tag{13}$$

and an invariant metric

$$G = (dM - TdS - \Phi dQ)^2 + (TS + \Phi Q)(-dT\,dS + d\Phi\,dQ). \tag{14}$$

The triplet $(\mathfrak{T},\Theta,G)$ defines a contact Riemannian manifold that plays an auxiliary role in GTD. It is used to properly handle the invariance with respect to Legendre transformations. In fact, for the charged black hole, a Legendre transformation involves in general all the thermodynamic variables M,S,Q,T, and Φ so that they must be independent from each other as they are in the phase space. We introduce also the geometric structure of the space of equilibrium states ε in the following manner: ε is a 2-dimensional submanifold of $\mathfrak{T}$ that is defined by the smooth embedding map $\varphi : \varepsilon \mapsto \mathfrak{T}$, satisfying the condition that the "projection" of the contact form Θ on ε vanishes, namely, $\varphi^*(\Theta) = 0$, where φ^* is the pullback of φ, and that G induces a Legendre invariant metric g on ε by means of ε. In principle, any 2-dimensional subset of the set of coordinates of $\mathfrak{T}$ can be used to label ε. For the sake of simplicity, we will use the set of extensive variables S and Q which in ordinary thermodynamics corresponds to the energy representation. Then, the embedding map for this specific choice is

$$\varphi : \{S,Q\} \longmapsto \left\{M(S,Q), S, Q, \frac{\partial M}{\partial S}, \frac{\partial M}{\partial Q}\right\}. \tag{15}$$

The condition $\varphi^*(\Theta) = 0$ is equivalent to (5) (the first law of thermodynamics) and (4), (6) (the conditions of thermodynamic equilibrium); the induced metric is obtained as follows:

$$g = \left(S\frac{\partial M}{\partial S} + Q\frac{\partial M}{\partial Q}\right)\left(-\frac{\partial^2 M}{\partial S^2}dS^2 + \frac{\partial^2 M}{\partial Q^2}dQ^2\right). \tag{16}$$

This metric determines all the geometric properties of the equilibrium space ε. We see that in order to obtain the explicit form of the metric, it is only necessary to specify the thermodynamic potential M as a function of S and Q. In ordinary thermodynamics, this function is usually referred to as the fundamental equation from which all the equations of state can be derived. From (3), the fundamental equation $M = M(S,Q)$ is given by

$$M(S,Q) = 4\Lambda\sqrt{S}\left(1 + \frac{Q^2}{\Lambda S}\right). \tag{17}$$

The first-order and the second-order partial differentials can be expressed, respectively, as

$$\begin{aligned} \frac{\partial M}{\partial S} &= \frac{2}{r_+}\left(\Lambda - \frac{Q^2}{r_+^2}\right), \qquad \frac{\partial M}{\partial Q} = \frac{8Q}{r_+}, \\ \frac{\partial^2 M}{\partial S^2} &= -\frac{1}{r_+^3}\left(\Lambda - \frac{3Q^2}{r_+^2}\right), \qquad \frac{\partial^2 M}{\partial Q^2} = \frac{8}{r_+}. \end{aligned} \tag{18}$$

Substituting (18) into (16), the lines of GTD for the 3D charged-dilaton black hole are written as

$$dS_G^2 = \frac{2}{r_+^6}\left(\Lambda r_+^4 - 9Q^4\right)dS^2 + \frac{16}{r_+^2}\left(\Lambda r_+^2 + 3Q^2\right)dQ^2, \tag{19}$$

where the index G denotes the geometrothermodynamics. Thus, the curvature scalar can be obtained by

$$\mathfrak{R}_G = \frac{9r_+r_-(r_+ - r_-)}{2\Lambda^4(3r_- - r_+)^2(r_+ + 3r_-)^3}. \tag{20}$$

We see in our setup that the scalar curvature $\mathfrak{R}_G$ vanishes only at the extremal limit to where $r_+ = r_-$. In a general case, the scalar curvature $\mathfrak{R}_G$ does not vanish and it goes positive infinity when $r_+ = 3r_-$, which stands for a kind of phase transition or long rang correlation of the system according to the Ruppeiner theory [28]. It is interesting to note that the divergence point of the scalar curvature is just the transition point of Davies [29]. In the fact, it is easy to check this by calculating the heat capacity with a fixed charge as follows:

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q = \frac{2r_+^2(r_+ - r_-)}{3r_- - r_+}, \tag{21}$$

which is singular at $r_+ = 3r_-$ corresponding to $M^2 = 266Q^2/3$ and indicates that the black hole has a second-order phase transition. Moreover, we see that all thermodynamic variables are well behaved, except perhaps in the extremal limit $r_+ = r_-$, at this point, changes sign and the scalar curvature diverge. Therefore, there will be a phase transition $\mathfrak{R}_G$.

4. Discussion and Conclusions

In this work, we investigated the Weinhold metric and the Ruppeiner metric as well as the geometrothermodynamics of a 3D Charged-Dilaton Black Hole. In all these cases, our results showed that the thermodynamic curvature is in general different, indicating the presence of thermodynamic interaction. For instance, the scalar curvature $\mathfrak{R}_W$ indicates the presence of second-order transition points in $r_+ = r_-$. Nevertheless, the scalar curvature $\mathfrak{R}_R$ is zero, indicating that no phase transitions can occur, and the scalar curvature $\mathfrak{R}_R$ lost the information about charge Q. Moreover, the scalar curvature $\mathfrak{R}_G$ indicates more physical interest since the first-order phase transition point and the second-order transition both occur in the extremal limit at $r_+ = r_-$ and $r_+ = 3r_-$.

In addition, the thermodynamic metric proposed in this work has been applied to the case of black hole configurations in four dimensions with and without the cosmological constant. It has been shown that this thermodynamic metric correctly describes the thermodynamic behavior of the corresponding black hole configurations. One additional advantage of this thermodynamic metric is its invariance with respect to total Legendre transformations. This means that the results are independent of the thermodynamic potential used to generate the thermodynamic metric. A very interesting result is that it can recreate the lost information in Ruppeiner metric by using Legendre transformation. In summary, all of the above thermodynamic geometries leading to different results indicate that it is still unresolved to introduce geometrical concepts into all kinds of black holes; we also expect that this unified geometry description may give more information about a thermodynamic system.

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