

Research Article

Further Results about Traveling Wave Exact Solutions of the Drinfeld-Sokolov Equations

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We employ the complex method to obtain all meromorphic exact solutions of complex Drinfeld-Sokolov equations (DS system of equations). The idea introduced in this paper can be applied to other nonlinear evolution equations. Our results show that all constant and simply periodic traveling wave exact solutions of the equations (DS) are solitary wave solutions, the complex method is simpler than other methods and there exist simply periodic solutions $v_{s,3}(z)$ which are not only new but also not degenerated successively by the elliptic function solutions. We believe that this method should play an important role for finding exact solutions in the mathematical physics. For these new traveling wave solutions, we give some computer simulations to illustrate our main results.

1. Introduction

In this work, we aim to further study the Drinfeld-Sokolov system (Drinfeld-Sokolov equations (DS system of equations)) [1-3]

$$\begin{aligned}u_t + (v^2)_x &= 0, \\v_t - av_{xxx} + 3dvu_x + 3kuv_x &= 0,\end{aligned}\tag{1}$$

where a , d , and k are constants. This system was introduced by Drinfeld and Sokolov as an example of a system of nonlinear equations possessing Lax pairs of a special form [1].

Wazwaz [4] used the sine-cosine method and the tanh method to stress the power of these methods to nonlinear equations. Wazwaz [4] also investigated the traveling wave solutions with compact and noncompact structures for the Drinfeld-Sokolov equations (DS system of equations).

To look for the traveling wave solution of (1), we use the transformation $u(x, t) = u(z)$, $z = x - ct$ (where c is a

complex constant number). Then in (1), the system is carried to a system of ordinary differential equation:

$$-cu' + (v^2)' = 0,\tag{2}$$

$$cv' + av''' - 3dvu' - 3kuv^2 = 0.\tag{3}$$

By integrating the first equation in the system and neglecting the constant of integration we find

$$cu = v^2.\tag{4}$$

By inserting (4) into (3) and integration we find

$$c^2v - (2d + k)v^3 + acv'' = 0.\tag{5}$$

In 2007, El-Wakil and Abdou [5] got two solutions of (5) via tanh-function method for finding exact solutions of (5).

In this paper, we employ the complex method which was introduced by Yuan et al. [6-8] to obtain the general solutions and some new solutions of (5). In order to state our results, we need some concepts and notations.

A meromorphic function $w(z)$ means that $w(z)$ is holomorphic in the complex plane \mathbb{C} except for poles. $\wp(z; g_2, g_3)$

is the Weierstrass elliptic function with invariants g_2 and g_3 . We say that a meromorphic function f belongs to the class W if f is an elliptic function, or a rational function of $e^{\alpha z}$, $\alpha \in \mathbb{C}$, or a rational function of z .

Our main result is the following theorem.

Theorem 1. *All meromorphic solutions u of (5) belong to the class W . Furthermore, (5) has the following two forms of solutions:*

(i) *the elliptic general solutions*

$$\begin{aligned}
 &v_d(z) \\
 &= \pm \sqrt{\frac{ac}{4d+2k}} \\
 &\quad \times ((-\wp + A)(4\wp A^2 + 4\wp^2 A + 2\wp' B - \wp g_2 - Ag_2)) \\
 &\quad \times \left(\left((12A^2 - g_2)\wp + 4A^3 - 3Ag_2 \right) \wp' \right. \\
 &\quad \left. + 4B\wp^3 + 12AB\wp^2 - 3Bg_2\wp - ABg_2 \right)^{-1}, \tag{6}
 \end{aligned}$$

where $g_3 = 0$, $B^2 = 4A^3 - g_2A$, g_2 and A are arbitrary constants;

(ii) *the simply periodic solutions, where $\xi = e^{\alpha z}$ are obtained, for $z_0 \in \mathbb{C}$, by*

$$v_{s,1}(\xi) = \pm c \frac{1}{\sqrt{2d+k}} \left(\coth \frac{\alpha}{2} (z - z_0) \right), \tag{7}$$

where $a = 2c/\alpha^2$;

$$v_{s,2}(\xi) = \pm \sqrt{2}c \sqrt{\frac{-1}{2d+k}} \frac{1}{\sinh \alpha (z - z_0)}, \tag{8}$$

where $a = -c/\alpha^2$;

$$v_{s,3}(\xi) = \pm c \sqrt{\frac{1}{2d+k}} \left(\frac{1}{\sinh \alpha (z - z_0)} + 1 \right), \tag{9}$$

where $a = c/2\alpha^2$.

Remark 2. Compared with the results of Wazwaz [4], we find that $v_{s,3}(z)$ is the new solution of this Drinfeld-Sokolov equations (DS system of equations).

2. Preliminary Lemmas and the Complex Method

In order to give complex method and the proof of Theorem 1, we need some notations and results.

Set $m \in \mathbb{N} := \{1, 2, 3, \dots\}$, $r_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $r = (r_0, r_1, \dots, r_m)$, and $j = 0, 1, \dots, m$. We define a differential monomial denoted by

$$M_r[w](z) := [w(z)]^{r_0} [w'(z)]^{r_1} [w''(z)]^{r_2} \dots [w^{(m)}(z)]^{r_m}. \tag{10}$$

$p(r) := r_0 + 2r_1 + \dots + (m+1)r_m$ and $\deg(M)$ are called the weight and degree of $M_r[w]$, respectively.

A differential polynomial $P(w, w', \dots, w^{(m)})$ is defined as follows:

$$P(w, w', \dots, w^{(m)}) := \sum_{r \in I} a_r M_r[w], \tag{11}$$

where a_r are complex constants and I is a finite index set. The total weight and degree of $P(w, w', \dots, w^{(m)})$ are defined by $W(P) := \max_{r \in I} \{p(r)\}$ and $\deg(P) := \max_{r \in I} \{\deg(M_r)\}$, respectively.

We will consider the following complex ordinary differential equations:

$$P(w, w', \dots, w^{(m)}) = bw^n + c, \tag{12}$$

where $b \neq 0$, c are complex constants, $n \in \mathbb{N}$.

Let $p, q \in \mathbb{N}$. Suppose that (12) has a meromorphic solution w with at least one pole; we say that (12) satisfies weak $\langle p, q \rangle$ condition if it substitutes Laurent series

$$w(z) = \sum_{k=-q}^{\infty} c_k z^k, \quad q > 0, c_{-q} \neq 0 \tag{13}$$

into (12). We can determine p distinct Laurent singular parts as follows:

$$\sum_{k=-q}^{-1} c_k z^k. \tag{14}$$

Lemma 3 (see [6–8]). *Let $p, l, m, n \in \mathbb{N}$, $\deg P(w, w^{(m)}) < n$. Suppose that an m -order Briot-Bouquet equation*

$$P(w^{(m)}, w) = bw^n + c \tag{15}$$

satisfies weak $\langle p, q \rangle$ condition; then all meromorphic solutions of (15) belong to the class W . If for some values of parameters such solution w exists, then other meromorphic solutions form a one-parametric family $w(z - z_0)$, $z_0 \in \mathbb{C}$. Furthermore, each elliptic solution with pole at $z = 0$ can be written as

$$\begin{aligned}
 &w(z) \\
 &= \sum_{i=1}^{l-1} \sum_{j=2}^{q_i} \frac{(-1)^j c_{-ij}}{(j-1)!} \frac{d^{j-2}}{dz^{j-2}} \left(\frac{1}{4} \left[\frac{\wp'(z) + B_i}{\wp(z) - A_i} \right]^2 - \wp(z) \right) \\
 &\quad + \sum_{i=1}^{l-1} \frac{c_{-i1}}{2} \frac{\wp'(z) + B_i}{\wp(z) - A_i} + \sum_{j=2}^{q_l} \frac{(-1)^j c_{-lj}}{(j-1)!} \frac{d^{j-2}}{dz^{j-2}} \wp(z) + c_0, \tag{16}
 \end{aligned}$$

where c_{-ij} are given by (13), $B_i^2 = 4A_i^3 - g_2A_i - g_3$, and $\sum_{i=1}^l c_{-i1} = 0$.

Each rational function solution $w := R(z)$ is of the form

$$R(z) = \sum_{i=1}^l \sum_{j=1}^q \frac{c_{ij}}{(z - z_i)^j} + c_0, \tag{17}$$

with $l(\leq p)$ distinct poles of multiplicity q .

Each simply periodic solution is a rational function $R(\xi)$ of $\xi = e^{\alpha z} (\alpha \in \mathbb{C})$. $R(\xi)$ has $l(\leq p)$ distinct poles of multiplicity q and is of the form

$$R(\xi) = \sum_{i=1}^l \sum_{j=1}^q \frac{c_{ij}}{(\xi - \xi_i)^j} + c_0. \tag{18}$$

In order to give the representations of elliptic solutions, we need some notations and results concerning elliptic function [9].

Let ω_1, ω_2 be two given complex numbers such that $\text{Im}(\omega_1/\omega_2) > 0$, and let $L = L[2\omega_1, 2\omega_2]$ be a discrete subset $L[2\omega_1, 2\omega_2] = \{\omega \mid \omega = 2n\omega_1 + 2m\omega_2, n, m \in \mathbb{Z}\}$, which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. The discriminant $\Delta = \Delta(c_1, c_2) := c_1^3 - 27c_2^2$ and

$$s_n = s_n(L) := \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^n}. \tag{19}$$

Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ is a meromorphic function with double periods $2\omega_1, 2\omega_2$ and satisfying the equation

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{20}$$

where $g_2 = 60s_4, g_3 = 140s_6$, and $\Delta(g_2, g_3) \neq 0$.

If we changed (18) to the form

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3), \tag{21}$$

we will have $e_1 = \wp(\omega_1), e_2 = \wp(\omega_2)$, and $e_3 = \wp(\omega_1 + \omega_2)$.

Inversely, given two complex numbers g_2 and g_3 such that $\Delta(g_2, g_3) \neq 0$, then there exists double periods $2\omega_1, 2\omega_2$ Weierstrass elliptic function $\wp(z)$ such that the previous results hold.

Lemma 4 (see [9, 10]). *Weierstrass elliptic functions $\wp(z) := \wp(z, g_2, g_3)$ have two successive degeneracies and addition formula:*

- (i) *degeneracy to simply periodic functions (i.e., rational functions of one exponential e^{kz}) according to*

$$\wp(z, 3d^2, -d^3) = 2d - \frac{3d}{2} \coth^2 \sqrt{\frac{3d}{2}} z, \tag{22}$$

if one root e_j is double ($\Delta(g_2, g_3) = 0$);

- (ii) *degeneracy to rational functions of z according to*

$$\wp(z, 0, 0) = \frac{1}{z^2} \tag{23}$$

if one root e_j is triple ($g_2 = g_3 = 0$);

- (iii) *addition formula*

$$\wp(z - z_0) = -\wp(z) - \wp(z_0) + \frac{1}{4} \left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)} \right]^2. \tag{24}$$

By the previous lemma and results, we can give a new method next say *complex method*, to find exact solutions of some PDEs.

Step 1. Substituting the transform $T : u(x, t) \rightarrow w(z), (x, t) \rightarrow z$ into a given PDE gives a nonlinear ordinary differential equations (12) or (15).

Step 2. Substitute (13) into (12) or (15) to determine that weak $\langle p, q \rangle$ condition holds.

Step 3. By determinant relation (16)–(18), we find the elliptic rational and simply periodic solutions $u(z)$ of (12) or (15) with pole at $z = 0$, respectively.

Step 4. By Lemmas 3 and 4, we obtain the all meromorphic solutions $w(z - z_0)$.

Step 5. By substituting the inverse transform T^{-1} into these meromorphic solutions $w(z - z_0)$, we get all the exact solutions $u(x, t)$ of the original given PDE.

3. Proof of Theorem 1

By substituting (13) into (5), we have $q = 1, p = 2, c_{-1} = \pm \sqrt{2ac/(2d+k)}, c_0 = 0, c_1 = \sqrt{2}c^2/6\sqrt{ac(2d+k)}$, and $c_2 = 0$. Hence, (5) satisfies weak $\langle 2, 1 \rangle$ condition and is a 2-order Briot-Bouquet differential equation. Obviously, (5) satisfies the dominant condition. So, by Lemma 3, we know that all meromorphic solutions of (5) belong to W . Now, we will give the forms of all meromorphic solutions of (5).

By (16), we infer the indeterminate rational solutions of (5) with pole at $z = 0$ that

$$v_r(z) = \frac{c_{11}}{z} + \frac{c_{12}}{z - z_1} + c_{10}. \tag{25}$$

By substituting $v_r(z)$ into (5), we get two distinct forms as follows:

$$\begin{aligned} v_{r,1}(z) &= 0, \\ v_{r,2}(z) &= \pm c \sqrt{\frac{1}{2d+k}}. \end{aligned} \tag{26}$$

We omit the constant solutions, and we obtain that there does not exist rational function solution.

In order to have simply periodic solutions, set $\xi = \exp(\alpha z)$ and put $v = v(\xi)$ into (5), then

$$a\alpha^2 (\xi^2 v'' + \xi v') - (2d+k)v^3 + c^2 v = 0. \tag{27}$$

By substituting

$$v_2(\xi) = \frac{c_2}{\xi - 1} + \frac{c_1}{\xi - \xi_1} + c_0 \tag{28}$$

into (5), we obtain the indeterminate simply periodic solutions of (27) with pole at $\xi = 1$ that

$$v_{s,1}(\xi) = \pm c \sqrt{\frac{1}{2d+k}} \left(\frac{2}{\xi - 1} + 1 \right), \tag{29}$$

where $a = 2c/\alpha^2$;

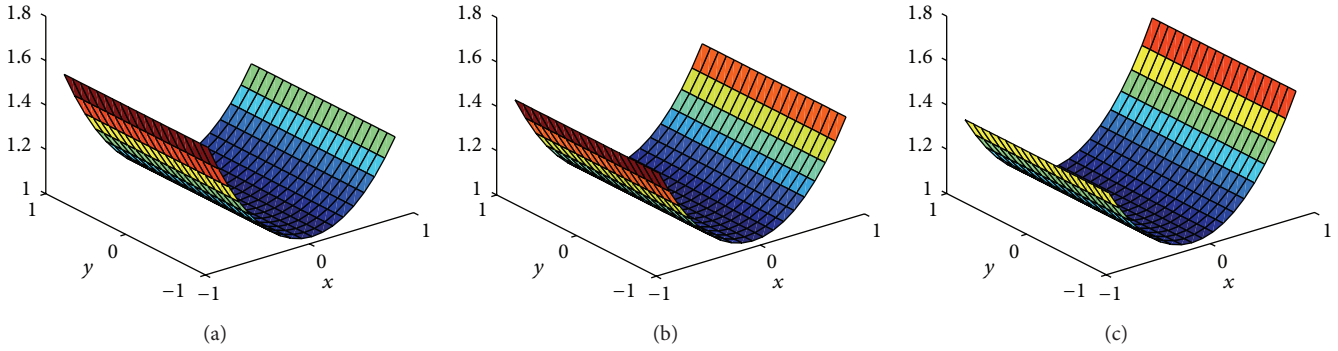


FIGURE 1: The solution of (DS) corresponding to $v_{s,1}$: take $c = 1, z_0 = 0, d = 1/2, k = 0, \alpha = 2$, (a) $t = -1/10$, (b) $t = 0$, and (c) $t = 1/10$.

$$v_{s1,2}(\xi) = \pm \sqrt{2c} \sqrt{\frac{-1}{2d+k}} \left(\frac{1}{\xi-1} + \frac{1}{\xi+1} \right), \quad (30)$$

where $a = -c/\alpha^2$;

$$v_{s1,3}(\xi) = \pm c \sqrt{\frac{1}{2d+k}} \left(\frac{1}{\xi-1} + \frac{1}{\xi+1} + 1 \right), \quad (31)$$

where $a = c/2\alpha^2$.

By substituting $\xi = e^{\alpha z}$ into the previous six relations, we get all simply periodic solutions of (5) with pole at $z = 0$:

$$v_{s0,1}(z) = \pm c \sqrt{\frac{1}{2d+k}} \left(\frac{2}{e^{\alpha z} - 1} + 1 \right), \quad (32)$$

where $a = 2c/\alpha^2$;

$$v_{s0,2}(z) = \pm \sqrt{2c} \sqrt{\frac{-1}{2d+k}} \left(\frac{1}{e^{\alpha z} - 1} + \frac{1}{e^{\alpha z} + 1} \right), \quad (33)$$

where $a = -c/\alpha^2$;

$$v_{s0,3}(z) = \pm c \sqrt{\frac{1}{2d+k}} \left(\frac{1}{e^{\alpha z} - 1} + \frac{1}{e^{\alpha z} + 1} + 1 \right), \quad (34)$$

where $a = c/2\alpha^2$.

So all simply periodic solutions of (5) are obtained, for $z_0 \in \mathbb{C}$, by

$$\begin{aligned} v_{s,1}(z) &= \pm c \sqrt{\frac{1}{2d+k}} \left(\frac{2}{e^{\alpha(z-z_0)} - 1} + 1 \right) \\ &= \pm c \frac{1}{\sqrt{2d+k}} \left(\coth \frac{\alpha}{2} (z - z_0) \right), \end{aligned} \quad (35)$$

where $a = 2c/\alpha^2$;

$$\begin{aligned} v_{s,2}(z) &= \pm \sqrt{2c} \sqrt{\frac{-1}{2d+k}} \left(\frac{1}{e^{\alpha(z-z_0)} - 1} + \frac{1}{e^{\alpha(z-z_0)} + 1} \right) \\ &= \pm \sqrt{2c} \sqrt{\frac{-1}{2d+k}} \frac{1}{\sinh \alpha (z - z_0)}, \end{aligned} \quad (36)$$

where $a = -c/\alpha^2$;

$$\begin{aligned} v_{s,3}(z) &= \pm c \sqrt{\frac{1}{2d+k}} \left(\frac{1}{e^{\alpha(z-z_0)} - 1} + \frac{1}{e^{\alpha(z-z_0)} + 1} + 1 \right) \\ &= \pm c \sqrt{\frac{1}{2d+k}} \left(\frac{1}{\sinh \alpha (z - z_0)} + 1 \right), \end{aligned} \quad (37)$$

where $a = c/2\alpha^2$.

From (15) in Lemma 3, we have indeterminate relations of elliptic solutions of (5) with pole at $z = 0$:

$$v_{d0}(z) = \pm \sqrt{\frac{ac}{4d+2k}} \frac{\wp'(z) + F}{\wp(z) - E} + c_{30}, \quad (38)$$

where $F^2 = 4E^3 - g_2E - g_3$. By applying the conclusion (ii) of Lemma 4 to $u_{d0}(z)$ and noting that the results of rational solutions obtained previously, we deduce that $c_{30} = 0, E = F = 0$, and $g_3 = 0$. Then, we get that

$$v_{d0}(z) = \pm \sqrt{\frac{ac}{4d+2k}} \wp'(z). \quad (39)$$

Therefore, all elliptic function solutions of (5) are as follows

$$v_{d0}(z) = \pm \sqrt{\frac{ac}{4d+2k}} \frac{\wp'(z - z_0)}{\wp(z - z_0)}. \quad (40)$$

Here $z_0 \in \mathbb{C}, g_3 = 0$. Making use of the addition of Lemma 3 we rewrite it to the form

$$\begin{aligned} v_d(z) &= \pm \sqrt{\frac{ac}{4d+2k}} \\ &\times ((-\wp + A)(4\wp^2 A + 4\wp^2 A + 2\wp' B - \wp g_2 - A g_2)) \\ &\times (((12A^2 - g_2)\wp + 4A^3 - 3A g_2)\wp' \\ &\quad + 4B\wp^3 + 12AB\wp^2 - 3B g_2\wp - AB g_2)^{-1}. \end{aligned} \quad (41)$$

Here $g_3 = 0, B^2 = 4A^3 - g_2 A, g_2$, and A are arbitrary constants.

This completes the proof of Theorem 1.

4. Computer Graphs for New Solutions

In this section, we give some computer graphs to illustrate our main results. Here, we take the simple periodic solutions $v_{s,1-3}(z)$ by Figures 1, 2, and 3.

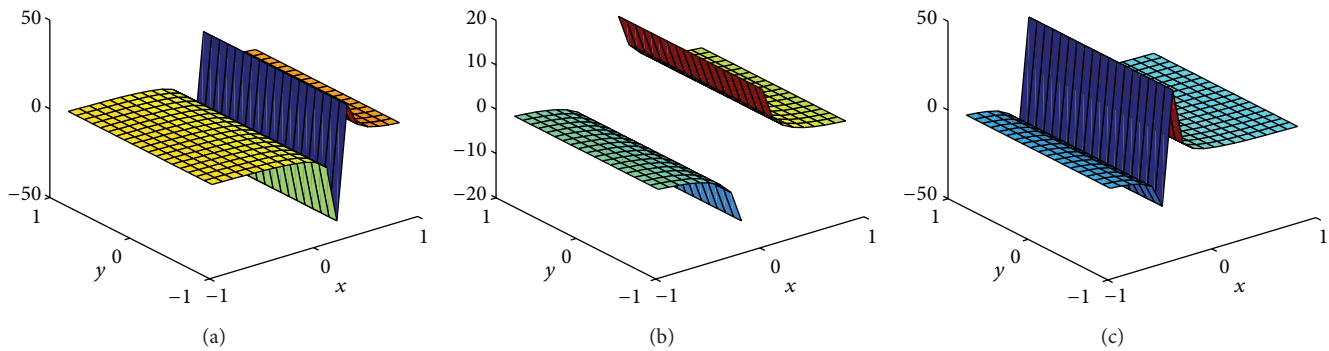


FIGURE 2: The solution of (DS) corresponding to $v_{s,2}$: take $c = 1$, $z_0 = 0$, $d = -1/2$, $k = 0$, $\alpha = 1$, (a) $t = -1/3$, (b) $t = 0$, and (c) $t = 1/3$.

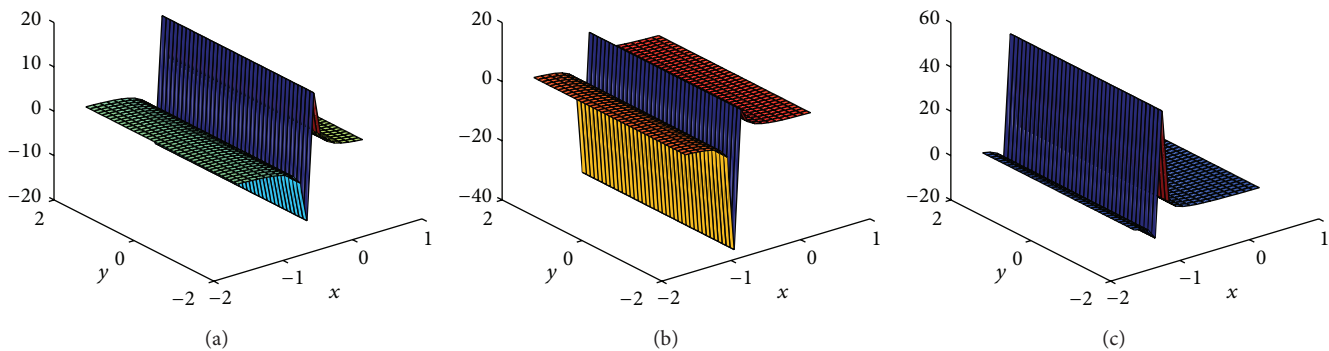


FIGURE 3: The solution of (DS) corresponding to $v_{s,3}$: take $c = 1$, $z_0 = 0$, $d = 1/2$, $k = 0$, $\alpha = 1$, (a) $t = -1/3$, (b) $t = 0$, and (c) $t = 1/3$.

5. Conclusions

Complex method is a very important tool in finding the exact solutions of nonlinear evolution equations, and the Drinfeld-Sokolov equations (DS system of equations) are a classic and simplest case of the nonlinear reaction-diffusion equation. In this paper, we employ the complex method to obtain the general meromorphic solutions of the Drinfeld-Sokolov equations (DS system of equations), which improves the corresponding result obtained by El-Wakil and Abdou, [5]. Our results show that simply periodic traveling wave exact solutions of the equations (DS) are solitary wave solutions, the complex method is simpler than other methods and there does not exist any rational solutions and there exist simply periodic solutions $v_{s,3}(z)$ which are not only new but also not degenerated successively by the elliptic function solutions. We believe that this method should play an important role in finding exact solutions in the mathematical physics. For these new traveling wave solutions, we give some computer simulations to illustrate our main results.

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