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Approximation on the hexagonal grid of the Dirichlet problem for Laplace's equation

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Turkey**Abstract**

The fourth order matching operator on the hexagonal grid is constructed. Its application to the interpolation problem of the numerical solution obtained by hexagonal grid approximation of Laplace's equation on a rectangular domain is investigated. Furthermore, the constructed matching operator is applied to justify a hexagonal version of the combined Block-Grid method for the Dirichlet problem with corner singularity. Numerical examples are illustrated to support the analysis made.

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1 Introduction

When solving PDEs, many approximate methods, such as overlapping versions of domain decomposition, composite grids and different types of combined methods, use the matching operator to connect the subsystems within them. Hence, the approximation of the solutions relies heavily on the order of accuracy of the matching operator, as well as on the order of accuracy of the subsystems. In [1] and [2], the second order matching operator is used to construct and justify the second order composite grid method for solving Laplace's boundary value problems. In [3], the fourth order matching operator is constructed and used for the fourth order composite grids and in [4] and [5] it is used for the fourth order Block-Grid method. In [6–9], the sixth order matching operator is constructed for the Block-Grid method and it is used for the sixth order composite grids in [10].

In all of the above mentioned papers, the fourth and sixth order matching operators were constructed on the basis of the 9-point finite difference solution of Laplace's equation on square grids. In this paper, the matching operator is constructed for the solution of the Dirichlet problem on a hexagonal grid. In order to approximate the given differential equations at each regular node P_0 on a hexagonal grid, the six equidistant nodes surrounding P_0 are used, and the truncation error obtained is $O(h^4)$. Thus, we obtain the same order of accuracy when using the 7-point scheme on the hexagonal grid, as we do when using the 9-point scheme on the rectangular grid (see [11]). This has many computational advantages such as (i) the matrix of the system will contain seven diagonals rather than nine and will lead to less use of memory space, (ii) the calculations will require less computational effort and (iii) the algorithm will be easier to implement. Hexagonal grids are favored in many applied problems in dynamical meteorology and dynamical oceanog-

raphy as well (see [12–14]), due to the benefits a hexagonal grid provides, compared to a rectangular grid.

Even though using a hexagonal grid has the above mentioned advantages, it has not been used before in methods such as composite grids, domain decomposition, and combined methods, as the fourth order matching operator for connecting the subsystems together was not constructed. In this paper, in Section 2, the approximate solution in a hexagonal grid on a rectangular domain is analyzed. In Section 3 a fourth order matching operator is constructed and its application to find the fourth order accurate approximate solution on the closed domain is considered. Section 4 contains the justification of using a hexagonal grid for the solution of Laplace’s equation on a staircase polygon, with the use of the Block-Grid method. This method requires the application of the matching operator constructed in Section 3 and gives an overall fourth order accuracy. Numerical examples are illustrated in Section 5 to support the analysis made.

2 Approximation in the hexagonal grid of the Dirichlet problem on a rectangle

Let $\Pi = \{(x, y) : 0 < x < a, 0 < y < b\}$ be a rectangle, $\gamma_j, j = 1, 2, 3, 4$, be its sides, including the ends, enumerated counterclockwise starting from left ($\gamma_0 \equiv \gamma_4, \gamma_1 \equiv \gamma_5$), $\gamma = \bigcup_{j=1}^4 \gamma_j$ be the boundary of Π , and let $A_j = \gamma_{j-1} \cap \gamma_j$ be the j th vertex. We consider the boundary value problem

$$\Delta u = 0 \quad \text{on } \Pi, \tag{2.1}$$

$$u = \varphi_j \quad \text{on } \gamma_j, j = 1, 2, 3, 4, \tag{2.2}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, φ_j is a given function of arclength s taken along γ , and

$$\varphi_j \in C^{6,\lambda}(\gamma_j), \quad 0 < \lambda < 1, j = 1, 2, 3, 4. \tag{2.3}$$

At the vertices $s = s_j$ (s_j is the beginning of γ_j), the conjugation conditions

$$\varphi_j^{(2q)}(s_j) = (-1)^q \varphi_{j-1}^{(2q)}(s_j), \quad q = 0, 1, 2, 3, \tag{2.4}$$

are satisfied.

Let $h > 0$, with $a/h \geq 2, b/\sqrt{3}h \geq 2$ integers. We assign Π^h a hexagonal grid on Π , with step size h , defined as the set of nodes

$$\Pi^h = \left\{ (x, y) \in \Pi : x = \frac{k-j}{2}h, y = \frac{\sqrt{3}(k+j)}{2}h, k = 1, 2, \dots; j = 0 \pm 1 \pm 2, \dots \right\}. \tag{2.5}$$

Let γ_j^h be the set of nodes on the interior of γ_j , and let $\dot{\gamma}_j^h = \gamma_j \cap \gamma_{j+1}, \gamma^h = \bigcup(\gamma_j^h \cup \dot{\gamma}_j^h), \bar{\Pi}^h = \Pi^h \cup \gamma^h$. Also let Π^{*h} denote the set of nodes whose distance from the boundary γ of $\bar{\Pi}$ is $\frac{h}{2}$ and $\Pi^{0h} = \Pi^h \setminus \Pi^{*h}$.

We consider the system of finite difference equations

$$u_h = Su_h \quad \text{on } \Pi^{0h}, \tag{2.6}$$

$$u_h = S_j^* u_h + E_{j_h}^*(\varphi_j) \quad \text{on } \Pi^{*h}, \tag{2.7}$$

$$u_h = \varphi_j \quad \text{on } \gamma_j^h, j = 1, 2, 3, 4, \tag{2.8}$$

where

$$\begin{aligned} Su(x, y) = & \frac{1}{6} \left(u(x+h, y) + u\left(x + \frac{h}{2}, y + \frac{\sqrt{3}}{2}h\right) + u\left(x - \frac{h}{2}, y + \frac{\sqrt{3}}{2}h\right) \right. \\ & \left. + u(x-h, y) + u\left(x - \frac{h}{2}, y - \frac{\sqrt{3}}{2}h\right) + u\left(x + \frac{h}{2}, y - \frac{\sqrt{3}}{2}h\right) \right), \end{aligned} \tag{2.9}$$

$$\begin{aligned} S_j^* u(x, y) = & \frac{1}{7} \left(u\left(x + \frac{h}{2}, y - \frac{\sqrt{3}}{2}h\right) + u(x+h, y) \right. \\ & \left. + u\left(x + \frac{h}{2}, y + \frac{\sqrt{3}}{2}h\right) \right), \end{aligned} \tag{2.10}$$

$$E_{jh}^*(\varphi_j) = \frac{1}{21} \left(2\varphi_j\left(y + \frac{\sqrt{3}}{2}h\right) + 8\varphi_j(y) + 2\varphi_j\left(y - \frac{\sqrt{3}}{2}h\right) \right). \tag{2.11}$$

From formulae (2.9) and (2.10) it follows that the coefficients of the expressions $Su(x, y)$ and $S_j^* u(x, y)$ are non-negative, and their sums do not exceed one. Hence, on the basis of the maximum principle, it follows that the solution of system (2.6)-(2.8) exists and it is unique (see [11]).

Everywhere below we will denote constants which are independent of h and of the cofactors on their right by c, c_0, c_1, \dots , generally using the same notation for different constants for simplicity.

Lemma 2.1 *Let*

$$v_1 = Sv_1 + f_h \quad \text{on } \Pi^{0h},$$

$$v_1 = S_j^* v_1 \quad \text{on } \Pi^{*h},$$

$$v_1 = 0 \quad \text{on } \gamma_h,$$

and

$$v_2 = Sv_2 + \bar{f}_h \quad \text{on } \Pi^{0h},$$

$$v_2 = S_j^* v_2 + \bar{f}_h^* \quad \text{on } \Pi^{*h},$$

$$v_2 = \bar{\eta}_h \quad \text{on } \gamma_h,$$

where $f_h, \bar{f}_h, \bar{f}_h^*$ and $\bar{\eta}_h$ are arbitrary grid functions. If the conditions

$$\bar{f}_h^* \geq 0, \quad |f_h| \leq \bar{f}_h, \quad \text{and} \quad \bar{\eta}_h \geq 0$$

are satisfied, then

$$|v_1| \leq v_2.$$

Proof The proof of this lemma is similar to the proof of the comparison theorem (see Ch. 4 in [11]). □

Theorem 2.2 *Let u be the solution of problem (2.1), (2.2) and u_h be the solution of system (2.6)-(2.8), then*

$$\max_{\bar{\Pi}^h} |u_h - u| \leq ch^4. \tag{2.12}$$

Proof Let

$$\epsilon_h = u_h - u,$$

where u is the trace of the solution of problem (2.1), (2.2) on $\bar{\Pi}^h$, and u_h is the solution of system (2.6)-(2.8). Then, the error function ϵ_h satisfies the following system:

$$\epsilon_h = S\epsilon_h + \Psi_h \quad \text{on } \Pi^{0h}, \tag{2.13}$$

$$\epsilon_h = S_j^* \epsilon_h + \Psi_h^* \quad \text{on } \Pi^{*h}, \tag{2.14}$$

$$\epsilon_h = 0 \quad \text{on } \gamma^h, \tag{2.15}$$

where

$$\Psi_h = Su - u, \tag{2.16}$$

$$\Psi_h^* = S_j^* u - u + E_{jh}^*(\varphi_j) \tag{2.17}$$

are the truncation errors of equations (2.6) and (2.7), respectively.

On the basis of conditions (2.3) and (2.4), from Theorem 3.1 in [15] it follows that $u \in C^{6,\lambda}(\bar{\Pi})$, $0 < \lambda < 1$. Then, by Taylor's formula, we obtain (see [16])

$$\max_{(x,y) \in \bar{\Pi}} |\Psi_h(x,y)| \leq c_1 h^6 M_6, \tag{2.18}$$

where

$$M_j = \sup_{(x,y) \in \bar{\Pi}} \left\{ \left| \frac{\partial^j u(x,y)}{\partial x^i \partial y^{j-i}} \right|, i = 0, 1, \dots, j \right\}.$$

We represent the solution of (2.13)-(2.15) as

$$\epsilon_h = \epsilon_h^1 + \epsilon_h^2, \tag{2.19}$$

where

$$\epsilon_h^1 = S\epsilon_h^1 + \Psi_h \quad \text{on } \Pi^{0h},$$

$$\epsilon_h^1 = S_j^* \epsilon_h^1 \quad \text{on } \Pi^{*h},$$

$$\epsilon_h^1 = 0 \quad \text{on } \gamma^h,$$

and

$$\epsilon_h^2 = S\epsilon_h^2 \quad \text{on } \Pi^{0h},$$

$$\begin{aligned} \epsilon_h^2 &= S_j^* \epsilon_h^2 + \Psi_h^* \quad \text{on } \Pi^{*h}, \\ \epsilon_h^2 &= 0 \quad \text{on } \gamma^h. \end{aligned}$$

By taking the function as $v_2 = h^4 c_1 M_6 (a^2 + b^2 - x^2 - y^2)$ in Lemma 2.1, we obtain

$$\max_{(x,y) \in \Pi^h} |\epsilon_h^1| \leq \max_{(x,y) \in \bar{\Pi}} |v_2| \leq c_2 h^4 M_6. \tag{2.20}$$

Using Taylor’s formula about each of the points $(\frac{h}{2}, y) \in \Pi^{*h}$ and from (2.17), we have

$$\max_{(x,y) \in \Pi^{*h}} |\Psi^*| \leq c_3 M_4 h^4.$$

On the basis of the maximum principle, we obtain

$$\max_{(x,y) \in \Pi^h} |\epsilon_h^2| \leq \frac{7}{4} \max_{(x,y) \in \Pi^{*h}} |\Psi_h^*| \leq c_4 M_4 h^4. \tag{2.21}$$

From (2.19), (2.20), and (2.21) it follows that

$$\max_{(x,y) \in \bar{\Pi}^h} |\epsilon_h| \leq ch^4. \tag{2.22}$$

□

Remark 2.3 Estimation (2.12) remains true when $E_{jh}^*(\varphi_j)$ in system (2.6)-(2.8) is replaced by

$$E_{j,h}^* = \frac{1}{7} \varphi_j \left(y - \frac{\sqrt{3}h}{2} \right) + \frac{1}{7} \varphi_j \left(y + \frac{\sqrt{3}h}{2} \right) + \frac{2}{7} \varphi_j(y) - \frac{h^2}{2!14} \varphi_j^{(2)}(y) + \frac{h^4}{4!56} \varphi_j^{(4)}(y).$$

3 Construction of the fourth order matching operator

Let $z = x + iy$ be a complex variable, and let $\Omega = \{z : |z| < 1\}$ be a unit circle. For a harmonic function u on Ω with $u \in C^{4,0}(\bar{\Omega})$, by Taylor’s formula, any point $(x, y) \in \Omega$ can be represented as

$$u(x, y) = \sum_{k=0}^3 a_k \operatorname{Re} z^k + \sum_{k=1}^3 b_k \operatorname{Im} z^k + O(r^4), \tag{3.1}$$

where $r = \sqrt{x^2 + y^2}$,

$$\begin{aligned} a_0 &= u(0, 0), & a_1 &= \frac{\partial u(0, 0)}{\partial x}, & a_2 &= \frac{1}{2} \frac{\partial^2 u(0, 0)}{\partial x^2}, & a_3 &= \frac{1}{3!} \frac{\partial^3 u(0, 0)}{\partial x^3}, \\ b_1 &= \frac{\partial u(0, 0)}{\partial y}, & b_2 &= \frac{1}{2} \frac{\partial^2 u(0, 0)}{\partial x \partial y}, & b_3 &= \frac{1}{3!} \frac{\partial^3 u(0, 0)}{\partial x^2 \partial y}. \end{aligned}$$

By analogy with the idea used in [8], we construct the operator S^4 from the condition that the expression

$$S^4 u = \sum \xi_k u_k,$$

where $u_k = u(P_k)$, P_k is a node of the hexagonal grid Π^h , gives the exact value of any harmonic polynomial

$$F_3(x, y) = \sum_{k=0}^3 a_k \operatorname{Re} z^k + \sum_{k=1}^3 b_k \operatorname{Im} z^k,$$

at each point $P \in \Pi$, and

$$\xi_k \geq 0, \quad \sum \xi_k \leq 1.$$

Let Π_0 denote the set of points $P \in \Pi$ such that all the nodes P_k to determine the expression $S^4 u$ belong to $\overline{\Pi}^h$, and Π_{01} contain the points P , where some of the nodes P_k emerge through the side γ_j , $j = 1, 2, 3, 4$. We construct the fourth order matching operator S^4 by considering the cases when the point P belongs to one of the sets Π_0 or Π_{01} .

Case 1. The point $P \in \Pi_0$ lies on the line connecting two neighboring grid nodes (a grid line).

We place the origin of the rectangular system of coordinates on the node P_0 and direct the positive axis of x along the grid line, so that $P = P(\delta h, 0)$, $0 < \delta \leq 1/2$, and take the nodes.

$$P_0(0, 0), \quad P_1(h, 0), \quad P_2\left(\frac{h}{2}, \frac{\sqrt{3}h}{2}\right), \quad P_3\left(-\frac{h}{2}, \frac{\sqrt{3}h}{2}\right), \\ P_4\left(\frac{h}{2}, -\frac{\sqrt{3}h}{2}\right), \quad P_5\left(-\frac{h}{2}, -\frac{\sqrt{3}h}{2}\right).$$

First, we find the coefficients λ'_j , $j = 0, 1, 2, 3$, such that the representation

$$u_0 = \lambda'_0 u + \lambda'_1 u_1 + \lambda'_2 u_2 + \lambda'_3 u_3 \tag{3.2}$$

is true for the harmonic polynomials $\operatorname{Re} z^n$, $n = 0, 1, 2, 3$, where $u = u(P)$, $u_k = u(P_k)$, $k = 0, 1, 2, 3$, $z = x + iy$. This gives the system

$$\begin{aligned} \lambda'_0 + \lambda'_1 + \lambda'_2 + \lambda'_3 &= 1, \\ \delta \lambda'_0 + \lambda'_1 + \frac{1}{2} \lambda'_2 - \frac{1}{2} \lambda'_3 &= 0, \\ \delta^2 \lambda'_0 + \lambda'_1 - \frac{1}{2} \lambda'_2 - \frac{1}{2} \lambda'_3 &= 0, \\ \delta^3 \lambda'_0 + \lambda'_1 - \lambda'_2 + \lambda'_3 &= 0. \end{aligned} \tag{3.3}$$

By solving system (3.3) and rearranging (3.2) for u , we obtain the equation

$$u = \frac{u_0}{\lambda'_0} - \frac{\lambda'_1}{\lambda'_0} u_1 - \frac{\lambda'_2}{\lambda'_0} u_2 - \frac{\lambda'_3}{\lambda'_0} u_3. \tag{3.4}$$

We now take into consideration the nodes $P_4(\frac{h}{2}, -\frac{\sqrt{3}h}{2})$ and $P_5(-\frac{h}{2}, -\frac{\sqrt{3}h}{2})$ which are symmetric to the points P_2 and P_3 , respectively, with respect to the x -axis. Since $\operatorname{Im} z^k = 0$,

$k = 1, 2, 3$ for $y = 0$, and odd with respect to y , and $\operatorname{Re} z^k$, $k = 0, 1, 2, 3$, is even with respect to y , from (3.4) we have

$$u = \frac{u_0}{\lambda'_0} - \frac{\lambda'_1}{\lambda'_0} u_1 - \frac{\lambda'_2}{2\lambda'_0} u_2 - \frac{\lambda'_3}{2\lambda'_0} u_3 - \frac{\lambda'_2}{2\lambda'_0} u_4 - \frac{\lambda'_3}{2\lambda'_0} u_5.$$

Hence, we obtain the matching operator S^4 , for which the expression

$$S^4 u = \sum_{k=0}^5 \lambda_k u_k \tag{3.5}$$

gives the exact value of the harmonic polynomial $F_3(x, y)$ at the point P , where

$$\begin{aligned} \lambda_0 &= -(-1 + \delta)(1 - \delta + \delta^2), & \lambda_1 &= \frac{2\delta + \delta^3}{3}, \\ \lambda_2 = \lambda_4 &= \frac{-(-1 + \delta)\delta}{2}, & \lambda_3 = \lambda_5 &= \frac{(-1 + \delta)(-\delta + 2\delta^2)}{6}. \end{aligned}$$

It is easy to check that

$$\lambda_0 > 0, \quad \lambda_j \geq 0, \quad j = 1, 2, 3, \text{ for } 0 < \delta \leq 1/2, \tag{3.6}$$

and

$$\sum_{k=0}^5 \lambda_k = 1. \tag{3.7}$$

Remark 3.1 When $1/2 < \delta < 1$, the node P_1 , which is closest to P , is taken as the origin.

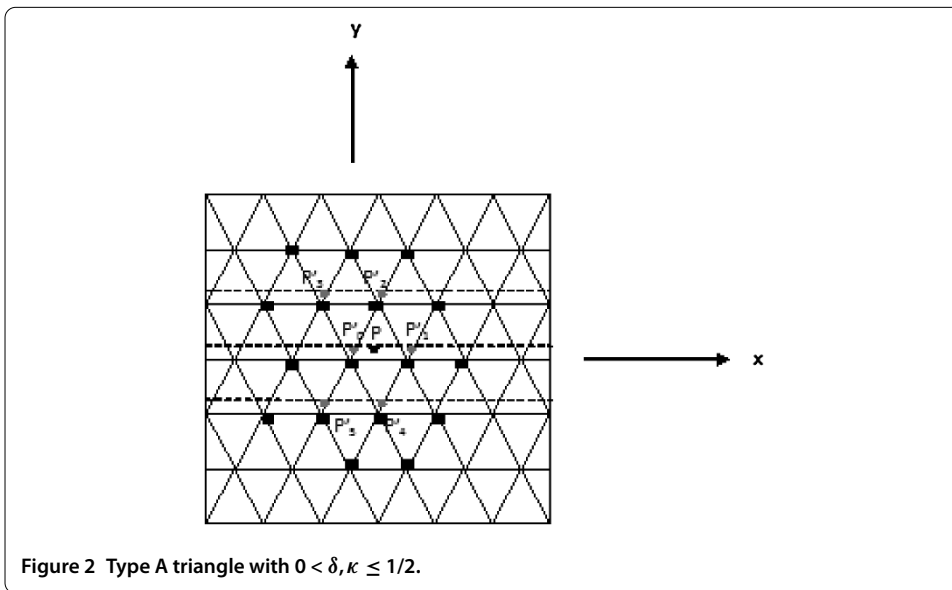
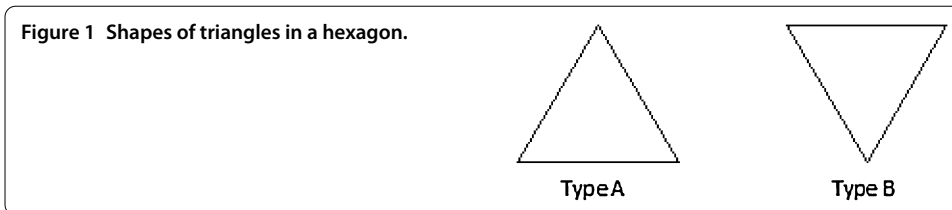
Case 2. The point $P \in \Pi_0$ lies inside a grid cell of the hexagonal grid.

Again, we place the origin of the rectangular system of coordinates at the node P_0 and direct the positive axis of x along the grid line, so that P has the coordinates $P(\delta h, \frac{\sqrt{3}h\kappa}{2})$, where $0 < \delta, \kappa \leq 1/2$. We form an artificial grid by taking the following points:

$$\begin{aligned} P'_0 &\left(\frac{\kappa h}{2}, \frac{\sqrt{3}h\kappa}{2}\right), & P'_1 &\left(h + \frac{\kappa h}{2}, \frac{\sqrt{3}h\kappa}{2}\right), & P'_2 &\left(\frac{h}{2} + \frac{\kappa h}{2}, \frac{\sqrt{3}h}{2} + \frac{\sqrt{3}h\kappa}{2}\right), \\ P'_3 &\left(-\frac{h}{2} + \frac{\kappa h}{2}, \frac{\sqrt{3}h}{2} + \frac{\sqrt{3}h\kappa}{2}\right), & P'_4 &\left(\frac{h}{2} + \frac{\kappa h}{2}, -\frac{\sqrt{3}h}{2} + \frac{\sqrt{3}h\kappa}{2}\right), \\ P'_5 &\left(-\frac{h}{2} + \frac{\kappa h}{2}, -\frac{\sqrt{3}h}{2} + \frac{\sqrt{3}h\kappa}{2}\right). \end{aligned}$$

Each of the nodes P'_k , $k = 0, 1, \dots, 5$, of the artificial grid falls on a grid line, and for the approximation of P the expression

$$S^4 u = \sum_{k=0}^5 \lambda_k u(P'_k)$$



is used. As $P'_k, k = 0, 1, \dots, 5$, all lie on grid lines, each of these points needs to be approximated using the matching operator as follows:

$$S^4 u = \sum_{k=0}^5 \lambda_k S^4 u(P'_k).$$

From the distribution of the nodes it becomes obvious that only 17 nodes are needed for this approximation (see Figure 2).

Hence, we form the matching operator as

$$S^4 u = \sum_{k=0}^{16} \xi_k u(P_k), \tag{3.8}$$

where $\xi_k, k = 0, \dots, 16$, are defined by the coefficients obtained earlier and

$$\xi_k \geq 0, \quad \sum_{k=0}^{16} \xi_k = 1. \tag{3.9}$$

For the approximation, it is also important to examine the structure of the hexagonal grid. There are two types of triangles in each hexagon, *Type A* and *Type B*, as shown in Figure 1.

We consider triangles of *Type A* with $0 < \delta, \kappa \leq 1/2$. The nodes used in $S^4 u$ are shown in Figure 2.

In the case $1/2 < \delta < 1$, $0 < \kappa \leq 1/2$, the 17 nodes used have the same distribution as the reflection of the nodes in Figure 2 about the line $x = 0$. In the cases $0 < \delta \leq 1/2$, $1/2 < \kappa < 1$ and $1/2 < \delta, \kappa < 1$, the nodes for S^4 are defined analogously.

In the case when P falls into a triangle of Type B, we rotate the artificial grids formed for Type A with an angle of 180° , for all four cases of δ and κ specified earlier.

Case 3. $P \in \Pi_{01}$, where $u = \varphi_j$ on the side γ_j , $j = 1, 2, 3, 4$, and $\varphi_j \in C^{4,\lambda}(\gamma_j)$, $0 < \lambda < 1$.

We position the origin of the rectangular system of coordinates on γ_j so that the point P lies on the positive y axis, and the x axis is in the direction of the vertex A_{j+1} along γ_j . It is obvious that $\sum_{k=1}^3 b_k \operatorname{Im} z^k = 0$ if $y = 0$, where $z = x + iy$. Hence, when the function $\varphi_j \in C^{4,\lambda}(\gamma_j)$, $0 < \lambda < 1$, is represented using Taylor's formula about the point $x = 0$ in the neighborhood $|z| \leq 4h$ of the origin, we define a_k , $k = 0, 1, 2, 3$, of (3.1) as

$$a_k = \frac{1}{k!} \frac{d^k \varphi_j(0)}{dx^k}.$$

We let

$$\tilde{u}(x, y) = u(x, y) - \sum_{k=0}^3 a_k \operatorname{Re} z^k = \sum_{k=1}^3 b_k \operatorname{Im} z^k + O(h^4)$$

for $y > 0$, and keeping in mind that $\operatorname{Im} z^k$ is odd extendable, we complete the definition with $\tilde{u}(x, y) = -\tilde{u}(x, -y)$ for $y < 0$. Clearly, in the given neighborhood, $\tilde{u}(x, y)$ is equal to the harmonic polynomial $\sum_{k=1}^3 b_k \operatorname{Im} z^k$, with an accuracy of $O(h^4)$. To form an expression for the matching operator $S^4 \tilde{u}$, we use

$$S^4 \tilde{u} = \sum_{0 \leq j \leq 16} \mu_j \left(u - \sum_{k=0}^3 a_k \operatorname{Re} z^k \right) (P_j) \tag{3.10}$$

or

$$S^4 \tilde{u} = \sum_{0 \leq j \leq 5} v_j \left(u - \sum_{k=0}^3 a_k \operatorname{Re} z^k \right) (P_j), \tag{3.11}$$

where

$$\mu_j \geq 0, \quad \sum_{0 \leq j \leq 16} \mu_j \leq 1; \quad v_j \geq 0, \quad \sum_{0 \leq j \leq 5} v_j \leq 1. \tag{3.12}$$

Hence using (3.10) or (3.11), with the addition of the term

$$\left(\sum_{k=0}^3 a_k \operatorname{Re} z^k \right) (P),$$

we have the following representation for the solution u of problem (2.1), (2.2) at any $P \in \Pi_{01}$

$$u = S^4 \tilde{u} + \left(\sum_{k=0}^3 a_k \operatorname{Re} z^k \right) (P) + O(h^4). \tag{3.13}$$

Remark 3.2 We obtain the representation (3.13), with a less number of grid nodes P_j in (3.10) or (3.11) for the points on the boundary γ of Π .

Let $\varphi = \{\varphi_j\}_{j=1}^4$. We express the matching operator S^4 as follows:

$$S^4(u, \varphi) = \begin{cases} S^4 u & \text{on } \Pi_0, \\ S^4(u - \sum_{k=0}^3 a_k \operatorname{Re} z^k) + (\sum_{k=0}^3 a_k \operatorname{Re} z^k)(P) & \text{on } \Pi_{01} \cup \gamma. \end{cases} \quad (3.14)$$

Theorem 3.3 *Let the boundary functions $\varphi_j, j = 1, 2, 3, 4$, in problem (2.1), (2.2) satisfy the conditions*

$$\varphi_j \in C^{4,\lambda}(\gamma_j), \quad 0 < \lambda < 1, \quad (3.15)$$

$$\varphi_j^{(2q)}(s_j) = (-1)^q \varphi_{j-1}^{(2q)}(s_j), \quad q = 0, 1, 2. \quad (3.16)$$

Then

$$\max_{\overline{\Pi}} |S^4(u, \varphi) - u| \leq c_5 h^4, \quad (3.17)$$

where u is the exact solution of problem (2.1), (2.2).

Proof According to Theorem 3.1 in [15], from conditions (3.15) and (3.16) it follows that $u \in C^{4,\lambda}(\overline{\Pi})$. Then on the basis of (3.1), (3.5), (3.8), (3.13), and Remark 3.2, we obtain inequality (3.17). \square

We define the function \widehat{u}_h as follows:

$$\widehat{u}_h = S^4(u_h, \varphi) \quad \text{on } \overline{\Pi}, \quad (3.18)$$

where u_h is the solution of the finite difference problem (2.6)-(2.8).

Theorem 3.4 *Let conditions (2.3) and (2.4) be satisfied. Then the function \widehat{u}_h is continuous on $\overline{\Pi}$, and*

$$\max_{(x,y) \in \overline{\Pi}} |\widehat{u}_h - u| \leq c_6 h^4, \quad (3.19)$$

where u is the solution of problem (2.1), (2.2).

Proof From the construction of the expression $S^4(u_h, \varphi)$ it follows that $\widehat{u}_h = u_h$ on Π^h , and $\widehat{u}_h = \varphi_j$ on $\gamma_j^h, j = 1, 2, 3, 4$. The continuity of \widehat{u}_h on Π follows from the continuity $S^4(u_h, \varphi)$ on each closed triangle Type A and Type B, and from the equality $\widehat{u}_h = u_h$ on Π^h . By Remark 3.2 and from the condition $\widehat{u}_h = \varphi_j$ on $\gamma_j^h, j = 1, 2, 3, 4$, the continuity of the function \widehat{u}_h on the closed rectangle $\overline{\Pi}$ follows. By virtue of (2.3) and (2.4) it follows that $u \in C^{6,\lambda}(\overline{\Pi}), 0 < \lambda < 1$ (see Theorem 3.1 in [15]). Then, on the basis of (3.6), (3.7), (3.9),

(3.11) Theorem 2.2, Theorem 3.3 and (3.18), we obtain

$$\begin{aligned} \max_{(x,y) \in \overline{\Pi}} |\widehat{u}_h - u| &\leq \max_{(x,y) \in \overline{\Pi}} |S^4(u, \varphi) - u| + \max_{(x,y) \in \overline{\Pi}} |S^4(u_h - u, 0)| \\ &\leq c_5 h^4 + \sum_{k=0}^{16} \xi_k \max_{(x,y) \in \overline{\Pi}^h} |u_h - u| \leq c_6 h^4. \end{aligned} \quad \square$$

4 An application of the matching operator in the Block-Grid method

Let G be an open simply connected staircase polygon, let $\gamma_j, j = 1, 2, \dots, N$, be its sides, including the ends, and let $\alpha_j \pi, \alpha_j \in \{\frac{1}{2}, 1, \frac{3}{2}, 2\}$, be the interior angle formed by the sides γ_{j-1} and γ_j ($\gamma_0 = \gamma_N$). Furthermore, let s be the arc length measured along the boundary of G in the positive direction and s_j be the value of s at the vertex $A_j = \gamma_{j-1} \cap \gamma_j$, (r_j, θ_j) be a polar system of coordinates with pole in A_j and the angle θ_j taken counterclockwise from the side γ_j .

We consider the boundary value problem

$$\Delta u = 0 \quad \text{on } G, \quad u = \varphi_j \quad \text{on } \gamma_j, j = 1, 2, \dots, N, \tag{4.1}$$

where φ_j are given functions, and

$$\varphi_j \in C^{6,\lambda}(\gamma_j), \quad 0 < \lambda < 1, 1 \leq j \leq N. \tag{4.2}$$

Moreover, at the vertices A_j for $\alpha_j = \frac{1}{2}$ the conjugation conditions

$$\varphi_j^{(2q)}(s_j) = (-1)^q \varphi_{j-1}^{(2q)}(s_j), \quad q = 0, 1, 2, 3, \tag{4.3}$$

are satisfied. At the vertices A_j for $\alpha_j \neq \frac{1}{2}$ no compatibility conditions for boundary functions are required; in particular the values of φ_{j-1} and φ_j at these vertices might be different. Additionally, it is required that when $\alpha_j \neq 1/2$, the boundary functions on γ_{j-1} and γ_j are given as algebraic polynomials of arclength s measured along γ .

Let $E = \{j : \alpha_j \neq 1/2, j = 1, 2, \dots, N\}$. We call the vertices $A_j, j \in E$, the singular vertices of the polygon G . We construct two fixed block sectors in the neighborhood of $A_j, j \in E$, denoted by $T_j^i = T_j(r_{ji}) \subset G, i = 1, 2$, where $0 < r_{j2} < r_{j1} < \min\{s_{j+1} - s_j, s_j - s_{j-1}\}$, $T_j(r) = \{(r_j, \theta_j) : 0 < r_j < r, 0 < \theta_j < \alpha_j \pi\}$. On the closed sector $\overline{T}_j^1, j \in E$, we consider the function $Q_j(r_j, \theta_j)$, which has the following properties:

- (i) $Q_j(r_j, \theta_j)$ is harmonic and bounded on the open sector T_j^1 ;
 - (ii) continuous everywhere on \overline{T}_j^1 apart from the point $A_j, j \in E$ when $\varphi_{j-1} \neq \varphi_j$;
 - (iii) continuously differentiable on $\overline{T}_j^1 \setminus A_j$;
 - (iv) satisfies the given boundary conditions on $\gamma_{j-1} \cap \overline{T}_j^1$ and $\gamma_j \cap \overline{T}_j^1, j \in E$.
- The function $Q_j(r_j, \theta_j)$ with properties (i)-(iv) is given in [17].

Let

$$R_j(r_j, \theta_j, \eta) = \frac{1}{\alpha_j} \sum_{k=0}^1 (-1)^k R \left(\left(\frac{r}{r_{j2}} \right)^{1/\alpha_j}, \frac{\theta}{\alpha_j}, (-1)^k \frac{\eta}{\alpha_j} \right), \quad j \in E, \tag{4.4}$$

where

$$R(r, \theta, \eta) = \frac{1 - r^2}{2\pi(1 - 2r \cos(\theta - \eta) + r^2)} \tag{4.5}$$

is the kernel of the Poisson integral for a unit circle.

The approximation of the integral representation given in the following lemma is used to construct an approximate solution of problem (4.1) around the singular vertices $A_j, j \in E$.

Lemma 4.1 *The solution u of problem (2.1), (2.2) can be represented on $\overline{T_j^2} \setminus V_j, j \in E$, in the form*

$$u(r_j, \theta_j) = Q_j(r_j, \theta_j) + \int_0^{\alpha_j \pi} (u(r_{j2}, \eta) - Q_j(r_{j2}, \eta)) R_j(r_j, \theta_j, \eta) d\eta, \tag{4.6}$$

where V_j is the curvilinear part of the boundary of the sector T_j^2 .

Proof The proof follows from Theorems 3.1 and 5.1 in [17]. □

We define the approximate solution in the whole polygon G by applying a version of the Block-Grid method introduced in [8] (see also [9]).

Let us consider, in addition to the sectors T_j^1, T_j^2 , the sectors T_j^3 and T_j^4 , which are also in the neighborhood of each vertex $A_j, j \in E$, of the polygon G , with $0 < r_{j4} < r_{j3} < r_{j2}$, $r_{j3} = (r_{j2} + r_{j4})/2$ and $T_k^3 \cap T_l^3 = \emptyset, k \neq l$, where $k, l \in E$. Furthermore, let $G_T = G \setminus (\bigcup_{j \in E} T_j^4)$. We give the description of the Block-Grid method on a hexagonal grid:

- (i) All singular corners $A_j, j \in E$, are separated by the double sectors $T_j^i = T_j(r_{ji}), i = 2, 3$, with $r_{j3} < r_{j2}, T_k^2 \cap T_l^2 = \emptyset, k \neq l$ and $k, l \in E$. The polygon is covered by overlapping rectangles $\Pi_k, k = 1, 2, \dots, M$, and sectors $T_j^3, j \in E$, such that the distance from Π_k to a singular point A_j is greater than r_{j4} for all $k = 1, 2, \dots, M$ and $j \in E$.
- (ii) On each rectangle Π_k , the seven point difference scheme for the approximation of Laplace's equation on a hexagonal grid is used, with step size $h_k \leq h, h$ is a parameter, and as an approximate solution on $\overline{T_j^3}, j \in E$, the harmonic function (4.6) is used.
- (iii) We use the matching operator S^4 constructed in Section 3 to connect the subsystems.

For obtaining the numerical solution of the algebraic system of equations (2.1), (2.2), we outline the procedure: Let $\Pi_k \subset G_T, k = 1, 2, \dots, M$, be certain fixed open rectangles with sides a_{1k} and a_{2k} parallel to the sides of G , and $G \subset (\bigcup_{k=1}^M \Pi_k) \cup (\bigcup_{j \in E} T_j^3) \subset G$. We use η_k to denote the boundary of the rectangle Π_k, V_j is the curvilinear part of the boundary of the sector T_j^2 and $t_j = (\bigcup_{k=1}^M \eta_k) \cap \overline{T_j^3}$.

The overlapping condition is imposed on the arrangement of the rectangles $\Pi_k, k = 1, 2, \dots, M$: any point P lying on $\eta_k \cap G_T, 1 \leq k \leq M$, or located on $V_j \cap G, j \in E$, falls inside at least one of the rectangles $\Pi_{k(P)}, 1 \leq k(P) \leq M$, where the distance from P to $G_T \cap \eta_{k(P)}$ is not less than some constant \varkappa_0 independent of P . The quantity \varkappa_0 is called the gluing depth of the rectangles $\Pi_k, k = 1, 2, \dots, M$.

We introduce the parameter $h \in (0, \varkappa_0/4]$ and consider a hexagonal grid on $\Pi_k, k = 1, 2, \dots, M$, with maximal possible step $h_k \leq \min\{h, \min\{a_{1k}, a_{2k}\}/4\}$. Let Π_k^h be the set of nodes on Π_k , let η_k^h be the set of nodes on η_k , and let $\overline{\Pi_k^h} = \Pi_k^h \cap \eta_k^h$. We denote the set

of nodes on the closure of $\eta_k \cap G_T$ by η_{k0}^h , and the set of nodes on Π_k^h whose distance from the boundary $\eta_k \cap G_T$ of Π_k is $\frac{h}{2}$ by η_{k0}^{*h} . We also have Π_k^{*h} denoting the set of nodes whose distance from the boundary η_{k1} of Π_k is $\frac{h}{2}$ and $\Pi_k^{0h} = \Pi_k^h \setminus (\Pi_k^{*h} \cup \eta_{k0}^{*h})$. Let t_j^h be the set of nodes on t_j , and let η_{k1}^h be the set of remaining nodes on η_k . We also specify a natural number $n \geq [\ln^{1+\varkappa} h^{-1}] + 1$, where $\varkappa > 0$ is a fixed number and the quantities $n(j) = \max\{4, [\alpha_j n]\}$, $\beta_j = \alpha_j \pi / n(j)$, and $\theta_j^m = (m - 1/2)\beta_j$, $j \in E$, $1 \leq m \leq n(j)$. On the arc V_j we choose the points (r_{j2}, θ_j^m) , $1 \leq m \leq n(j)$, and denote the set of these points by V_j^n . Finally, let

$$\omega^{h,n} = \left(\bigcup_{k=1}^M \eta_{k0}^h \right) \cup \left(\bigcup_{k=1}^M \eta_{k0}^{*h} \right) \cup \left(\bigcup_{j \in E} V_j^n \right), \quad \bar{G}_*^{h,n} = \omega^{h,n} \cup \left(\bigcup_{k=1}^M \bar{\Pi}_k^h \right).$$

Consider the system of equations

$$u_h = Su_h \quad \text{on } \Pi_k^{0h}, \tag{4.7}$$

$$u_h = S_m^* u_h + E_{mh}^*(\varphi_m) \quad \text{on } \Pi_k^{*h}, \eta_{k1}^h \cap \gamma_m \neq \emptyset, \tag{4.8}$$

$$u_h = \varphi_m \quad \text{on } \eta_{k1}^h \cap \gamma_m, \tag{4.9}$$

$$u_h(r_j, \theta_j) = Q_j(r_j, \theta_j) + \beta_j \sum_{k=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^k) (u_h(r_{j2}, \theta_j^k) - Q_j(r_{j2}, \theta_j^k)) \quad \text{on } t_j^h, \tag{4.10}$$

$$u_h = S^4(u_h, \varphi) \quad \text{on } \omega^{h,n}, \tag{4.11}$$

where $1 \leq k, m \leq M$, $j \in E$, $\varphi = \{\varphi_j\}_{j=1}^N$; Su_h , $S_m^* u_h$ and $E_{mh}^*(\varphi_m)$ are defined as equations (2.9), (2.10), and (2.11) in Section 2, respectively.

The solution of the system of equations (4.7)-(4.11) is a numerical solution of problem (2.1), (2.2) on \bar{G}_T ('nonsingular' part of the polygon G).

Theorem 4.2 *There is a natural number n_0 such that for all $n \geq n_0$ and $h \in (0, \frac{\varkappa_0}{4}]$, where \varkappa_0 is the gluing depth, the system of equations (4.7)-(4.11) has a unique solution.*

Proof Let v_h be a solution of the system of equations

$$u_h = Su_h \quad \text{on } \Pi_k^{0h},$$

$$u_h = S_m^* u_h \quad \text{on } \Pi_k^{*h}, \eta_{k1}^h \cap \gamma_m \neq \emptyset,$$

$$u_h = 0 \quad \text{on } \eta_{k1}^h \cap \gamma_m, \tag{4.12}$$

$$u_h(r_j, \theta_j) = \beta_j \sum_{k=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^k) u_h(r_{j2}, \theta_j^k) \quad \text{on } t_j^h, \tag{4.13}$$

$$u_h = S^4 u_h \quad \text{on } \omega^{h,n},$$

where $1 \leq k, m \leq M$, $j \in E$. To prove the given theorem, we show that $\max_{\bar{G}_*^{h,n}} |v_h| = 0$. On the basis of the structure of operators S and S_j^* , and the forms (3.5), (3.6), (3.7), (3.8), (3.9), and (3.10)-(3.11) of the matching operator S^4 and by the maximum principle (see Ch. 4,

[11]) it follows that the nonzero maximum value of the function v_h can be at the points on $\bigcup_{j \in E} t_j^h$. From estimation (2.29) in [18] the existence of the positive constants n_0 and $\sigma > 0$ such that for $n \geq n_0$

$$\max_{(r_j, \theta_j) \in \overline{T_j^3}} \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q) \leq \sigma < 1 \tag{4.14}$$

follows. However, taking (4.14) into account in (4.13) we have that the nonzero maximum value can not be at the points on $\bigcup_{j \in E} t_j^h$ either. Since the set $\overline{G_*^{h,n}}$ is connected, from equation (4.12) it follows that $\max_{\overline{G_*^{h,n}}} |v_h| = 0$. \square

Let u_h be the solution of the system of equations (4.7)-(4.11). The function

$$U_h(r_j, \theta_j) = Q_j(r_j, \theta_j) + \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q) (u_h(r_{j2}, \theta_j^q) - Q_j(r_{j2}, \theta_j^q)) \tag{4.15}$$

is the approximation of the integral representation (4.6) with the use of the composite mid-point rule. We use the function $U_h(r_j, \theta_j)$ as an approximate solution of problem (2.1), (2.2) on the closed block $\overline{T_j^3}$, $j \in E$ ('singular' parts of the polygon G).

Let

$$\epsilon_h = u_h - u, \tag{4.16}$$

where u_h is the solution of system (4.7)-(4.11) and u is the trace of the solution of (2.1), (2.2) on $\overline{G_*^{h,n}}$. On the basis of (2.1), (2.2), (4.7)-(4.11), and (4.16), ϵ_h satisfies the following difference equations:

$$\epsilon_h = S\epsilon_h + r_h^1 \quad \text{on } \Pi_k^{0h}, \tag{4.17}$$

$$\epsilon_h = S_m^* \epsilon_h + r_h^2 \quad \text{on } \Pi_k^{*h}, \eta_{k1}^h \cap \gamma_m \neq \emptyset, \quad \epsilon_h = 0 \quad \text{on } \eta_{k1}^h \cap \gamma_m, \tag{4.18}$$

$$\epsilon_h(r_j, \theta_j) = \beta_j \sum_{k=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^k) \epsilon_h(r_{j2}, \theta_j^k) + r_{jh}^3, \quad (r_j, \theta_j) \in t_{kj}^h, \tag{4.19}$$

$$\epsilon_h = S^4 \epsilon_h + r_h^4 \quad \text{on } \omega^{h,n}, \tag{4.20}$$

where $1 \leq k, m \leq M, j \in E$ and

$$r_h^1 = Su - u \quad \text{on } \bigcup_{k=1}^M \Pi_k^{0h}, \quad r_h^2 = S_m^* u + E_{mh}^*(\varphi_m) - u \quad \text{on } \bigcup_{1 \leq k \leq M} \Pi_k^{*h}, \tag{4.21}$$

$$r_{jh}^3 = \beta_j \sum_{k=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^k) (u(r_{j2}, \theta_j^k) - Q_j(r_{j2}, \theta_j^k)) - (u(r_j, \theta_j) - Q_j(r_j, \theta_j)) \quad \text{on } \bigcup_{j \in E} t_j^h, \tag{4.22}$$

$$r_h^4 = S^4(u, \varphi) - u \quad \text{on } \omega^{h,n}. \tag{4.22}$$

Since all the rectangles $\Pi_k, k = 1, 2, \dots, M$ are located away from the singular vertices $A_j, j \in E$ of the polygon G , at a distance greater than $r_{j4} > 0$ independent of h , by virtue of the conditions (2.3) and (2.4), up to sixth order derivatives of the solution of problem (2.1), (2.2) are bounded on $\bigcup_{k=1}^M \Pi_k$. Then, by Taylor's formula, from (4.21) we obtain

$$\max_{\bigcup_{k=1}^M \Pi_k^{0h}} |r_h^1| \leq c_1 h^6, \quad \max_{\bigcup_{k=1}^M \Pi_k^{*h}} |r_h^2| \leq c_2 h^4. \tag{4.23}$$

Furthermore, as $\omega^{h,n} \subset \bigcup_{k=1}^M \Pi_k$ from (4.22) and Theorem 3.3, we have

$$\max_{\omega^{h,n}} |r_h^4| \leq c_3 h^4. \tag{4.24}$$

By analogy to the proof of Lemma 6.2 in [9], it is shown that there exists a natural number n_0 , such that for all $n \geq \max\{n_0, [\ln^{1+\varkappa} h^{-1}] + 1\}$, $\varkappa > 0$ being a fixed number,

$$\max_{j \in E} |r_{jh}^3| \leq c_4 h^4. \tag{4.25}$$

Theorem 4.3 *There exists a natural number n_0 such that for all $n \geq \max\{n_0, [\ln^{1+\varkappa} h^{-1}]\}$, $\varkappa > 0$ being a fixed number,*

$$\max_{\overline{G}_*^{h,n}} |u_h - u| \leq c h^4. \tag{4.26}$$

Proof The proof follows from estimations (4.23)-(4.25) and the principle of maximum by analogy to the proof of Theorem 6.1 in [9]. □

Theorem 4.4 *Let u_h be the solution of the system of equations (4.7)-(4.11), and let an approximate solution of problem (2.1), (2.2) be found on blocks $\overline{T}_j^3, j \in E$, by (4.15). There is a natural number n_0 such that for all $n \geq \max\{n_0, [\ln^{1+\varkappa} h^{-1}]\}$, $\varkappa > 0$ being a fixed number, the following estimations hold:*

$$|U_h(r_j, \theta_j) - u(r_j, \theta_j)| \leq c_0 h^4 \quad \text{on } \overline{T}_j^3, j \in E, \tag{4.27}$$

$$\left| \frac{\partial^p}{\partial x^{p-q} \partial y^q} (U_h(r_j, \theta_j) - u(r_j, \theta_j)) \right| \leq c_p h^4 / r_j^{p-1/\alpha_j} \quad \text{on } \overline{T}_j^3 \setminus A_j, j \in E, \tag{4.28}$$

where $0 \leq q \leq p, p = 0, 1, \dots$

Proof Estimation (4.27) is obtained from the integral representation (4.6) and formula (4.15) by using estimations (4.25) and (4.26). Estimation (4.28) for $p = 0, 1, \dots$ is obtained by using inequality (4.27) and Lemma 6.12 in [17]. □

5 Numerical results and discussion

To support the theoretical results, numerical examples have been solved in two different domains.

Example 5.1 Approximation in a rectangular domain. Consider the rectangular domain

$$\Pi = \left\{ (x, y) \in D : 0 < x < 1, 0 < y < \frac{\sqrt{3}}{2} \right\},$$

Table 1 Approximation in a rectangle with smooth exact solution

h	$\ \epsilon_h\ _{\Pi^h}$	$R_{\Pi^h}^m$
2^{-3}	1.15727×10^{-7}	
2^{-4}	7.33698×10^{-9}	15.7731
2^{-5}	4.58658×10^{-10}	15.9966
2^{-6}	2.89896×10^{-11}	15.4765
2^{-7}	2.02482×10^{-12}	14.3171

Table 2 Approximation in a rectangle with less smooth exact solution

h	$\ \epsilon_h\ _{\Pi^h}$	$R_{\Pi^h}^m$
2^{-3}	1.9285677×10^{-4}	
2^{-4}	1.1998304×10^{-5}	16.0737
2^{-5}	7.4809403×10^{-7}	16.0385
2^{-6}	$4.67808169 \times 10^{-8}$	15.9915
2^{-7}	2.922653×10^{-9}	16.0063

with the boundary γ . The hexagonal grid (2.5), denoted Π^h , is assigned to the grid Π , where γ^h denotes the set of nodes on the boundary γ .

We consider the problem

$$\begin{aligned} \Delta u &= 0 \quad \text{on } \Pi^h, \\ u &= v(x, y) \quad \text{on } \gamma^h, \end{aligned}$$

where

$$v(x, y) = e^y \sin x \tag{5.1}$$

is the exact solution in the rectangular domain.

This example is solved using the incomplete LU-decomposition method (see [19], Ch. 5), and all the calculations are carried out in double precision. As a convergence test, we request the maximum residual error to be 10^{-12} , and as a starting point $v_h = 0$ is used.

Table 1 gives the values obtained in the maximum norm of the difference between the exact and the approximate solutions, for the values of $h = 2^{-k}$, $k = 3, 4, 5, 6, 7$, i.e., $\|\epsilon_h\|_{\Pi^h} = \max_{\Pi^h} |v - v_h|$. The ratios $R_{\Pi^h}^m = \frac{\|v - v_{2^{-m}}\|_{\Pi^h}}{\|v - v_{2^{-(m+1)}}\|_{\Pi^h}}$ have also been included, where $O(h^4)$ order of accuracy corresponds to 2^4 of the value $R_{\Pi^h}^m$.

Example 5.2 Less smooth function. We consider the same problem as in Example 5.1 with the exact solution

$$v(x, y) = \frac{1}{2} \ln(x^2 + y^2) \operatorname{Re} z^7 - \tan^{-1}\left(\frac{y}{x}\right) \operatorname{Im} z^7, \tag{5.2}$$

which is less smooth than (5.1). The results obtained are consistent with the theoretical results and are summarized in Table 2.

Example 5.3 The matching operator. Examples of the matching operator have also been considered in the domain Π . The coordinate $P_1(0.55, 0.4387)$ is chosen, where $P_1 \in \Pi_0$.

Table 3 Results for approximation of inner points with the matching operator

$S^4 u$	1.56912199976621
Exact	1.56912199014188
$ \epsilon_h(P_1) $	9.624329×10^{-9}

Table 4 Results for approximation of near boundary points with the matching operator

h	$ \epsilon_h(P_2) $	$ \epsilon_h(P_3) $
2^{-4}	1.716286×10^{-8}	$3.85255412 \times 10^{-8}$
2^{-5}	5.385032×10^{-10}	4.3619541×10^{-9}
2^{-6}	$2.2436186 \times 10^{-10}$	$3.41679468 \times 10^{-10}$
2^{-7}	$2.4942270 \times 10^{-11}$	$2.8927971 \times 10^{-12}$

The harmonic function

$$u(x, y) = e^x \cos y \tag{5.3}$$

is assumed to be the exact solution. The result in Table 3 is obtained using $h = 2^{-4}$ and demonstrates high accuracy of the above constructed matching operator.

The second coordinate considered demonstrates the accuracy of the approximation of near-boundary points. The point chosen is $P_2(0.195938, 0.02)$, where $P_2 \in \Pi_{01}$, and equation (3.13) is used for approximation. Again, the harmonic function (5.3) is used as the exact solution. Lastly, a point near one of the corners of the domain $P_3(0.005, 0.005)$ has been considered, where the nodes of evaluation emerge outside of the domain from both adjacent sides of the corner. The function

$$u(x, y) = e^y \cos x$$

is used as the exact solution. The results obtained are summarized in Table 4.

Example 5.4 Approximation in an L-shaped domain. The final example is solved in an L-shaped domain with an angle singularity at the origin, where $\alpha_1\pi = 3\pi/2$. The domain is defined by

$$\Omega = \left\{ (x, y) : -1 \leq x \leq 1, -\frac{\sqrt{3}}{2} \leq y \leq \frac{\sqrt{3}}{2} \right\} \setminus \Omega_1,$$

where $\Omega_1 = \{(x, y) : 0 \leq x \leq 1, -\frac{\sqrt{3}}{2} \leq y \leq 0\}$, and is covered by four overlapping rectangles and a sector. The singular part is defined to be the region

$$\Omega^S = \left\{ (x, y) : -\frac{1}{2} \leq x \leq \frac{1}{2}, -\frac{\sqrt{3}}{4} \leq y \leq \frac{\sqrt{3}}{4} \right\} \setminus \Omega_1^S,$$

where $\Omega_1^S = \{(x, y) : 0 \leq x \leq \frac{1}{2}, -\frac{\sqrt{3}}{4} \leq y \leq 0\}$, and the nonsingular part is $\Omega^{NS} = \Omega/\Omega^S$. The system of Block-Grid equations is solved by Schwarz's alternating method. The quadrature nodes on the circular arc, whose radius is taken as 0.75, and the overlapping boundaries of the rectangles are renewed after each Schwarz's iteration. The nodes on the circular arc, the inner boundaries of the overlapping rectangles and the nodes in the set $\bigcup_{k=1}^4 \eta_{k0}^{*h}$ are

Table 5 The order of convergence in ‘nonsingular’ part

(h, N)	$\ \epsilon_h\ _{\Omega^{NS}}$	$R_{\Omega^{NS}}^m$
$(2^{-4}, 40)$	9.9742×10^{-4}	27.9237
$(2^{-5}, 60)$	3.57195×10^{-5}	
$(2^{-5}, 100)$	8.2649×10^{-7}	15.3796
$(2^{-6}, 100)$	5.373923×10^{-8}	
$(2^{-6}, 100)$	5.373923×10^{-8}	15.7215
$(2^{-7}, 125)$	3.418192×10^{-9}	

Table 6 The order of convergence in ‘singular’ part

(h, N)	$\ \epsilon_h\ _{\Omega^S}$	$R_{\Omega^S}^m$
$(2^{-4}, 40)$	9.9742×10^{-4}	11.7569
$(2^{-5}, 60)$	3.57195×10^{-5}	
$(2^{-5}, 100)$	8.2649×10^{-7}	14.9731
$(2^{-6}, 100)$	5.373923×10^{-8}	
$(2^{-6}, 100)$	5.373923×10^{-8}	16.5665
$(2^{-7}, 125)$	3.418192×10^{-9}	

renewed using the matching operator constructed above. Since the boundary functions are harmonic polynomials on the sides γ_1 and $\gamma_0 \equiv \gamma_6$, the nodes whose neighbors emerge outside of the domain from these sides are approximated using the function $u - Q_1$. Finally, the solution on the singular part is approximated using the integral representation [3, 9].

The problem considered is

$$\begin{aligned} \Delta u &= 0 && \text{on } \Omega^h, \\ u &= v(x, y) && \text{on } \gamma^h, \end{aligned}$$

where

$$v(x, y) = \theta + r^{2/3} \sin\left(\frac{2}{3}\theta\right) + \operatorname{Re} z^5 + \operatorname{Im} z^5,$$

is the exact solution. Accordingly, the function $Q_1(x, y)$ used in the integral representation is constructed as

$$Q_1(x, y) = \theta + r^5 (\cos(5\theta) + \sin(5\theta)).$$

The results in Tables 5 and 6 show the solution for different pairs (h, N) , where N is the number of quadrature nodes, h is the mesh size of the hexagonal grid.

6 Conclusions

The fourth order classical 7-point scheme on a hexagonal grid is applied on a rectangular domain. This leads to some of the nodes emerging from the two parallel sides while approximating points whose distance from the boundary is $\frac{h}{2}$. This problem was overcome by devising an approximating equation for near-boundary nodes, which included the use of three inner nodes around the point of evaluation and three points lying on the boundary. The fourth order matching operator has been constructed on the hexagonal

grid functions. It is applied to construct a fourth order accurate interpolating function, on the closed rectangle, for the numerical solution of Laplace's equation on the hexagonal grids. Further, the matching operator and the hexagonal grid approximation in a rectangle are used to obtain and justify the Block-Grid method in solving the Dirichlet problem for Laplace's equation on staircase polygons.

Numerical examples have been provided as an illustration of the theoretical results mentioned above.

The matching operator constructed can be applied to many other forms of domain decomposition or combined methods. It will also be an interesting study to extend the approximation of the methods to using mixed or Neumann boundary conditions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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