Rumyantsev *Boundary Value Problems* 2014, 2014:53
http://www.boundaryvalueproblems.com/content/2014/1/53

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

On the constructive investigation of a class of linear boundary value problems for nth order differential equations with deviating arguments

Aleksandr Rumyantsev*

*Correspondence:
arumyanzev@gmail.com
Department of Applied
Mathematics and Informatics, Perm
State University, Bukireva street, 15,
Perm, 614990, Russia

Abstract

A constructive technique for the study of boundary value problems for nth order differential equations with deviating arguments is described. A computer-assisted proof of the correct solvability of the considered problem is given.

MSC: Primary 34B05; 34K06; 65L10; secondary 34K10; 65L70

Keywords: differential equations with deviating arguments; boundary value problems; constructive methods; unique solvability; approximate solution

Introduction

Let us cite some facts from the theory of functional differential equations [1] and the general description of the constructive approach to the investigation of boundary value problems for such equations [2, 3].

Consider the linear boundary value problem

$$\mathcal{L}y = f, \qquad \ell y = \beta, \tag{1}$$

where $\mathcal{L} : DS_p^n[0,T](m) \to L_p^n[0,T]$ is a bounded linear operator, $\ell : DS_p^n[0,T](m) \to R^n$ is a bounded linear vector functional, and $f \in L_p^n[0,T]$, $\beta \in R^{mn+n}$. R^n denotes the linear space of real columns α, where $\alpha = \operatorname{col}\{\alpha_1,\dots,\alpha_n\}$ with the norm $\|\alpha\|_n = \max_{1\le i\le n}|\alpha_i|$; $\lfloor\alpha\rfloor \stackrel{\text{def}}{=} \operatorname{col}\{|\alpha_1|,\dots,|\alpha_n|\}$; $L_p^n[0,T]$, $1 \le p < \infty$, denotes the Banach space of measurable functions $z : [0,T] \to R^n$, $z(\cdot) = \operatorname{col}\{z_1(\cdot),\dots,z_n(\cdot)\}$, such that

$$\|z\|_{L_p^n[0,T]} = \max_{1\le i\le n}\left(\int_0^T |z_i(s)|^p\,ds\right)^{\frac{1}{p}} < +\infty;$$

let us fix a collection of points $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$. Put $B_q = [t_{q-1}, t_q)$, $q = 1,\dots,m$; $B_{m+1} = [t_m, T]$;

$$\chi_q(t) = \begin{cases} 1, & \text{if } t \in B_q, \\ 0, & \text{if } t \notin B_q; \end{cases}$$

$DS_p^n[0,T](m)$ denotes the Banach space of all functions $y:[0,T]\to R^n$ with $\dot{y}\in L_p^n[0,T]$ and the representation:

$$y(t)=y(0)+\int_0^t \dot{y}(s)\,ds+\sum_{q=1}^{m}\chi_{[t_q,T]}(t)\Delta y(t_q),$$

where $\Delta y(t_q)=y(t_q)-y(t_q-0)$; $\chi_{[t_q,T]}(t)=\left\{\begin{smallmatrix}1, & \text{if } t\in[t_q,T],\\ 0, & \text{if } t\notin[t_q,T];\end{smallmatrix}\right.$ $\Delta y \overset{\text{def}}{=} \operatorname{col}\{y(0),\Delta y(t_1),\ldots,\Delta y(t_m)\}$, with the norm

$$\|y\|_{DS_p^n[0,T]}(m)=\|\dot{y}\|_{L_p^n[0,T]}+\|\Delta y\|_{mn+n}.$$

We assume that the principal boundary value problem

$$\mathcal{L}y=f,\qquad \Delta y=\alpha,\quad f\in L_p^n[0,T],\alpha\in R^{mn+n},\tag{2}$$

is correctly solvable. Then under these assumptions there exists an $(mn+n)\times(mn+n)$ fundamental matrix $\mathbf{Y}$ for the homogeneous equation:

$$\mathcal{L}y=0.\tag{3}$$

The following statement holds: the problem (1) is uniquely solvable for any f, α if and only if the $(mn+n)\times(mn+n)$ matrix $\boldsymbol{\Gamma}$,

$$\boldsymbol{\Gamma}\overset{\text{def}}{=}\ell\mathbf{Y}\tag{4}$$

(each column of the $(mn+n)\times(mn+n)$ matrix $\boldsymbol{\Gamma}$ is the result of applying of the functional ℓ to the corresponding column of the matrix $\mathbf{Y}$) is invertible. The problem (2) is correctly solvable for a broad class of equations, including

- the ordinary differential equation

$$\mathcal{L}y\overset{\text{def}}{=}\dot{y}(t)+P(t)y(t)=f(t),\quad t\in[0,T];$$
$$P(\cdot)=\{p(\cdot)_{ij}\}_{i,j=1}^{n},\qquad p_{ij}\in L_p^1[0,T];$$

- and a differential equation with concentrated delays

$$\mathcal{L}y\overset{\text{def}}{=}\dot{y}_i(t)+\sum_{j=1}^{n}p_{ij}(t)y_j\big[h_{ij}(t)\big]=f_i(t),\quad t\in[0,T];$$
$$h_{ij}(\xi)=0,\quad \xi<0,$$
$$p_{ij}\in L_p^1[0,T],\qquad h_{ij}\le t,\quad i,j=1,\ldots n.$$

It should be noted that in the more general case of a differential equation with deviating arguments, the problem (2) does not have this property and some further investigation is required. For example, define the space $DS_1^1(1)$ on the partition $[0,\frac{1}{2},1]$ and consider the principal boundary value problem

$$\dot{x}(t)=x(1)+f(t),\quad t\in[0,1],\qquad \Delta x=\operatorname{col}\{\alpha,0\},\quad f\in L_1^1[0,1],\alpha\in R.$$

It easy to see that this problem has a unique solution only for $\alpha=-\int_0^1 f(s)\,ds$.

The key idea of the constructive study of the solvability of the problem (1) is as follows.

- Two $(mn+n) \times (mn+n)$ matrices, ${}^a\mathbf{\Gamma}$ and ${}^v\mathbf{\Gamma}$, with rational elements are constructed according to a specially developed procedure based on a computer-assisted proof, such that

$$\lfloor \mathbf{\Gamma} - {}^a\mathbf{\Gamma} \rfloor \leq \lfloor {}^v\mathbf{\Gamma} \rfloor;$$

  let $R^{n\times n}$ denotes the linear space of real $n \times n$-matrices $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ with the norm $\|\mathbf{A}\|_{R^{n\times n}} = \max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}|$; $\lfloor \mathbf{A} \rfloor \stackrel{\text{def}}{=} \{|a_{ij}|\}_{i,j=1}^n$.
- The invertibility of the matrix ${}^a\mathbf{\Gamma}$ is verified using exact arithmetic.
- If there exists an inverse matrix ${}^a\mathbf{\Gamma}^{-1}$, it should be checked whether

$$\|{}^v\mathbf{\Gamma}\|_{R^{mn+n}} < \frac{1}{\|{}^a\mathbf{\Gamma}^{-1}\|_{R^{mn+n}}} \tag{5}$$

  holds, from which, by the theorem on the inverse operator [4, p.207], it follows that the matrix $\mathbf{\Gamma}$ is invertible, *i.e.*, the boundary value problem (1) is correctly solvable.

Further, the suggested general scheme of the constructive investigation will be applied to the boundary value problem for the nth order differential equation with deviating arguments.

A class of functions and operators

The constructive techniques for the study of equations with deviating arguments described below are based on a specific approximation of original problems within the class of computable functions and operators [2]. In what follows, we assume that the spaces $DS_p^n[0,T](m)$ and $WS_p^n[0,T](m)$ are constructed by means of the partition

$$0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T, \tag{6}$$

where t_q, $q = 1,\ldots,m+1$, are rational numbers. The sets $B_q = [t_{q-1}, t_q)$, $q = 1,\ldots,m$; $B_{m+1} = [t_m, T]$, and the corresponding characteristic functions $\chi_q(\cdot)$ are defined with respect to the same partition. $WS_p^n[0,T](m)$ denotes the Banach space of functions $y : [0,T] \to R^1$, with $y^{(n)} \in L_p^1[0,T]$, $y^{(i)} \in DS_p^1[0,T](m)$, $i = 0,\ldots,n-1$, and the representation

$$\begin{aligned}
y(t) &= \int_0^t \frac{(t-s)^{(n-1)}}{(n-1)!} y^{(n)}(s)\,ds + \sum_{i=0}^{n-1} \frac{t^i}{i!} y^{(i)}(0) \\
&\quad + \sum_{i=0}^{n-1} \sum_{q=1}^{m} \frac{(t-t_q)^i}{i!} \chi_{[t_q,T]}(t) \Delta y^{(i)}(t_q); \\
\Delta y^{(i)}(t_q) &= y^{(i)}(t_q) - y^{(i)}(t_q - 0), \\
\Delta^n y &= \operatorname{col}\bigl\{y(0), y^{(1)}(0), \ldots, y^{(n-1)}(0), \Delta y(t_1), \Delta y^{(1)}(t_1), \ldots, \\
&\qquad \Delta y^{(n-1)}(t_1), \ldots, \Delta y^{(1)}(t_m), \Delta y^{(2)}(t_m), \ldots, \Delta y^{(n-1)}(t_m)\bigr\},
\end{aligned}$$

with the norm

$$\|y\|_{WS_p^n[0,T](m)} = \|\dot{y}\|_{L_p^1[0,T]} + \left\|\Delta^n y\right\|_{mn+n}.$$

$D_p^n[0,T]$ denotes the Banach space of absolutely continuous functions $x:[0,T]\to R^n$ such that $\dot{x}\in L_p^n[0,T]$, with the norm

$$\|x\|_{D_p^n[0,T]}=\|x(0)\|_n+\|\dot{x}\|_{L_p^n[0,T]}.$$

Definition 1 A function $y\in DS_p^n[0,T](m)$ is said to possess the property $\mathcal{C}$ (*is computable*) if its components as well as the components of the functions $\dot{y}(\cdot)$ and $\int_0^{(\cdot)}y(s)\,ds$ take rational values at rational values of their arguments.

Let $y\in DS_p^n[0,T](m)$. The property $\mathcal{C}$ is satisfied by functions y of the form

$$y(t)=\sum_{q=1}^{m}\chi_q(t)p_q(t),\quad t\in[0,T],\tag{7}$$

where the components $p_q:[0,T]\to R^n$, $q=1,\dots,m$ are polynomials with rational coefficients. We denote by $\mathcal{P}_m^n$ the set of all $y\in DS_p^n[0,T](m)$ of the form (7).

Definition 2 A function $y\in WS_p^n[0,T](m)$ is said to possess the property $\mathcal{C}$ (*is computable*) if this function and the functions $y^{(i)}(\cdot)$, $i=1,\dots,n$ and $\int_0^{(\cdot)}y(s)\,ds$ take rational values at rational values of their arguments.

Obviously, the functions $y\in WS_p^n[0,T](m)$ with $y^{(n)}\in\mathcal{P}_m^n$ possess the property $\mathcal{C}$.

Definition 3 A function $h:[0,T]\to R^1$ is said to be computable over the partition (6) if h possesses the property $\mathcal{C}$ and for every $j=1,\dots,m$ there exists an integer q_j, $0\le q_j\le j$, such that $h(t)\in B_{q_j}$ as $t\in B_j$.

An example of a function that is computable over the partition (6) is $h:[0,T]\to R^1$ such that

$$h(t)=\sum_{q=1}^{m+1}\chi_q(t)h_q,\quad 0\le h_q\le t_q, t\in[0,T],$$

where h_q, $q=1,\dots,m+1$, are rational constants.

Definition 4 A function $h:[0,T]\to R^1$ is said to be computable over the partition (6) in the generalized sense if h possesses the property $\mathcal{C}$, and for every $j=1,\dots,m$, there exists an integer q_j, $0\le q_j\le m+1$, such that $h(t)\in B_{q_j}$, as $t\in B_j$.

An example of a function that is computable over the partition (6) in the generalized sense is $h:[1,T]\to R^1$ such that

$$h(t)=\sum_{q=1}^{m+1}\chi_q(t)h_q,\quad t\in[0,T],$$

where $h_q\in[0,T]$, $q=1,\dots,m+1$, are rational constants.

Definition 5 A bounded linear operator $\mathcal{L} : WS_p^n[0,T](m) \to L_p^n[0,T]$ is said to possess the property $\mathcal{C}$ (*is computable*) if it maps $\mathcal{P}_m^n$ into itself.

An example of an operator that is computable is

$$\begin{aligned}&(\mathcal{L}^n y)(t) \equiv y^{(n)}(t) + \sum_{i=0}^{n-1}\sum_{j=1}^{n_i} p_{ij}(t) y^{(i)}[h_{ij}(t)] = f(t),\\ &y^{(i)}(\xi) = 0, \quad \xi \notin [0,T], t \in [0,T],\end{aligned}$$

if the coefficients p_{ij} are the elements of the set $\mathcal{P}_m^n$ and the functions h_{ij} are computable over the partition (6) in the generalized sense.

Problem setting

Consider the linear boundary value problem

$$\begin{aligned}&(\mathcal{L}^n y)(t) \equiv y^{(n)}(t) + \sum_{i=0}^{n-1}\sum_{j=1}^{n_i} p_{ij}(t) y^{(i)}[h_{ij}(t)] = f(t), \qquad (8)\\ &y^{(i)}(\xi) = \begin{cases} \phi_i^0(\xi), & \xi < 0, \\ \phi_i^T(\xi), & \xi > T, \end{cases} \quad t \in [0,T],\\ &\ell^k y \equiv \int_0^T \varphi_k(s) y^{(n)}(s)\, ds + \sum_{i=0}^{n-1} \psi_{i0}^k y^{(i)}(0)\\ &\qquad + \sum_{i=0}^{n-1}\sum_{q=1}^{m} \psi_{iq}^k \Delta y^{(i)}(t_q) = \beta_k, \quad k = 1,\dots,mn+n,\end{aligned}$$

where $y \in WS_p^n[0,T](m)$, $1 \le p < \infty$, p_{ij}, and $f \in L_p^1[0,T]$, the $h_{ij}(\cdot)$ are continuous and strictly monotonic functions over every B_q, $i = 0,\dots,n-1$, $j = 1,\dots,n_i$,

$$\varphi_k \in \begin{cases} CS^1[0,T](m), & p = 1, \\ L_{p'}^1[0,T], & p > 1, \end{cases}$$

β_k and $\psi_{iq}^k \in R^1$, $k = 1,\dots,mn+n$, $q = 0,\dots,m$. $CS^n[0,T](m)$ denotes the Banach space of functions $x : [0,T] \to R^n$, defined by the equality

$$\begin{aligned}&x(t) = \sum_{q=1}^{m+1} \chi_q(t) x_q(t), \quad t \in [0,T],\\ &x_q(\cdot) = \operatorname{col}\{x_q^1(\cdot),\dots,x_q^n(\cdot)\},\end{aligned}$$

where x_q^i, $q = 1,\dots,m+1$, $i = 1,\dots,n$, are continuous functions, and

$$\|x\|_{CS^n[0,T](m)} = \max_{1\le i\le n}\sup_{t\in[0,T]} |x^i(t)|;$$

p' denotes the adjoint index to p:

$$p' = \begin{cases} \frac{p}{p-1}, & \text{if } p > 1, \\ \infty, & \text{if } p = 1. \end{cases}$$

Consider the principal boundary value problem corresponding to (8):

$$\begin{aligned}
&(\mathcal{L}^n y)(t) \equiv y^{(n)}(t) + \sum_{i=0}^{n-1}\sum_{j=1}^{n_i} p_{ij}(t)y^{(i)}[h_{ij}(t)] = f(t), \qquad (9)\\
&y^{(i)}(\xi) = \begin{cases} \varphi_i^0(\xi), & \xi < 0, \\ \varphi_i^T(\xi), & \xi > T, \end{cases} \qquad t \in [0, T],\\
&\Delta^n y = \alpha, \quad \alpha = \operatorname{col}\{\alpha_1, \dots, \alpha_{mn+n}\},
\end{aligned}$$

under the same assumptions on the problem parameters.

The procedure for the constructive study of the problem (8) consists of the following steps:

- approximation of the problem (8) within the class of computable functions and operators,
- study of the principal boundary value problem (9),
- analysis of its solvability.

Approximation of the problem within the class of computable operators

Fix $q = 1, \dots, m+1$. Approximate p_{ij} and f on the set B_q by polynomials ${}^a p_{ij}^q$ and ${}^a f_q$ with rational coefficients and define the rational error bounds:

$$^{\nu}p_{ij}^q \geq \|p_{ij} - {}^a p_{ij}^q\|_{L_p^1[t_{q-1},t_q]}, \qquad {}^{\nu}f_q \geq \|f - {}^a f_q\|_{L_p^1[t_{q-1},t_q]}.$$

Now define

$$^a p_{ij}(t) = \sum_{q=1}^{m+1} \chi_q(t)\, {}^a p_{ij}^q(t), \qquad {}^a f(t) = \sum_{q=1}^{m+1} \chi_q(t)\, {}^a f_q(t), \quad t \in [0, T],$$

for $i = 0, \dots, n-1$ and $j = 1, \dots, n_i$.

Approximation of the h_{ij}

Find rational approximations ${}^a\bar{h}_{ij}^q$ of $h_{ij}(t_q)$, $i = 0, \dots, n-1$, $j = 1, \dots, n_i$, $q = 1, \dots, m+1$, and define rational ${}^{\nu}h$ such that

$$^a\bar{h}_{ij}^q - {}^{\nu}h \leq h_{ij}(t_q) \leq {}^a\bar{h}_{ij}^q + {}^{\nu}h,$$

and define a rational constant $h_\triangle = GCD\{{}^a\bar{h}_{ij}^q\}$. Fix $i = 0, \dots, n-1$, $j = 1, \dots, n_i$, and construct the collection of points $\{{}^a h_{ij}^{\nu}\}$ by the rule:

$$\begin{aligned}
&^a h_{ij}^{\nu} = \min_{0\leq q\leq m+1}\{{}^a\bar{h}_{ij}^q\} + \nu\frac{\max_{0\leq q\leq m+1}\{{}^a\bar{h}_{ij}^q\} - \min_{0\leq q\leq m+1}\{{}^a\bar{h}_{ij}^q\}}{m_{ij}},\\
&\quad \nu = 0, \dots, m_{ij}, m_{ij} = h_\triangle \bar{\nu}_{ij};
\end{aligned}$$

here the positive integer parameter $\bar{\nu}_{ij}$ defines the accuracy of the approximation to h_{ij}, assuming that ${}^a h_{ij}^{\nu-1} + {}^{\nu}h < {}^a h_{ij}^{\nu} - {}^{\nu}h$, $\nu = 1, \dots, m_{ij}$. We denote by ν_{ij}^q, $q = 0, \dots, m+1$, a value of ν such that ${}^a h_{ij}^{\nu_{ij}^q} = {}^a\bar{h}_{ij}^q$. For $q = 1,\dots,m+1$, we denote $m_{ij}^q + 1$ for the number of values ${}^a h_{ij}^\nu$ that belong reliably to the inverse image of h_{ij} in the set B_q. We define the elements of the set $\mathcal{I}_{ij}^q = \{{}^a\tilde{h}_{ij}^{q\nu}\}_{\nu=0}^{m_{ij}^q}$ as follows:

1. ${}^a\tilde{h}_{ij}^{q\nu} = {}^a h_{ij}^{\nu_{ij}^{q-1}+\nu}$, $\nu = 0,\dots,m_{ij}^q$, if h_{ij} is strictly increasing on B_q;
2. ${}^a\tilde{h}_{ij}^{q\nu} = {}^a h_{ij}^{\nu_{ij}^{q}-\nu}$, $\nu = 0,\dots,m_{ij}^q$, if h_{ij} is strictly decreasing on B_q.

Next define the pairs of constants $[{}_1^a t_{ij}^{q\nu}, {}_1^\nu t_{ij}^{q\nu}]$, $[{}_2^a t_{ij}^{q\nu}, {}_2^\nu t_{ij}^{q\nu}]$, $\nu = 0,\dots,m_{ij}^q$, such that

$$h_{ij}^{-1}\left({}^a\tilde{h}_{ij}^{q\nu}\right) \in \left[{}_1^a t_{ij}^{q\nu}, {}_1^a t_{ij}^{q\nu} + {}_1^\nu t_{ij}^{q\nu}\right];$$
$$h_{ij}^{-1}\left({}^a\tilde{h}_{ij}^{q\nu} + {}^\nu h\right) \in \left[{}_2^a t_{ij}^{q\nu}, {}_2^a t_{ij}^{q\nu} + {}_2^\nu t_{ij}^{q\nu}\right].$$

We also assume that the following conditions hold:

$${}_1^a t_{ij}^{q\nu} + {}_1^\nu t_{ij}^{q\nu} < {}_2^a t_{ij}^{q\nu}; \qquad t_{q-1} < {}_1^a t_{ij}^{q1}; \qquad {}_2^a t_{ij}^{q(m_{ij}^q-1)} + {}_2^\nu t_{ij}^{q(m_{ij}^q-1)} < T,$$

which can always be satisfied by means of the accuracy of the calculation of the function h_{ij}^{-1} for the given points. Further, we construct the set $\mathcal{J}_{ij}^q = \{t_{ij}^{q\nu}\}$, $\nu = 0,\dots,m_{ij}^q$,

$$t_{ij}^{q\nu} = \frac{{}_1^a t_{ij}^{q\nu} + {}_1^\nu t_{ij}^{q\nu} + {}_2^a t_{ij}^{q\nu}}{2}.$$

From the order of the points $t_{ij}^{q\nu}$, $\nu = 0,\dots,m_{ij}^q$, it follows that

$$t_{q-1} = t_{ij}^{q0} < t_{ij}^{q1} < \cdots < t_{ij}^{q(m_{ij}^q-1)} < t_{ij}^{qm_{ij}^q} = t_q; \tag{10}$$
$${}^a\tilde{h}_{ij}^{q\nu} - {}^\nu h \le h\left(t_{ij}^{q\nu}\right) \le {}^a\tilde{h}_{ij}^{q\nu} + {}^\nu h, \quad \nu = 0,\dots,m_{ij}^q.$$

Next put

$$B_{ij}^{q\nu} = \left[t_{ij}^{q(\nu-1)}, t_{ij}^{q\nu}\right), \quad \nu = 1,\dots,m_{ij}^q - 1;$$
$$B_{ij}^{qm_{ij}^q} = \begin{cases} [t_{ij}^{q(m_{ij}^q-1)}, t_q), & q < m+1, \\ [t_{ij}^{q({}^q m_{ij}^k-1)}, T], & q = m+1, \end{cases}$$

and denote by $\chi_{ij}^{q\nu}(\cdot)$ the characteristic function of $B_{ij}^{q\nu}$, $\nu = 1,\dots,m_{ij}^q$. We construct the function ${}^a h_{ij}(\cdot)$ by the following rule:

$${}^a h_{ij}(t) = \sum_{q=1}^{m+1}\sum_{\nu=1}^{m_{ij}^q} \chi_{ij}^{q\nu}(t)\,{}^a\tilde{h}_{ij}^{q(\nu-1)}, \quad t \in [0,T]. \tag{11}$$

Define the set of points $\mathcal{J}$ by

$$\mathcal{J} = \bigcup_{i=0}^{n-1}\bigcup_{j=1}^{n_i}\bigcup_{q=1}^{m+1} \mathcal{J}_{ij}^q \tag{12}$$

(excluding duplicate elements). By construction, the ${}^a h_{ij}$, $i = 0,\dots,n-1$, $j = 1,\dots,n_i$, are computable over the partition (12) in the generalized sense.

Below suppose that $\phi_i^0 \in D_p^1[h_0^*, 0]$, $\phi_i^T \in D_p^1[T, h_T^*]$, $i = 0,\dots,n-1$, and define the constants h_0^* and h_T^* by the equalities

$$\begin{aligned}
&h_0^* = \min_{\substack{0\le i\le n-1\\ 1\le j\le n_i}} \{b_{ij}^0\},\\
&\quad b_{ij}^0 = \min\Big\{\min_{t\in[0,T]}\{h_{ij}(t)\}, \min_{0\le\nu\le m_{ij}}\{{}^a h_{ij}^\nu - {}^\nu h\}\Big\},\\
&h_T^* = \max_{\substack{0\le i\le n-1\\ 1\le j\le n_i}} \{b_{ij}^T\},\\
&\quad b_{ij}^T = \max\Big\{\min_{t\in[0,T]}\{h_{ij}(t)\}, \max_{0\le\nu\le m_{ij}}\{{}^a h_{ij}^\nu + {}^\nu h\}\Big\}.
\end{aligned}$$

We approximate the functions ϕ_i^0, ϕ_i^T by polynomials ${}^a\phi_i^0$, ${}^a\phi_i^T$ with rational coefficients and with rational error bounds $\{{}_0^\nu\phi_i^0, {}_1^\nu\phi_i^0\}$, $\{{}_0^\nu\phi_i^T, {}_1^\nu\phi_i^T\}$:

$$\begin{aligned}
&{}_0^\nu\phi_i^0 \ge \big|\phi_i^0(h_0^*) - {}^a\phi_i^0(h_0^*)\big|, \qquad {}_1^\nu\phi_i^0 \ge \big\|\dot\phi_i^0 - {}^a\dot\phi_i^0\big\|_{L_p^1[h_0^*,0]},\\
&{}_0^\nu\phi_i^T \ge \big|\phi_i^T(T) - {}^a\phi_i^T(T)\big|, \qquad {}_1^\nu\phi_i^T \ge \big\|\dot\phi_i^T - {}^a\dot\phi_i^T\big\|_{L_p^1[T,h_T^*]}.
\end{aligned}$$

Approximation of the functional ℓ^k

The real numbers β_i, $i = 1,\dots,mn+n$, are approximated by rational numbers ${}^a\beta_i$ with rational error bounds ${}^\nu\beta_i \ge |\beta_i - {}^a\beta_i|$. The constants ψ_{iq}^k are approximated by rational numbers ${}^a\psi_{iq}^k$ with rational error bounds ${}^\nu\psi_{iq}^k \ge |\psi_{iq}^k - {}^a\psi_{iq}^k|$, $i = 0,\dots,n-1$; $q = 0,\dots,m$; $k = 1,\dots,mn+n$. On each B_q, $q = 1,\dots,m+1$, the functions ϕ_j, $j = 1,\dots,mn+n$, are approximated by the polynomials ${}^a\phi_j^q$ with rational coefficients and with rational error bounds ${}^\nu\phi_j^q \ge \|{}^a\phi_j^q - \phi_j^q\|_{L_{p'}^1[t_{q-1},t_q]}$. Define the functions ${}^a\phi_j(\cdot)$ by the equalities

$${}^a\phi_j(t) = \sum_{q=1}^m \chi_q(t)\,{}^a\phi_j^q(t), \quad t \in [0,T], j = 1,\dots,mn+n.$$

Let us write the boundary value problem approximating the problem (8) as follows:

$$\begin{aligned}
&({}^a\mathcal{L}^n y)(t) \equiv y^{(n)}(t) + \sum_{i=0}^{n-1}\sum_{j=1}^{n_i} {}^a p_{ij}(t) y^{(i)}\big[{}^a h_{ij}(t)\big] = {}^a f(t), \qquad (13)\\
&y^{(i)}(\xi) = \begin{cases} {}^a\phi_i^0(\xi), & \xi < 0,\\ {}^a\phi_i^T(\xi), & \xi > T, \end{cases} \quad t \in [0,T],\\
&{}^a\ell^k y \equiv \int_0^T {}^a\varphi_k(s) y^{(n)}(s)\,ds + \sum_{i=0}^{n-1} {}^a\psi_{i0}^k y^{(i)}(0)\\
&\qquad + \sum_{i=0}^{n-1}\sum_{q=1}^m {}^a\psi_{iq}^k \Delta y^{(i)}(t_q) = {}^a\beta_k, \quad k = 1,\dots,mn+n.
\end{aligned}$$

Note that the operators ${}^a\mathcal{L}^n$ and ${}^a\ell^k$ are computable by construction.

Study of the principal boundary value problem

The aim of this study is to check whether the problem (9) is correctly solvable, having in mind the computer-assisted proof techniques. This issue is described in detail in [5]. Below suppose that the problem (9) is correctly solvable. Then there exists a fundamental system y_k, $k = 1, \dots, mn + n$ of the homogeneous equation

$$\begin{aligned} &(\mathcal{L}^n y)(t) \equiv y^{(n)}(t) + \sum_{i=0}^{n-1} \sum_{j=1}^{n_i} p_{ij}(t) y^{(i)}\bigl[h_{ij}(t)\bigr] = 0, \qquad (14) \\ &y^{(i)}(\xi) = 0, \quad \xi \notin [0,T]; t \in [0,T], \end{aligned}$$

and a fundamental system $\tilde{y}_k$, $k = 1, \dots, mn + n$, of the homogeneous equation

$$\begin{aligned} &({}^a\mathcal{L}^n y)(t) \equiv y^{(n)}(t) + \sum_{i=0}^{n-1} \sum_{j=1}^{n_i} {}^a p_{ij}(t) y^{(i)}\bigl[{}^a h_{ij}(t)\bigr] = 0, \qquad (15) \\ &y^{(i)}(\xi) = 0, \quad \xi \notin [0,T]; t \in [0,T]. \end{aligned}$$

Every function y_k is defined as a solution of the principal boundary value problem

$$\begin{aligned} &y^{(n)}(t) + \sum_{i=0}^{n-1} \sum_{j=1}^{n_i} p_{ij}(t) y^{(i)}\bigl[h_{ij}(t)\bigr] = 0, \qquad (16) \\ &y^{(i)}(\xi) = 0, \quad \xi \notin [0,T]; t \in [0,T], \\ &\Delta^n y = \delta_k, \quad \delta_k = \{\delta_{kq}\}_{q=1}^{mn+n}, \delta_{kq} = \begin{cases} 1, & k = q, \\ 0, & k \neq q, \end{cases} \end{aligned}$$

and every function $\tilde{y}_k$ is defined as a solution of the principal boundary value problem

$$\begin{aligned} &y^{(n)}(t) + \sum_{i=0}^{n-1} \sum_{j=1}^{n_i} {}^a p_{ij}(t) y^{(i)}\bigl[{}^a h_{ij}(t)\bigr] = 0, \qquad (17) \\ &y^{(i)}(\xi) = 0, \quad \xi \notin [0,T]; t \in [0,T], \\ &\Delta^n y = \delta_k, \quad \delta_k = \{\delta_{kq}\}_{q=1}^{mn+n}, \delta_{kq} = \begin{cases} 1, & k = q, \\ 0, & k \neq q, \end{cases} \end{aligned}$$

$k = 1, \dots, mn + n$. Denote by ${}^a y_k$ an approximation to the functions $\tilde{y}_k$ and by ${}^v y_k$ the approximation error bounds:

$$\begin{aligned} &{}^a y_k(t) = \sum_{q=1}^{m+1} \chi_q(t) {}^a_q y_k(t), \qquad {}^v y_k(t) = \sum_{q=1}^{m+1} \chi_q(t) {}^v_q y_k, \quad t \in [0,T], \\ &{}^v_q y_k \geq \bigl\| \tilde{y}_k^{(n)} - {}^a_q y_k^{(n)} \bigr\|_{L_p^1[t_{q-1}, t_q]}, \quad k = 1, \dots, mn + n. \end{aligned}$$

A detailed description of the construction of the functions ${}^a y_k$ and the estimations of ${}^v y_k$, $k = 1, \dots, mn + n$, is given in [5].

Analysis of solvability

Denote the elements of the matrices $\mathbf{\Gamma} = \{\gamma_{ij}\}_{i,j=1}^{mn+n}$, ${}^{a}\mathbf{\Gamma} = \{{}^{a}\gamma_{kj}\}_{k,j=1}^{mn+n}$ and ${}^{v}\mathbf{\Gamma} = \{{}^{v}\gamma_{kj}\}_{k,j=1}^{mn+n}$ as follows:

$$\gamma_{kj} \stackrel{\text{def}}{=} \ell^{k} y_{j} = \int_{0}^{T} \phi_{k}(s) y_{j}^{(n)}(s)\,ds + \psi_{i_j\tau_j}^{k}; \tag{18}$$

$${}^{a}\gamma_{kj} \stackrel{\text{def}}{=} \int_{0}^{T} {}^{a}\phi_{k}(s) {}^{a}y_{j}^{(n)}(s)\,ds + {}^{a}\psi_{i_j\tau_j}^{k};$$

$${}^{v}\gamma_{kj} \stackrel{\text{def}}{\geq} \left|{}^{v}\psi_{i_j\tau_j}\right| + \sum_{q=1}^{m} \Big\{ \left\|{}^{a}\psi_{k}^{q}\right\|_{L_{p'}^{1}[t_{q-1},t_q]} {}^{v}_{q}y_{j} + {}^{v}\phi_{k}^{q} \times {}^{v}_{q}y_{j} + {}^{v}\phi_{k}^{q} \left\|{}^{a}_{q}y_{j}^{(n)}\right\|_{L_{p}^{1}[t_{q-1},t_q]} \Big\};$$

$$i_j = \begin{cases} j, & 1 \leq j \leq n; \\ j-n, & n+1 \leq j \leq 2n; \\ \vdots & \\ j-mn, & mn+1 \leq j \leq mn+n; \end{cases}$$

$$\tau_j = \begin{cases} 0, & 1 \leq j \leq n; \\ 1, & n+1 \leq j \leq 2n; \\ \vdots & \\ m, & mn+1 \leq j \leq mn+n. \end{cases}$$

From definition (18) it follows that

$${}^{v}\gamma_{kj} \geq \left|\gamma_{kj} - {}^{a}\gamma_{kj}\right|, \qquad \left\|\mathbf{\Gamma} - {}^{a}\mathbf{\Gamma}\right\|_{R^{(mn+n)\times(mn+n)}} \leq \left\|{}^{v}\mathbf{\Gamma}\right\|_{R^{(mn+n)\times(mn+n)}},$$

where the matrix $\mathbf{\Gamma}$ is defined by (4). Thus we arrived at the following.

Theorem 1 *Let the matrix* ${}^{a}\mathbf{\Gamma}$ *defined by* (18) *be invertible, and let the inequality*

$$\left\|{}^{v}\mathbf{\Gamma}\right\|_{R^{(mn+n)\times(mn+n)}} < \frac{1}{\|{}^{a}\mathbf{\Gamma}^{-1}\|_{R^{(mn+n)\times(mn+n)}}} \tag{19}$$

hold. *Then the boundary value problem* (8) *is correctly solvable*.

Proof Under the condition of the theorem the inequality

$$\left\|\mathbf{\Gamma} - {}^{a}\mathbf{\Gamma}\right\|_{R^{(mn+n)\times(mn+n)}} < \frac{1}{\|{}^{a}\mathbf{\Gamma}^{-1}\|_{R^{(mn+n)\times(mn+n)}}} \tag{20}$$

holds. By the theorem on the inverse operator the matrix $\mathbf{\Gamma}$ is defined by (4) to be invertible, *i.e.*, the problem (8) is correctly solvable. □

Competing interests
The author declares that he has not competing interests.

Acknowledgements
The author expresses his sincere thanks to the referees for the careful and noteworthy reading of the manuscript and helpful suggestions that improved the manuscript.

Received: 12 December 2013 Accepted: 21 February 2014 Published: 12 Mar 2014

References

1. Azbelev, NV, Maksimov, VP, Rakhmatullina, LF: Introduction to the Theory of Functional Differential Equations: Methods and Applications. Hindawi Publishing Corporation, New York (2007)
2. Rumyantsev, AN: Reliable computing experiment in the study of boundary problems. Perm State University, Perm (1999) (in Russian)
3. Rumyantsev, AN: The reliable computing experiment in the study of boundary value problems for functional differential equations. Funct. Differ. Equ. **26**(3-4), 499-519 (2002)
4. Akilov, GP, Kantorovich, LV: Functional Analysis. Nauka, Moscow (1978) (in Russian)
5. Rumyantsev, AN: The constructive study of differential equations with deviated argument. Mem. Differ. Equ. Math. Phys. **26**, 91-136 (2002)

10.1186/1687-2770-2014-53
Cite this article as: Rumyantsev: On the constructive investigation of a class of linear boundary value problems for *n*th order differential equations with deviating arguments. *Boundary Value Problems* 2014, 2014:53