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Some equilibrium, stability, instability and oscillatory results for an extended discrete epidemic model with evolution memory

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Abstract

This paper investigates the existence and potential uniqueness of equilibrium points and some stability, instability and oscillatory properties of a discrete nonlinear epidemic model, which generalises previous Stevic's model, whose solutions possess memory from a finite chain of preceding samples. An application example is provided where the proposed model is 'ad hoc' adapted to a class of SIS models widely used in epidemiology.

1 Introduction

In this paper, some properties of equilibrium points as well as some stability and instability properties of the following nonlinear discrete epidemic model are investigated:

$$x_{n+1} = \max\left(0, \left(1 - \sum_{j=0}^{k-1} F(x_{n-j})\right)\left(1 - M(\bar{x}_n)e^{-AG(\bar{x}_n)}\right)\right), \quad n \in \mathbf{N}_0, \tag{1.1}$$

with $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, where $\bar{x}_n = (x_n, x_{n-1}, \ldots, x_{n-q+1})$ is a q ($\in \mathbf{N}$)-tuple of values of the solution sequence previous to its $(n+1)$th value and $F : \mathbf{R}_{0+} \to \mathbf{R}$ and $M, G : \mathbf{R}_{0+}^q \to \mathbf{R}$ under any set of initial conditions $x_i \geq 0$; $i = 1-p, 2-p, \ldots, 0$, where $p = \max(k, q)$, and $A \in \mathbf{R}_+$ is a weighting forgetting factor in the model. Note that such sets of initial conditions guarantee that the associated solution sequence $\{x_n\}$, $n \in \mathbf{N}_0 \cup (-\overline{p})$, where $\overline{p} = \{1, 2, \ldots, p\}$, is nonnegative by construction. It has to be pointed out that Stevic studied negative solutions of the epidemic model

$$x_{n+1} = \max\left(0, \left(1 - \sum_{j=0}^{k-1} x_{n-j}\right)\left(1 - e^{-Ax_n}\right)\right) \tag{1.2}$$

for $A \in (0, \infty)$ and also described and interpreted prior work on nonnegative solutions of the same epidemic model and their stability, instability and oscillation properties [1]. See also [2–5] for some related work. Later on, the discrete epidemic model was extended to include two coupled extended difference equations [6]. It turns out that both continuous-type and discrete-type epidemic models are of great importance in research nowadays because of its intrinsic interest in medical applications and because of their rich dynamics which make them also attractive to mathematicians involved in the investigations of nonlinear differential and difference equations. See, for instance, [7–11]. Note that discrete modelling techniques are very relevant in the study of ecology and biology problems as well, like, for instance, the discretization of logistic equations [12] leading to discrete models such as those related to the well-known Riecker, Beverton-Holt and Hassel equations (see, for instance, [13–21] and references therein). A way of establishing a direct generalisation of Zhang-Shi's [5], Stevic's [1], and Papaschinopoulos *et al.*'s [6] analysis for a related epidemic model is to restrict the codomains of the various functions to be nonnegative by defining them as $F:\mathbf{R}_{0+}\to\mathbf{R}_{0+}$; $M,G:\mathbf{R}_{0+}^q\to\mathbf{R}_{0+}$, where $\mathbf{R}_{0+}=\{z\in\mathbf{R}:z\geq 0\}=\mathbf{R}_+\cup\{0\}$ with $\mathbf{R}_+=\{z\in\mathbf{R}:z>0\}$. However, under a more general discussion, it can be allowed for the functions to have ranges in $\mathbf{R}$. The function $F:\mathbf{R}_{0+}\to\mathbf{R}$ is assumed to be upper-bounded and lower-bounded by known polynomials in x of degree q. The constraint $\max(\sum_{j=0}^{k-1}F(x_{n-j}),M(\bar{x}_n)e^{-AG(\bar{x}_n)})\leq 1$ for $j\in\mathbf{N}_{0+}$ is not assumed so that both sequences $(1-\sum_{j=0}^{k-1}F(x_{n-j}))$, $(1-M(\bar{x}_n)e^{-AG(\bar{x}_n)})$ may be eventually negative in (1.1) while generating a nonnegative solution sequence. Note that the solution evolution of the proposed model can be interpreted as the propagation of the infection (say, roughly speaking, the infected population or a normalised value for it) from certain initial conditions and subject to weighting factor parameterization and a number of discrete delays. A considerable freedom of implementation of the proposed model in terms of choices of the parameterization structures F, M, A and the number of terms in the parameterization sequences is allowed. The above model remembers, in a much more general context and discretized version, the original Bernoulli proposal to introduce a simple epidemic model with just a single variable being the solution of a scalar equation. However, this model can be useful for situations involving two (or even three) variables as, for instance, the SI-epidemic model when the total population remains constant for all time during the illness cycle, *i.e.* for the case when mortality caused by the disease is not expected, or even for models of three variables. Related interpretations and practical use are addressed in the simulated examples, and it has to be pointed out that simple structures for epidemic models are often preferred in medical structures compared to complex alternative structures. A simulated example is also provided with a detailed study combining the proposed epidemic models with a class of SIS epidemic models that have been previously known in the background literature [16, 17]. Finally, it has to be pointed out that the properties of stability, instability and oscillatory behaviour of the solutions are very relevant issues in the study of epidemic models. See, for instance, [16–21] and references therein. Thus, the study and associated discussion provided concerning the proposed model pay a special attention to them.

2 Equilibrium points

To study the existence of potential equilibrium points, we set the values of solution sequence (1.1) to a constant one x, which yields

$$x=\max\bigl(0,\bigl(1-kF(x)\bigr)\bigl(1-M(\bar{x})e^{-AG(\bar{x})}\bigr)\bigr),\tag{2.1}$$

where $\bar{x}=(x,x,\dots,x)$ is in $\mathbf{R}^q$. Define $g:\mathbf{R}_{0+}\to\mathbf{R}$ as follows:

$$g(x)=\max\bigl(0,\bigl(1-kF(x)\bigr)\bigl(1-M(\bar{x})e^{-AG(\bar{x})}\bigr)\bigr)-x.\tag{2.2}$$

Note that $x \in \mathbf{R}_{0+}$ is an equilibrium point of (1.1) if and only if $g(x) = 0$. The existence of equilibrium points of (1.1) are subject to the following direct result.

Theorem 2.1 *The following properties hold.*

(i) $x = 0$ *is an equilibrium point of* (1.1) *if and only if* $g(0) = 0$, *equivalently if and only if* $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) \leq 0$, *where* $0 \in \mathbf{R}$ *and* $\bar{0} = (0, 0, \dots, 0) \in \mathbf{R}^q$, *equivalently if and only if*

$$\big(F(0) = 1/k\big) \vee \big(M(\bar{0}) = e^{AG(\bar{0})}\big) \\ \vee \big(\big[\big(F(0) \neq 1/k\big) \wedge M(\bar{0}) \neq e^{AG(\bar{0})}\big] \Leftrightarrow \big[\big(1 - kF(0)\big)\big(1 - M(\bar{0})e^{-AG(\bar{0})}\big) < 0\big]\big),$$

where the symbols '$\vee$' and '$\wedge$' stand for 'or' and 'and' (*i.e. for logic disjunction and conjunction*, *respectively*).

(ii) $x \in \mathbf{R}_+$ (*respectively*, $x \in \mathbf{R}_{0+}$) *is a positive* (*respectively, a nonnegative*) *equilibrium point of* (1.1) *if and only* $g(x) = 0$ *for such* $x \in \mathbf{R}_+$ (*respectively, for such* $x \in \mathbf{R}_{0+}$). *If* $g(x) \neq 0 \Leftrightarrow (1 - kF(x))(1 - M(\bar{x})e^{-AG(\bar{x})}) \neq x > 0$; $\forall x \in [x_1, x_2] \subset \mathbf{R}_{0+}$, *then there is no equilibrium point of* (1.1) *within the real interval* $[x_1, x_2]$. *Finally, a necessary condition for* $x \in \mathbf{R}_{0+}$ (*respectively, for* $x \in \mathbf{R}_+$) *to be an equilibrium point of* (1.1) *is that* $(1 - kF(x))(1 - M(\bar{x})e^{-AG(\bar{x})}) \geq 0$ (*respectively*, $(1 - kF(x))(1 - M(\bar{x})e^{-AG(\bar{x})}) > 0$).

(iii) *There is no positive* (*respectively, no nonnegative*) *equilibrium point of* (1.1) *if and only if* $g(x) \neq 0$; $\forall x \in \mathbf{R}_+$ (*respectively, if and only if* $g(x) \neq 0$; $\forall x \in \mathbf{R}_{0+}$).

(iv) $x = 0$ *is the unique nonnegative equilibrium point of* (1.1) *if and only if* $(1 - kF(x))(1 - M(\bar{x})e^{-AG(\bar{x})}) \leq 0$; $\forall x \in \mathbf{R}_{0+}$.

Proof It follows that $x = 0$ is an equilibrium point of (1.1) if and only if $g(0) = 0 \Leftrightarrow (1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) \leq 0$ from (2.1)-(2.2). This proves directly Property (i). To prove Property (ii), note that the logic proposition $g(x) \neq 0 \Leftrightarrow (1 - kF(x))(1 - M(\bar{x})e^{-AG(\bar{x})}) \neq x > 0$ for any $x \in [x_1, x_2] \subset \mathbf{R}_+$ is equivalent to its contrapositive logic proposition $\neg\exists x \in [x_1, x_2]$ such that $(1 - kF(x))(1 - M(\bar{x})e^{-AG(\bar{x})}) \leq 0 \Leftrightarrow g(x) = 0$ ('$\neg$' stands for logic negation) so that there is no equilibrium point of (1.1) in $[x_1, x_2]$. This proves the first part of Property (ii). The last part of Property (ii) follows by contradiction since $(1 - kF(x))(1 - M(\bar{x})e^{-AG(\bar{x})}) \leq 0$ implies $g(x) = -x < 0$; $\forall x \in \mathbf{R}_+$, so that there is no positive equilibrium point of (1.1), and, on the other hand, $g(x) = 0$ if and only if $x = 0$ is an equilibrium point, which is impossible since $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) > 0$. Hence, Property (ii). Property (iii) is direct since any equilibrium point of (1.1) in $\mathbf{R}_{0+}$ implies and is implied by the condition $g(x) = 0$. The sufficiency part of Property (iv) follows since the given condition implies that $g(0) = 0$, then $x = 0$ is an equilibrium point of (1.1) and $g(x) = -x < 0$ for $x \in \mathbf{R}_+$, so that there is no positive equilibrium point of (1.1). The necessity part of Property (iv) follows since $x = 0$ is an equilibrium point of (1.1) only if $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) \leq 0$, which is unique in $\mathbf{R}_{0+}$, since any $x \in \mathbf{R}_+$ is an equilibrium point of (1.1) only if $(1 - kF(x))(1 - M(\bar{x})e^{-AG(\bar{x})}) > 0$ from Property (ii). □

Some parts of the subsequent analysis are simplified subject to the following assumption.

Assumption 2.2 $F : \mathbf{R}_{0+} \to \mathbf{R}$ is differentiable on $\mathbf{R}_{0+}$ and $M, G : \mathbf{R}^q_{0+} \to \mathbf{R}$ are differentiable in $\mathbf{R}^q_{0+}$.

Note that if Assumption 2.2 holds, then $g : \mathbf{R}_{0+} \to \mathbf{R}$ is also everywhere differentiable. The following result is a direct conclusion of Theorem 2.1.

Theorem 2.3 *A sufficient condition for $x_1 \in \mathbf{R}_{0+}$ to be the unique equilibrium point of* (1.1) *is that $g(x_1) = 0$ and $g : \mathbf{R}_{0+} \to \mathbf{R}$ fulfils one of the conditions below*:

(a) *strictly monotone in $\mathbf{R}_{0+}$*;

(b) *non-decreasing* (*or, respectively, decreasing*) *in $[x_1, \infty)$ and, furthermore, strictly increasing* (*or, respectively, strictly decreasing*) *in some real subinterval $[x_1, x_3) \subseteq [x_1, \infty)$ of nonzero measure and, in addition if $x_1 \in \mathbf{R}_+$, either strictly decreasing* (*or, respectively, strictly increasing*) *in some interval $[x_2, x_1] \subset \mathbf{R}_{0+}$ of nonzero measure and, correspondingly, either decreasing* (*or, respectively, non-decreasing*) *in some* (*eventually being of zero measure*) *interval $[0, x_2]$*;

(c) *non-decreasing* (*or, respectively, decreasing*) *in $[0, \infty)$ and, furthermore, strictly increasing* (*or, respectively, strictly decreasing*) *in some real subinterval $[x_2, x_1) \subseteq [0, x_1)$ of nonzero measure.*

The result also holds under Assumption 2.2 *if $g' : \mathbf{R}_+ \to \mathbf{R}$ is continuous, nonzero, and has a constant sign in some real intervals $[x_2, x_1) \subseteq [0, x_1)$ and $(x_1, x_3) \subseteq [x_1, \infty)$ of nonzero measure while being identically zero in $[0, x_2) \cup [x_3, \infty)$.*

Proof If $g(x_1) = 0$ for some $x_1 \in \mathbf{R}_{0+}$ and $g : \mathbf{R}_{0+} \to \mathbf{R}$ is strictly monotone, then $g(x) \neq 0$; $\forall x\ (\neq x_1) \in \mathbf{R}_{0+}$ so that there is no equilibrium point of (1.1) other than x_1 on $\mathbf{R}_{0+}$. The result has been proven under Condition (a). Now, assume that $g : \mathbf{R}_{0+} \to \mathbf{R}$ is strictly increasing in $[x_1, x_3) \subseteq [x_1, \infty)$ and non-decreasing in $[x_1, \infty)$. Then, $g(x) \geq g(x_3) > g(y) > g(x_1) = 0$; $\forall x \in (x_3, \infty)$, $\forall y \in (x_1, x_3)$. If $g : \mathbf{R}_{0+} \to \mathbf{R}$ is strictly decreasing in $[x_1, x_3) \subseteq [x_1, \infty)$ and decreasing in $[x_1, \infty)$, then $0 = g(x_1) > g(y) > g(x_3) \geq g(x)$; $\forall y \in (x_1, x_3)$, $\forall x \in (x_3, \infty)$. Thus, x_1 is the unique equilibrium point of (1.1) in $[x_1, \infty)$. The result under Condition (b) is already proven for $x_1 = 0$ but not yet for $x_1 \in \mathbf{R}_+$. Thus, assume now that $x_1 \in \mathbf{R}_+$, that the above conditions hold and that, in addition, $g : \mathbf{R}_{0+} \to \mathbf{R}$ is either strictly decreasing (or, respectively, strictly increasing) in some interval $[x_2, x_1] \subset \mathbf{R}_{0+}$ of nonzero measure and, correspondingly, is either decreasing (or, respectively, non-decreasing) within an interval $[0, x_2]$ being of zero or nonzero measure. Then, there is no $x\ (\neq x_1) \in \mathbf{R}_{0+}$ being an equilibrium point of (1.1). Hence, the result fully follows under Condition (b). A particular case occurs when the function is continuously differentiable and either strictly increasing or strictly decreasing in each of the real intervals $[0, x_1)$ and (x_1, ∞). Hence, the theorem. □

Theorem 2.4 *Assume that $F : \mathbf{R}_{0+} \to \mathbf{R}$ satisfies $\{x \in \mathbf{R}_+ : F(x) = 1/k\} = \varnothing$ and that Assumption* 2.2 *holds with all the derivatives referred to as being everywhere continuous in their definition domains. Define $m' : \mathbf{R}_{0+} \to \mathbf{R}$ as*

$$m'(x) = \left(AG'(x) + \frac{kF'(x)}{1 - kF(x)}\right)M(\bar{x}) - \frac{1 + kF'(x)}{1 - kF(x)} e^{AG(\bar{x})}; \quad \forall x \in \mathbf{R}_{0+}. \tag{2.3}$$

Then the following properties hold.

(i) *Assume that $F : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ and $M, G : \mathbf{R}_{0+}^q \to \mathbf{R}_{0+}$ satisfy $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) \leq 0$, and*

$$M'(x) = m'(x) + \varepsilon(x); \quad \forall x \in \mathbf{R}_{0+} \tag{2.4}$$

for some given everywhere continuous $\varepsilon : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *if* $g(0) < 0$, *where*

$$g'(0) = \begin{cases} -1 & \text{if } (1-kF(0))(1-M(\bar{0})e^{-AG(\bar{0})}) < 0, \\ [M(\bar{0})AG'(\bar{0}) - M'(\bar{0})](1-kF(0))e^{-AG(\bar{0})} & \\ \quad + kF'(0)(M(\bar{0})e^{-AG(\bar{0})} - 1) - 1 & \text{if } (1-kF(0))(1-M(\bar{0})e^{-AG(\bar{0})}) = 0. \end{cases} \tag{2.5}$$

Then $x = 0$ *is the unique nonnegative equilibrium point of* (1.1) *in* $\mathbf{R}_{0+}$.

(ii) *Property* (i) *also holds if* $(1-kF(0))(1-M(\bar{0})e^{-AG(\bar{0})}) = 0$ *and if* $M'(x) = m'(x) - \varepsilon(x)$; $\forall x \in \mathbf{R}_{0+}$ *with* $g'(0) > 0$, *where*

$$g'(0) = \left[M(\bar{0})AG'(\bar{0}) - M'(\bar{0})\right]\left(1-kF(0)\right)e^{-AG(\bar{0})} + kF'(0)\left(M(\bar{0})e^{-AG(\bar{0})} - 1\right) - 1 \tag{2.6}$$

and $\varepsilon : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *is everywhere continuous.*

(iii) *Assume that* $F : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *and* $M, G : \mathbf{R}_{0+}^q \to \mathbf{R}_{0+}$ *satisfy* $(1-kF(0))(1-M(\bar{0})e^{-AG(\bar{0})}) > 0$. *Then* $\exists x_1 \in \mathbf{R}_+$, *which satisfies* $g(x_1) = 0$, *is a unique nonnegative equilibrium point of* (1.1) *if* $M'(x) = m'(x) + \varepsilon(x)$ *subject to* (2.5) *with* $\varepsilon : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *being everywhere continuous,* (*strictly*) *positive in* $[0, x_1 + \varepsilon_0]$ *for some sufficiently large* $\varepsilon_0 \in \mathbf{R}_{0+}$, *and decreasing in* $\mathbf{R}_{0+}$.

(iv) *Assume that* $F : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *and* $M, G : \mathbf{R}_{0+}^q \to \mathbf{R}_{0+}$ *satisfy* $g(0) = (1-kF(0))(1-M(\bar{0})e^{-AG(\bar{0})}) > 0$. *Then,* $\neg\exists x_1 \in \mathbf{R}_{0+}$ *is a nonnegative equilibrium point of* (1.1) *if* $M'(x) = m'(x) - \varepsilon(x)$ *with* $\varepsilon : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *is everywhere continuous.*

Proof Property (i) is first proven. Direct calculation from (2.2) under Assumption 2.2 yields

$$\begin{aligned} g'(x) &= \left(1-kF(x)\right)\left(AG'(\bar{x})M'(\bar{x}) - M(\bar{x})\right)e^{-AG(\bar{x})} - kF'(x)\left(1 - M(\bar{x})e^{-AG(\bar{x})}\right) - 1 \\ &= \left[M(\bar{x})\left(AG'(\bar{x})\left(1-kF(x)\right) + kF'(x)\right) - M'(\bar{x})\left(1-kF(x)\right)\right]e^{-AG(\bar{x})} - kF'(x) - 1 \\ &= \left[M(\bar{x})AG'(\bar{x}) - M'(\bar{x})\right]\left(1-kF(x)\right)e^{-AG(\bar{x})} + kF'(x)\left(M(\bar{x})e^{-AG(\bar{x})} - 1\right) - 1 \end{aligned} \tag{2.7}$$

for any $x \in \mathbf{R}_{0+}$ such that $(1-kF(x))(1-M(\bar{x})e^{-AG(\bar{x})}) \geq 0$, and $g'(x) = -1$ for any $x \in \mathbf{R}_{0+}$ such that $(1-kF(x))(1-M(\bar{x})e^{-AG(\bar{x})}) \leq 0$. It is assumed that $g'(0) = a - 1 < g(0) = 0$, obtained from (2.5), where $a = a(0)$ is defined by

$$a = \begin{cases} 0 & \text{if } (1-kF(0))(1-M(\bar{0})e^{-AG(\bar{0})}) < 0, \\ [M(\bar{0})AG'(\bar{0}) - M'(\bar{0})](1-kF(0))e^{-AG(\bar{0})} & \\ \quad + kF'(0)(M(\bar{0})e^{-AG(\bar{0})} - 1) & \text{otherwise.} \end{cases} \tag{2.8}$$

Define $m' : \mathbf{R}_{0+} \to \mathbf{R}$ as

$$m'(x) = \left(AG'(x) + \frac{kF'(x)}{1-kF(x)}\right)M(\bar{x}) - \frac{1+kF'(x)}{1-kF(x)}e^{AG(\bar{x})}; \quad \forall x \in \mathbf{R}_{0+}, \tag{2.9}$$

which is everywhere continuous in $\mathbf{R}_+$ since the subset of its second-class discontinuity points is empty by hypothesis. Then, from (2.4), one has for any $x \in \mathbf{R}_+$ such that $F(x) \neq 1/k$, then for any $x \in \mathbf{R}_+$ since $\{x \in \mathbf{R}_+ : F(x) = 1/k\} = \varnothing$, that

$$g'(x) \leq 0 \quad \Leftrightarrow \quad M'(x) \geq m'(x); \qquad g'(x) < 0 \quad \Leftrightarrow \quad M'(x) > m'(x);$$
$$g'(x) = 0 \quad \Leftrightarrow \quad M'(x) = m'(x); \quad \forall x \in \mathbf{R}_+, \tag{2.10a}$$

$$g'(x) \geq 0 \quad \Leftrightarrow \quad M'(x) \leq m'(x);$$
$$g'(x) > 0 \quad \Leftrightarrow \quad M'(x) < m'(x); \quad \forall x \in \mathbf{R}_+. \tag{2.10b}$$

Now, $x = 0$ is an equilibrium point of (1.1), equivalently, $g(0) = 0$, with $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) \leq 0$ and $g'(0) = a - 1 < 0$ with $a = 0$ if $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) < 0$ and $a = a(0) = (M(\bar{0})AG'(\bar{0}) - M'(\bar{0}))(1 - kF(0))e^{-AG(\bar{0})} + kF'(0)(M(\bar{0})e^{-AG(\bar{0})} - 1) < 1$ if $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) = 0$. In both cases, $g'(0) < 0$ and, since $g' : \mathbf{R}_{0+} \to \mathbf{R}$ is continuous from (2.3)-(2.4), since $\{x \in \mathbf{R}_+ : F(x) = 1/k\} = \varnothing$ implies that $m' : \mathbf{R}_{0+} \to \mathbf{R}$ is continuous, there is some $x_1 \in \mathbf{R}_+$ such that $g : \mathbf{R}_{0+} \to \mathbf{R}$ is strictly decreasing on $[0, x_1)$ from (2.4)-(2.5). Thus, $0 \neq g(x) < g(0) = 0$; $\forall x \in [0, x_1)$ and $g(x) \leq g(x_1) < g(0) = 0$; $\forall x \in \mathbf{R}_+$ since $g : \mathbf{R}_{0+} \to \mathbf{R}$ is decreasing in $\mathbf{R}_{0+}$ if (2.10a) holds. Hence, Property (i). Property (ii) is a dual property of (i) with $g'(0) = a - 1 > g(0) = 0$ and $g : \mathbf{R}_{0+} \to \mathbf{R}$ being non-decreasing in $\mathbf{R}_{0+}$ under (2.10b).

Property (iii) holds since $g(0) = (1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) > 0$ (so that $x = 0$ is not an equilibrium point of (1.1)) with $g : \mathbf{R}_{0+} \to \mathbf{R}$ being strictly decreasing in $[0, x_1 + \varepsilon_0]$ and decreasing on $[0, \infty)$ from (2.10a) for some $x_1 \in \mathbf{R}_+$ (since $\varepsilon : \mathbf{R}_{0+} \backslash [0, x_1 + \varepsilon_0] \to \mathbf{R}_+$) which exists so that $g(x) \leq g(x_1 + \varepsilon) < g(x_1) = 0 < g(y)$; $\forall x\,(> x_1), y\,(< x_1) \in \mathbf{R}_+$. Then x_1 is the unique equilibrium point of (1.1) in $\mathbf{R}_{0+}$ if ε_0 is large enough. Property (iv) holds since $g(0) > 0$ (so that $x = 0$ is not an equilibrium point of (1.1)) with $g : \mathbf{R}_{0+} \to \mathbf{R}$ being non-decreasing in $[0, \infty)$ from (2.10b). Then there is no equilibrium point of (1.1) in $\mathbf{R}_{0+}$. □

The existence of the zero equilibrium and another positive equilibrium point of (1.1) can be given by a direct extension of Theorem 2.4(i)-(ii) as follows.

Theorem 2.5 *Assume that* $F : \mathbf{R}_{0+} \to \mathbf{R}$ *satisfies* $\{x \in \mathbf{R}_+ : F(x) = 1/k\} = \varnothing$ *and that Assumption* 2.2 *holds with all the derivatives referred to as being everywhere continuous in their definition domains. Then the following properties hold.*

(i) *Assume, in addition, that* $F : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *and* $M, G : \mathbf{R}_{0+}^q \to \mathbf{R}_{0+}$ *satisfy* $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) \leq 0$, $M'(x) = m'(x) + \varepsilon(x)$; $\forall x \in [0, x_1)$ *and* $M'(x) = m'(x) - \varepsilon(x)$; $\forall x \in [x_1, \infty)$ *for some* $x_1 \in \mathbf{R}_+$, *any continuous* $\varepsilon : \mathbf{R}_{0+} \to \mathbf{R}_+$ *with* $g'(0) < 0$ *satisfying* (2.5) (*i.e.* $g'(0) = a - 1 < 0$ *if* $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) = 0$ *and* $g'(0) = -1 < 0$ *if* $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) < 0$). *Then,* $x = 0$ *and* $x = x_2 > x_1$ *for some* $x_2 \in \mathbf{R}_+$ *are the only two nonnegative equilibrium points of* (1.1) *in* $\mathbf{R}_{0+}$.

(ii) *Property* (i) *also holds if* $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) = 0$ *and if* $M'(x) = m'(x) - \varepsilon(x)$; $\forall x \in [0, x_1)$ *and* $M'(x) = m'(x) + \varepsilon(x)$; $\forall x \in [x_1, \infty)$ *for some* $x_1 \in \mathbf{R}_+$ *and* $g'(0) > 0$, *according to* (2.6), *for any continuous* $\varepsilon : \mathbf{R}_{0+} \to \mathbf{R}_+$.

Outline of proof The proof of Property (i) follows with $g : \mathbf{R}_{0+} \to \mathbf{R}$ being strictly decreasing on some interval $[0, x_1)$ with $g'(0) < 0$ and strictly increasing on $[x_1, \infty)$ since $g'(0) = -1 < 0$ if $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) < 0$ and $g'(0) = a - 1 < 0$ if $g(0) = (1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) = 0$. Then there is $x_2\,(> x_1) \in \mathbf{R}_+$ such that $0 = g(0) > g(x) > g(x_1) < g(y) < g(x_2) = 0 < g(z)$; $\forall x \in (0, x_1)$, $x \in (x_1, x_2)$, $\forall z \in (x_2, \infty)$. Then there is no $x \in \mathbf{R}_{0+}$ other than

$x = 0$ and $x = x_2$ such that $g(x) = 0$. Hence, Property (i). The proof of Property (ii) is similar by noting that $0 = g(0) < g(x) < g(x_1) > g(y) > g(x_2) = 0 > g(z)$ with $g(x)$ being strictly increasing in $[0, x_1)$ and strictly decreasing in (x_1, ∞). Hence, the result. □

Note that the conditions of Theorem 2.5 can be relaxed in the sense that it is not necessary for $g : \mathbf{R}_{0+} \to \mathbf{R}$ to be strictly monotone in (x_2, ∞) but non-decreasing for Part (i) and decreasing for Part (ii).

Examples 2.6 (1) If we consider the particular case $p = 1$, $x = \bar{x}$, $F(x) = G(x) = x$, then $F(0) = G(0) = 0$, $M(x) = M(0) = 1$ so that $F'(x) = G'(x) = F'(0) = G'(0) = 1$, $M'(x) = M'(0) = 0$, one gets from Theorem 2.4(i) that the condition $M'(0) = 0 \geq A - 1$, equivalently, $0 < A \leq 1$, leads to $x = 0$ being an equilibrium point of (1.1). Also, one has again from Theorem 2.4(i) that $g(0) = 0$ and the following condition holds for $x \in [0, 1/k)$ and $0 < A \leq 1$:

$$0 = M'(x) \geq m'(x) = A - \frac{1}{1 - kx}\big(e^{Ax} + k\big(e^{Ax} - 1\big)\big).$$

Then $x = 0$ is the only equilibrium point of (1.1) if $0 < A \leq 1$ for any $k \in \mathbf{N}$, since $g : \mathbf{R}_{0+} \to \mathbf{R}_+$ is decreasing for $x \in [0, 1/k)$ from the above inequality and $g'(0) < 0$ implies that $\neg\exists x_1 \in \mathbf{R}_+$ such that $g(x_1) = 0$. This coincides with former results obtained in [1, 2] for $k = 1$.

(2) Consider the equilibrium equation $x = (1 + kx)(e^{Ax} + 1)$ of $x_{n+1} = (1 + \sum_{j=0}^{k-1} x_{n-j})(1 + e^{Ax_n})$ with $\min(x_{1-i} : i \in \overline{p}) \geq 0$. Thus, $F(x) = G(x) = -x$, $F'(x) = G'(x) = -1$, $M(x) = -1$, $M'(x) = 0$, $g(0) = 2$, $g'(0) = A + 2k - 1 > 0$, and $g : \mathbf{R}_{0+} \to \mathbf{R}_+$ is not decreasing for $A \geq 0$ since

$$0 = M'(x) \geq m'(x) = A + \frac{1}{1 + kx}\big(k\big(e^{-Ax} + 1\big) - e^{-Ax}\big)$$

cannot hold for $x \in \mathbf{R}_{0+}$. Thus, $\neg\exists x \in \mathbf{R}_{0+}$ is an equilibrium point for any $A \geq 0$.

Theorems 2.3, 2.4 and 2.5 may be extended by removing the condition $\{x \in \mathbf{R}_+ : F(x) = 1/k\} = \varnothing$ and the continuity of the derivatives of the vector functions in Assumption 2.2 by allowing such derivatives to be impulsive at the points of $x \in \mathbf{R}_+$ such that $F(x) = 1/k$, if any. The sign of eventual impulses should be such that they do not change the needed non-decreasing, decreasing or strictly monotone properties of $g : \mathbf{R}_{0+} \to \mathbf{R}$. We denote in the following by $f(x^-)$ the left limits and by $f(x)$ the right limits of functions at points of the functions $f : \mathbf{R}_{0+} \to \mathbf{R}$ which are distinct if such functions are discontinuous at x. Such an extension is as follows.

Theorem 2.7 *Let Assumption* 2.2 *hold with $F : \mathbf{R}_{0+} \to \mathbf{R}$ being continuous on $\mathbf{R}_{0+} \backslash S_{\mathrm{imp}}$ and impulsive on S_{imp}, and $M', G' : \mathbf{R}_{0+}^p \to \mathbf{R}$ being continuous on $\mathbf{R}_{0+}^p \backslash \overline{S}_{\mathrm{imp}}$ and impulsive on $\overline{S}_{\mathrm{imp}}$, where $S_{\mathrm{imp}} = \{x \in \mathbf{R}_+ : F(x) = 1/k\}$ and $\overline{S}_{\mathrm{imp}} = \{\bar{x} = (x, x, \dots, x) \in \mathbf{R}_{0+}^p : x \in S_{\mathrm{imp}}\}$ which can be empty or nonempty* (*note that $\bar{x} \in \overline{S}_{\mathrm{imp}}$ if and only if $x \in S_{\mathrm{imp}}$*). *Let the function $m' : \mathbf{R}_{0+} \to \mathbf{R}$ be defined in* (2.3). *Then the following properties hold.*

(i) *Assume that $F : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ and $M, G : \mathbf{R}_{0+}^q \to \mathbf{R}_{0+}$ satisfy $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) \leq 0$, and $M'(x) = m'(x) + \varepsilon(x)$; $\forall x \in \mathbf{R}_{0+}$ for some continuous $\varepsilon : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ being everywhere continuous and satisfying* (2.5). *Then $x = 0$ is the unique nonnegative equilibrium point of* (1.1) *in $\mathbf{R}_{0+}$ provided that either $F'(x)M(\bar{x}) - (1 + kF'(x))e^{AG(\bar{x})} = 0$ or $\operatorname{sgn}(F'(x)M(\bar{x}) - (1 + kF'(x))e^{AG(\bar{x})}) < 0$; $\forall x \in S_{\mathrm{imp}}$.*

(ii) *Property* (i) *also holds if* $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) = 0$ *and if* $M'(x) = m'(x) - \varepsilon(x)$; $\forall x \in \mathbf{R}_{0+}$ *with* $\varepsilon : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *being everywhere continuous and satisfying* (2.6) *and* $g'(0) > 0$ *provided that either* $F'(x)M(\bar{x}) - (1 + kF'(x))e^{AG(\bar{x})} = 0$ *or* $\operatorname{sgn}(F'(x)M(\bar{x}) - (1 + kF'(x))e^{AG(\bar{x})}) > 0$; $\forall x \in S_{\mathrm{imp}}$.

(iii) *Assume that* $F : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *and* $M, G : \mathbf{R}_{0+}^q \to \mathbf{R}_{0+}$ *satisfy* $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) > 0$. *Then*, $\exists x_1 \in \mathbf{R}_+$, *which satisfies* $g(x_1) = 0$, *is then a unique nonnegative equilibrium point of* (1.1) *if* $M'(x) = m'(x) + \varepsilon(x)$, *with* $\varepsilon : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *being everywhere continuous and strictly decreasing in* $[0, x_1]$ *and decreasing in* (x_1, ∞), *provided that* $g'(0)$ *fulfils* (2.5), $F'(x)M(\bar{x}) - (1 + kF'(x))e^{AG(\bar{x})} = 0$, *or* $\operatorname{sgn}(F'(x)M(\bar{x}) - (1 + kF'(x))e^{AG(\bar{x})}) < 0$ *with* $\frac{F'(x)M(\bar{x}) - (1+kF'(x))e^{AG(\bar{x})}}{1-KF(x)} \le -g(x^-)$; $\forall x \in S_{\mathrm{imp}}$. *If* $\frac{F'(x)M(\bar{x}) - (1+kF'(x))e^{AG(\bar{x})}}{1-KF(x)} = -g(x^-)$, *then* $x_1 = x$.

(iv) *Assume that* $F : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *and* $M, G : \mathbf{R}_{0+}^q \to \mathbf{R}_{0+}$ *satisfy* $(1 - kF(0))(1 - M(\bar{0})e^{-AG(\bar{0})}) > 0$. *Then* $\neg\exists x_1 \in \mathbf{R}_{0+}$ *is a nonnegative equilibrium point of* (1.1) *if* $M'(x) = m'(x) - \varepsilon(x)$ *with* $\varepsilon : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ *being everywhere continuous, and* $g'(0)$ *subject to* (2.6), *provided that either* $F'(x)M(\bar{x}) - (1 + kF'(x))e^{AG(\bar{x})} = 0$ *or* $\operatorname{sgn}(F'(x)M(\bar{x}) - (1 + kF'(x)e)^{AG(\bar{x})}) > 0$; $\forall x \in S_{\mathrm{imp}}$.

Outline of proof Note that the various extended conditions in Theorem 2.7(i)-(iv) with respect to those of Theorem 2.4 imply that $g(x) \le g(x^-)$ (respectively, $g(x) \ge g(x^-)$) if $x \in S_{\mathrm{imp}}$ and $g : (x - \varepsilon, x) \to \mathbf{R}$ is decreasing (respectively, non-decreasing) for some $\varepsilon \in \mathbf{R}_+$ since the constraints $M'(x) = m'(x) \pm \varepsilon(x)$ of Theorem 2.4 also hold at the discontinuities of $m'(x)$ at $x = -1/k$. □

3 Stability results

A result on boundedness of the solutions of (1.1) and then its stability under certain parametrical constraints is now given as follows.

Theorem 3.1 *Assume the following*:

(1) *The real sequences* $\{F_n\}$ *and* $\{M_n\}$ *of respective general terms* $F_n = F(x_n)$ *and* $M_n = M(\bar{x}_n)$ *satisfy the constraints* $\nu_n x_n \le F_n \le \mu_n x_n$ *and* $\delta_n \max_{n-p+1 \le i \le n} x_i \le M_n \le \omega_n \max_{n-p+1 \le i \le n} x_i$ *for some nonnegative real sequences* $\{\nu_n\}$, $\{\mu_n\}$, $\{\delta_n\}$ *and* $\{\omega_n\}$ *with* $\mu_n \ge \nu_n$ *and* $\omega_n \ge \delta_n$; $\forall n \in \mathbf{N}_0$.

(2) *There are* $a, b\ (\ge a) \in \mathbf{R}_{0+}$ *such that* $b \ge x_i \ge a$; $i = 1 - p, 2 - p, \dots, 0$.

(3) $G : \mathbf{R}_{0+}^q \to \mathbf{R}$, *which generates the real sequence* $\{G_n\}$ *of general terms* $G_n = G(x_n)$ *satisfies the following constraints on* $\mathbf{R}_{0+}$:

$$a \le G_n(x) \le b \quad \text{if } A \ge 0; \qquad a \ge G_n(x) \ge b \quad \text{if } A < 0.$$

Then the following properties hold:

(i) $b \ge x_i \ge a$; $\forall i \in \{1 - p, 2 - p, \dots, 0\} \cup \mathbf{N}$ *if the sequences* $\{\nu_n\}$, $\{\mu_n\}$ *satisfy*

$$\begin{aligned} &\frac{e^{Ab}}{a^2(\sum_{j=0}^{k-1} \nu_{n-j})}\left[a + b\left(\omega_n e^{-Aa} + \sum_{j=0}^{k-1} \mu_{n-j}\right) - 1\right] \\ &\quad \le \delta_n \le \omega_n \le \frac{e^{Aa}}{b^2(\sum_{j=0}^{k-1} \mu_{n-j})}\left[b + a\left(\delta_n e^{-Ab} + \sum_{j=0}^{k-1} \nu_{n-j}\right) - 1\right]; \end{aligned} \tag{3.1}$$

$\forall n \in \mathbf{N}_0$ *for given* a, b, $\{\nu_n\}$ *and* $\{\mu_n\}$.

(ii) *A sufficient condition for* (3.1) *to hold is*

$$\begin{aligned}
&\left(1-\frac{1}{ab(\sum_{j=0}^{k-1}\nu_{n-j})(\sum_{j=0}^{k-1}\mu_{n-j})}\right)^{-1}\frac{e^{Ab}}{a^2(\sum_{j=0}^{k-1}\nu_{n-j})}\\
&\qquad\times\left[a-1+b\left(\sum_{j=0}^{k-1}\mu_{n-j}\right)+\frac{1}{\sum_{j=0}^{k-1}\nu_{n-j}}+\frac{b(\sum_{j=0}^{k-1}\mu_{n-j}-1)}{a(\sum_{j=0}^{k-1}\nu_{n-j})}\right]\\
&\quad\le\delta_n\le\omega_n\\
&\quad\le\left(1-\frac{1}{ab(\sum_{j=0}^{k-1}\nu_{n-j})(\sum_{j=0}^{k-1}\mu_{n-j})}\right)^{-1}\frac{e^{Aa}}{b^2(\sum_{j=0}^{k-1}\mu_{n-j})}\\
&\qquad\times\left[b-1+a\left(\sum_{j=0}^{k-1}\nu_{n-j}\right)+\frac{1}{\sum_{j=0}^{k-1}\mu_{n-j}}+\frac{a(\sum_{j=0}^{k-1}\nu_{n-j}-1)}{b(\sum_{j=0}^{k-1}\mu_{n-j})}\right];
\end{aligned}$$

$\forall n\in\mathbf{N}_0$ *provided that* $(\sum_{j=0}^{k-1}\nu_{n-j})(\sum_{j=0}^{k-1}\mu_{n-j})\neq\frac{1}{ab}$.

Proof Note that $1-\omega_n\bar{x}_n e^{-Aa}\le 1-M_n e^{-AG_n}\le 1-\delta_n\bar{x}_n e^{-Ab}$; $\forall n\in\mathbf{N}_0$ since

$$\begin{aligned}
&\bigl[\bigl(a\le G_n(x)\le b\text{ if }A\ge 0\bigr)\wedge\bigl(a\ge G_n(x)\ge b\text{ if }A<0\bigr)\bigr]\\
&\quad\Rightarrow\quad\bigl[e^{Aa}\le e^{AG_n}\le e^{Ab}\bigr];\quad\forall n\in\mathbf{N}_0.
\end{aligned}$$

Assume that $0\le a\le x_n\le b<\infty$ for some $a,b\ (\ge a)\in\mathbf{R}_+$ and $p\ (\ge n)\in\mathbf{N}_0$. Proceed now by complete induction by assuming that if there is some $n\ (\ge p)\in\mathbf{N}$ such that $b\ge x_i\ge a$; $\forall i\in\bar{n}$, then $b\ge x_{n+1}\ge a$. If $x_{n+1}=0$, this holds trivially for any $a,b\ (\ge a)\in\mathbf{R}_{0+}$. Note that for $i\in\bar{n}$,

$$\nu_i a\le F_i\le\mu_i b;\qquad e^{Aa}\le e^{AG_i}\le e^{Ab},\qquad b\omega_i e^{-Aa}\ge M_i e^{-AG_i}\ge a\delta_i e^{-Ab},\tag{3.2}$$

$$1-b\omega_i e^{-Aa}\le 1-\omega_i\bar{x}_i e^{-Aa}\le 1-M_i e^{-AG_i}\le 1-\delta_i\bar{x}_i e^{-Ab}\le 1-a\delta_i e^{-Ab}.\tag{3.3}$$

If $x_{n+1}\neq 0$ then it satisfies from (1.1) by using (3.2)-(3.3) that

$$\begin{aligned}
a&\le a^2\delta_n\left(\sum_{j=0}^{k-1}\nu_{n-j}\right)e^{-Ab}-b\left(\omega_n e^{-Aa}+\sum_{j=0}^{k-1}\mu_{n-j}\right)+1\\
&\le x_{n+1}=\left(1-\sum_{j=0}^{k-1}F_{n-j}\right)\bigl(1-M_n e^{-AG_n}\bigr)\\
&=-F_n+F_nM_ne^{-AG_n}+\left(\sum_{j=1}^{k-1}F_{n-j}\right)M_ne^{-AG_n}-\left(\sum_{j=1}^{k-1}F_{n-j}\right)-M_ne^{-AG_n}+1\\
&\le b^2\omega_n\left(\sum_{j=0}^{k-1}\mu_{n-j}\right)e^{-Aa}-a\left(\delta_n e^{-Ab}+\sum_{j=0}^{k-1}\nu_{n-j}\right)+1\le b;
\end{aligned}\tag{3.4}$$

$\forall n\in\mathbf{N}_0$ provided that (3.1) holds. Thus, if $b\ge x_i\ge a$; $i=1-p,2-p,\dots,0$, then $b\ge x_i\ge a$; $\forall i\in\{1-p,2-p,\dots,0\}\cup\mathbf{N}$. Hence, Property (i).

Property (ii) follows by replacing the lower-bound of δ_n of (3.1) in the upper-bound of ω_n to get a more stringent upper-bound $\overline{\omega}_n$ of ω_n, which does not depend on δ_n since

$$\begin{aligned}\omega_n &\le \overline{\omega}_n \\ &= \frac{e^{Aa}}{b^2(\sum_{j=0}^{k-1}\mu_{n-j})}\left[b-1+a\left(\sum_{j=0}^{k-1}\nu_{n-j}\right)\right. \\ &\quad \left. + \frac{1}{a(\sum_{j=0}^{k-1}\nu_{n-j})}\left(a+b\omega_n e^{-Aa}+b\sum_{j=0}^{k-1}\mu_{n-j}-1\right)\right]; \quad \forall n\in\mathbf{N}_0.\end{aligned}$$

In a close way, we can get a more stringent lower-bound $\underline{\delta}_n$ of δ_n, which does not depend on ω_n since

$$\begin{aligned}\delta_n &\ge \underline{\delta}_n \\ &= \frac{e^{Ab}}{a^2(\sum_{j=0}^{k-1}\nu_{n-j})}\left[a-1+b\left(\sum_{j=0}^{k-1}\mu_{n-j}\right)\right. \\ &\quad \left. + \frac{1}{b(\sum_{j=0}^{k-1}\mu_{n-j})}\left(b+a\delta_n e^{-Ab}+a\sum_{j=0}^{k-1}\nu_{n-j}-1\right)\right]; \quad \forall n\in\mathbf{N}_0.\end{aligned}$$

□

Remark 3.2 (Brief historical note) It can be pointed out that assumptions of the type in the assumption in Theorem 3.1 are very relevant to some classical control problems for both continuous-time or discrete-time descriptions of dynamic systems, like, for instance, those of absolute stability in the Lure and Popov (following Vasile Mihai Popov - Galati, Romania, 1928) senses or Popovian hyperstability [22]. Basically, if there are uncertainties in parameterization, which is a very common drawback from fabrication dispersion of components for devices constructed for applications, a robust regulator or controller, in general, has to be able to stabilise all the particular elements of the whole series within some error margin, not just the theoretically nominal one. The lack in appropriately formulating that problem implied during Second World War II a lack of well-regulated equilibrium positioning of guns of some Soviet military tanks with the associate lack of effectiveness in military operations. This was the initial point of the theory of Popov's absolute stability later on being generalised to hyperstability after including the phenomena of existence of unmodelled dynamics. This was apparently one of the reasons for Popov's decision of to investigate the simultaneous stabilization of devices of the same family subject to parametrical dispersion of components related to a theoretical nominal ideal device (source: old private communication by PhD supervisor ID Landau, a former Popov's collaborator and later on a relevant researcher in the field, to the first author of this paper). See also [23] for relevant content honouring Popov's work. In the context of this paper, we can attribute the unmodelled or parametrical errors to a non-exact parameterization of the epidemic model for each possible situation.

A set of stability and instability properties, implying, furthermore, that any nontrivial solution of (1.1) is strictly monotone for $n\in\mathbf{N}_0$, are presented in the next two results. Some of the properties depend on parametrical conditions of lower- and upper-bounding sequences of $\{F_n\}$.

Theorem 3.3 *Assume that the real sequence* $\{F_n\}$ *satisfies the constraints* $\sum_{i=0}^{m} \nu_{ni} x_n^i \le F_n \le \sum_{i=0}^{m} \mu_{ni} x_n^i$ *for some real sequences* $\{\nu_{ni}\}$, $\{\mu_{ni}\}$ *with* $\mu_{ni} \ge \nu_{ni}$; $i \in \overline{m} \cup \{0\}$, $\forall n \in \mathbf{N}_0$. *The following properties hold.*

(i) *Any nontrivial nonnegative solution* $\{x_n\}$ *of* (1.1) *is uniformly bounded and strictly decreasing for* $n \in \mathbf{N}_0$ *and initial conditions* $1 \ge \min(1, \min_{n\in\mathbf{N}_{0+}}(\frac{1-\sum_{j=n-k+1}^{n}\mu_{j0}}{\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\mu_{ji}})) \ge x_{-p+1} \ge \cdots \ge x_{-1} \ge x_0$, *and then it converges to the zero equilibrium point under the following condition*:

$$\left(\left[\left(\sum_{j=n-k+1}^{n} \mu_{j0} < 1\right) \wedge \left(\sum_{j=n-k+1}^{n}\sum_{i=1}^{m} \mu_{ji} \ge 0\right)\right] \vee \left[\left(\sum_{j=n-k+1}^{n} \mu_{j0} > 1\right) \wedge \left(\sum_{j=n-k+1}^{n}\sum_{i=1}^{m} \mu_{ji} \le 0\right)\right]\right) \wedge \left(M_n \ge e^{AG_n}\right); \quad \forall n \in \mathbf{N}_0. \tag{3.5}$$

(ii) *Any nontrivial nonnegative solution* $\{x_n\}$ *of* (1.1) *is uniformly bounded and strictly decreasing for* $n \in \mathbf{N}_0$ *and initial conditions* $\min(1, \min_{n\in\mathbf{N}_{0+}}(\frac{\sum_{j=n-k+1}^{n}\nu_{j0}-1}{|\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\nu_{ji}|})) > x_{-p+1} \ge \cdots \ge x_{-1} \ge x_0$, *and then it converges to the zero equilibrium point under the following condition*:

$$\left[\left(\sum_{j=n-k+1}^{n} \nu_{j0} < 1\right) \wedge (\nu_{ji} \le 0; \forall j \in \mathbf{N}_{0+}, \forall i \in \overline{m})\right] \wedge \left(M_n \le e^{AG_n}\right); \quad \forall n \in \mathbf{N}_0. \tag{3.6}$$

Proof Define $\lambda_n = 1 - M_n e^{-AG_n}$. Thus, one gets

$$x_{n+1} - x_n = \left(1 - \sum_{j=n-k+1}^{n} F_j\right)\lambda_n - x_n < 0; \quad \forall n \in \mathbf{N}_0 \tag{3.7}$$

from (1.1) if $x_n > 0$, $\sum_{j=n-k+1}^{n} F_j \le 1$ and $\lambda_n \le 0$, which is guaranteed with $M_n \ge e^{AG_n}$; $\forall n \in \mathbf{N}_0$, if for any $n \in \mathbf{N}_0$, the constraints $0 < x_n < \cdots < x_0 \le x_{-1} \le \cdots \le x_{-p+1} \le \min(1, \min_{n\in\mathbf{N}_{0+}}(\frac{1-\sum_{j=n-k+1}^{n}\mu_{j0}}{\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\mu_{ji}}))$ hold for any given $n \in \mathbf{N}_{0+}$, together with

$$\begin{aligned}\sum_{j=n-k+1}^{n} F_j &\le \sum_{j=n-k+1}^{n} \mu_{j0} + \sum_{j=n-k+1}^{n}\sum_{i=1}^{m} \mu_{ji} x_j^i \le \sum_{j=n-k+1}^{n} \mu_{j0} + \sum_{j=n-k+1}^{n}\sum_{i=1}^{m} \mu_{ji} \max\left(x_j, x_j^m\right) \\ &\le \sum_{j=n-k+1}^{n} \mu_{j0} + \sum_{j=n-k+1}^{n}\sum_{i=1}^{m} \mu_{ji} x_j \le \sum_{j=n-k+1}^{n} \mu_{j0} + \left(\sum_{j=n-k+1}^{n}\sum_{i=1}^{m} \mu_{ji}\right) x_{-p+1} \\ &\le 1\end{aligned} \tag{3.8a}$$

so that

$$\begin{aligned}0 &< x_{n+1} < x_n < \cdots < x_0 \le x_{-p+1} \\ &\le \min\left(1, \min_{n\in\mathbf{N}_{0+}}\left(\frac{1-\sum_{j=n-k+1}^{n}\mu_{j0}}{\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\mu_{ji}}\right)\right); \quad \forall n \in \mathbf{N}_0\end{aligned} \tag{3.8b}$$

provided that $\min(1, \min_{n\in\mathbf{N}_{0+}}(\frac{1-\sum_{j=n-k+1}^{n}\mu_{j0}}{\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\mu_{ji}})) \geq x_{-p+1} \geq \cdots \geq x_{-1} \geq x_0$. It has been proven by complete induction that for any given $n \in \mathbf{N}_0$,

$$\begin{aligned}
&0 < x_n < \cdots < x_0 \leq x_{-1} \leq \cdots \leq x_{-p+1} \leq \min\left(1, \min_{n\in\mathbf{N}_{0+}}\left(\frac{1-\sum_{j=n-k+1}^{n}\mu_{j0}}{\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\mu_{ji}}\right)\right) \leq 1 \\
&\quad\Rightarrow\quad 0 < x_{n+1} < x_n < \cdots < x_0 \leq x_{-p+1} \\
&\qquad\quad \leq \min\left(1, \min_{n\in\mathbf{N}_{0+}}\left(\frac{1-\sum_{j=n-k+1}^{n}\mu_{j0}}{\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\mu_{ji}}\right)\right) \leq 1. \qquad (3.9)
\end{aligned}$$

A close result also holds involving non-strict inequalities if $x_n = 0$, $\sum_{j=n-k+1}^{n} F_j \leq 1$ and $\lambda_n \leq 0$ or if $x_n \geq 0$, $\sum_{j=n-k+1}^{n} F_j > 1$ and $\lambda_n \geq 0$, then $x_j = 0$; $\forall j\ (\geq n+1) \in \mathbf{N}_0$ from (1.1). Then (3.5) guarantees the convergence to zero of the sequence $\{x_n\}$; $n \in \mathbf{N}_0$ which is, furthermore, strictly decreasing. Hence, Property (i).

To prove Property (ii), note that (3.7) also holds from (1.1) for $x_n > 0$ if $\sum_{j=n-k+1}^{n} F_j \geq 1$ and $\lambda_n \geq 0$, which is guaranteed with $M_n \leq e^{AG_n}$; $\forall n \in \mathbf{N}_{0+}$, if $\sum_{j=n-k+1}^{n}\sum_{i=1}^{m} \nu_{ji} \leq 0$, and then the complete induction method gives

$$\begin{aligned}
&\min\left(1, \min_{n\in\mathbf{N}_{0+}}\left(\frac{\sum_{j=n-k+1}^{n}\nu_{j0} - 1}{|\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\nu_{ji}|}\right)\right) \\
&\quad > x_{-p+1} \geq \cdots \geq x_{-1} \geq x_0 > x_n \quad \text{for any given } n \in \mathbf{N}_0,
\end{aligned}$$

together with

$$\begin{aligned}
\sum_{j=n-k+1}^{n} F_j &\geq \sum_{j=n-k+1}^{n} \nu_{j0} - \left|\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\nu_{ji}\right| \max\left(x_j, x_j^m\right) \\
&\geq \sum_{j=n-k+1}^{n} \nu_{j0} - \left|\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\nu_{ji}\right| x_{-p+1} \geq 1 \\
\Rightarrow\quad &\min\left(1, \min_{n\in\mathbf{N}_{0+}}\left(\frac{\sum_{j=n-k+1}^{n}\nu_{j0} - 1}{|\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\nu_{ji}|}\right)\right) \\
&\quad > x_{-p+1} \geq \cdots \geq x_{-1} \geq x_0 > x_n > x_{n+1}; \quad \forall n \in \mathbf{N}_0 \qquad (3.10)
\end{aligned}$$

provided that $\min(1, \min_{n\in\mathbf{N}_{0+}}(\frac{\sum_{j=n-k+1}^{n}\nu_{j0}-1}{|\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\nu_{ji}|})) > x_{-p+1} \geq \cdots \geq x_{-1} \geq x_0$. On the other hand, if $x_n \geq 0$ with $\sum_{j=n-k+1}^{n} F_j \geq 1$ and $\lambda_n \geq 0$ or if $x_n \geq 0$, $\sum_{j=n-k+1}^{n} F_j \leq 1$ and $\lambda_n \leq 0$, then $x_j = 0$; $\forall j\ (\geq n+1) \in \mathbf{N}_0$ from (1.1). Then (3.6) guarantees the convergence to zero of the sequence $\{x_n\}$; $n \in \mathbf{N}_0$ which is, furthermore, strictly decreasing. Hence, Property (ii). □

A dual result to Theorem 3.3 concerned with the instability situations follows.

Theorem 3.4 *The following properties hold.*

(i) *Any nontrivial nonnegative solution* $\{x_n\}$ *of* (1.1) *is strictly increasing for* $n \in \mathbf{N}_0$ *and initial conditions* $x_{-p+1} \leq \cdots \leq x_{-1} \leq x_0 > 0$, *and then it tends to* $+\infty$ *under the following* *condition*:

$$\left(e^{AG_n}\left(1+\frac{1}{\sum_{j=n-k+1}^{n}\nu_{j1}}\right) > M_n \geq e^{AG_n}\right) \wedge \left(\sum_{j=n-k+1}^{n}\nu_{j0} \geq 1\right) \wedge (\nu_{ji} \geq 0; \forall j \in \mathbf{N}_{0+}, \forall i \in \overline{m}); \quad \forall n \in \mathbf{N}_0 \tag{3.11}$$

with $M_n > e^{AG_n}$ *if* $x_n \neq 0$ *and* $\sum_{j=n-k+1}^{n}\mu_{j1} > 0$ *with* $\{F_n\}$ *satisfying the same constraints as those of Theorem* 3.3.

(ii) *Any nontrivial nonnegative solution* $\{x_n\}$ *of* (1.1) *is strictly increasing for* $n \in \mathbf{N}_0$ *and initial conditions* $x_{-p+1} \leq \cdots \leq x_{-1} \leq x_0 > 0$, *and then it tends to* $+\infty$ *under the following condition*:

$$\left(M_n \leq e^{AG_n}\right) \wedge \left(\sum_{j=n-k+1}^{n}\mu_{j0} \leq 1\right) \wedge (\mu_{ji} \leq 0; \forall j \in \mathbf{N}_{0+}, \forall i \in \overline{m}); \quad \forall n \in \mathbf{N}_0 \tag{3.12}$$

with $M_n > e^{AG_n}$ *if* $x_n \neq 0$ *and* $\sum_{j=n-k+1}^{n}\mu_{j1} < 0$.

Proof Assume that $x_j > x_{j-1} > \cdots > x_0 \geq x_{-1} \geq \cdots \geq x_{-p+1}$ for any given $j\ (\leq n) \in \mathbf{N}_0$. If $x_{n+1} \neq 0$, then it follows that

$$\begin{aligned} x_{n+1} &= \left(1-\sum_{j=n-k+1}^{n}F_j\right)\left(1-M_n e^{-AG_n}\right) = \left(\sum_{j=n-k+1}^{n}F_j - 1\right)\left(M_n e^{-AG_n}-1\right) \\ &\geq \left(\sum_{j=n-k+1}^{n}\nu_{j0} - 1 + \sum_{j=n-k+1}^{n}\nu_{j1}x_j + \sum_{j=n-k+1}^{n}\sum_{i=2}^{m}\nu_{ji}x_j^i\right)\left(M_n e^{-AG_n}-1\right) > x_n \\ &\geq \left(\sum_{j=n-k+1}^{n}\nu_{j0} - 1 + \sum_{j=n-k+1}^{n}\nu_{j1}x_j + \sum_{j=n-k+1}^{n}\sum_{i=2}^{m}\nu_{ji}\min\left(x_n, x_n^m\right)\right)\left(M_n e^{-AG_n}-1\right) > x_n \end{aligned}$$

if either

$$\begin{aligned} &M_n \geq e^{AG_n}; \qquad \sum_{j=n-k+1}^{n}\nu_{j1} > \frac{1}{1-M_n e^{-AG_n}}; \\ &\sum_{j=n-k+1}^{n}\nu_{j0} \geq 1; \quad \nu_{ji} \geq 0; \forall j, n \in \mathbf{N}_0, \forall i \in \overline{m} \end{aligned} \tag{3.13}$$

with the first inequality being strict for any $x_n \neq 0$, or

$$\begin{aligned} &M_n \leq e^{AG_n}; \qquad \sum_{j=n-k+1}^{n}|\nu_{j1}| > \frac{1}{1-M_n e^{-AG_n}}; \\ &\sum_{j=n-k+1}^{n}\nu_{j0} \leq 1; \quad \nu_{ji} \leq 0; \forall j, n \in \mathbf{N}_0, \forall i \in \overline{m} \end{aligned} \tag{3.14}$$

with the first inequality being strict for any $x_n \neq 0$, and the proof of Property (i) follows by complete induction after combining the first two conditions of (3.13). The proof of

Property (ii) follows from (3.14) in the same way after combining the first two conditions which reduce to the first one. □

The constraints $\sum_{i=0}^{m} \nu_{ni} x_n^i \leq F_n \leq \sum_{i=0}^{m} \mu_{ni} x_n^i$; $\forall n \in \mathbf{N}_0$ have been used in order to simplify the discussion. More general constraints can be used instead as, for instance, $\sum_{i=0}^{m_n} \nu_{ni} x_n^i \leq F_n \leq \sum_{i=0}^{m_n} \mu_{ni} x_n^i$ or $\sum_{i\in I_n} \nu_{ni} x_n^i \leq F_n \leq \sum_{i\in I_n} \mu_{ni} x_n^i$, where $\{m_n\}$ is a nonnegative sequence of polynomial degrees subject to $\underline{m} \leq m_n \leq \overline{m}$; $\forall n \in \mathbf{N}_0$ and $\{I_n\}$ is a nonnegative real sequence of sets of cardinal $(m_n + 1)$; $\forall n \in \mathbf{N}_0$. The above second constraints are not necessarily of polynomial type. Theorems 3.3-3.4 have the following direct extensions such that the stability and instability conditions depend not only of the parameters but on the solution sequence as well.

Theorem 3.5 *The following results hold.*

(i) *Assume that the sequence $\{M_n\}$ satisfies the constraint*

$$M_n > e^{AG_n}\left(1 + \frac{x_n}{(\sum_{j=n-k+1}^{n} \nu_{j0} - 1) + (\sum_{j=n-k+1}^{n} \sum_{i=1}^{m} \nu_{ji}) \min(x_n, x_n^m)}\right); \quad \forall n \in \mathbf{N}_0. \tag{3.15}$$

Thus, any nontrivial nonnegative solution $\{x_n\}$ of (1.1) under initial conditions $x_{-p+1} \geq \cdots \geq x_{-1} \geq x_0 > 0$ is uniformly bounded and strictly decreasing and then converges to the zero equilibrium point. If the above inequality is non-strict, then any nontrivial solution still is uniformly bounded satisfying $x_n \leq x_0 \leq x_{-p+1} < +\infty$, $\forall n \in \mathbf{N}_0$.

(ii) *Assume that the sequence $\{M_n\}$ satisfies the constraint*

$$M_n < e^{AG_n}\left(1 - \frac{x_n}{(\sum_{j=n-k+1}^{n} \mu_{j0} - 1) + (\sum_{j=n-k+1}^{n} \sum_{i=1}^{m} \mu_{ji}) \max(x_n, x_n^m)}\right); \quad \forall n \in \mathbf{N}_0. \tag{3.16}$$

Thus, any nontrivial nonnegative solution $\{x_n\}$ of (1.1) under initial conditions $x_{-p+1} \leq \cdots \leq x_{-1} \leq x_0 > 0$ is strictly increasing and then tends to $+\infty$. If the above inequality is non-strict, then any nontrivial solution is bounded from below satisfying $x_{n+1} \geq x_n \geq x_0 \geq x_{-p+1} > 0$, $\forall n \in \mathbf{N}_0$.

Proof Note that for $x_n > 0$ and $x_{n+1} \geq 0$, one has

$$\begin{aligned} x_{n+1} &= \left(1 - \sum_{j=n-k+1}^{n} F_j\right)\left(1 - M_n e^{-AG_n}\right) \leq x_n \\ &\Leftrightarrow \quad M_n \geq e^{AG_n}\left(1 + \frac{x_n}{\sum_{j=n-k+1}^{n} F_j - 1}\right); \quad \forall n \in \mathbf{N}_0. \end{aligned} \tag{3.17}$$

If $x_n = 0$ then $x_j = 0$; $\forall j\ (\geq n) \in \mathbf{N}_0$ from (1.1). Thus, Eq. (3.15) is a sufficient condition for (3.17) to hold. This proves the first part of Property (i) and the solution sequence is strictly decreasing. If the inequality in (3.15) is non-strict, then $x_{n+1} \leq x_n \leq x_0 \leq x_{-p+1}$; $\forall n \in \mathbf{N}_0$ leading to the second part of Property (i). Property (ii) follows by proving that

(3.16) guarantees that

$$x_{n+1} = \max\left(0, \left(1 - \sum_{j=n-k+1}^{n} F_j\right)\left(1 - M_n e^{-AG_n}\right)\right) > (\geq)\, x_n; \quad \forall n \in \mathbf{N}_0. \qquad \square$$

Theorems 3.3-3.4 may be combined to give mixed conditions for the solution to be oscillatory while being uniformly bounded as follows.

Theorem 3.6 *Assume that the real sequence $\{F_n\}$ satisfies the constraints: $\sum_{i=0}^{m} \nu_{ni} x_n^i \leq F_n \leq \sum_{i=0}^{m} \mu_{ni} x_n^i$ for some nonnegative real sequences $\{\nu_{ni}\}$, $\{\mu_{ni}\}$ with $\mu_{ni} \geq \nu_{ni}$; $i \in \overline{m} \cup \{0\}$, $\forall n \in \mathbf{N}_0$. Then the following properties hold.*

(i) *Any nontrivial nonnegative solution $\{x_n\}$ of* (1.1) *for initial conditions* $\min(1, \min_{n \in \mathbf{N}_{0+}}(\frac{1-\sum_{j=n-k+1}^{n} \mu_{j0}}{\sum_{j=n-k+1}^{n} \sum_{i=1}^{m} \mu_{ji}})) \geq x_{-p+1} \geq \cdots \geq x_{-1} \geq x_0$ *is uniformly bounded and oscillatory if it satisfies, for each two consecutive intervals on nonnegative integers, the following constraints*:

$$\begin{aligned}
&\left(\left[\left(\sum_{j=n-k+1}^{n} \mu_{j0} < 1\right) \wedge \left(\sum_{j=n-k+1}^{n} \sum_{i=1}^{m} \mu_{ji} \geq 0\right)\right]\right. \\
&\quad \left. \vee \left[\left(\sum_{j=n-k+1}^{n} \mu_{j0} > 1\right) \wedge \left(\sum_{j=n-k+1}^{n} \sum_{i=1}^{m} \mu_{ji} \leq 0\right)\right]\right) \\
&\quad \wedge \left(M_n \geq e^{AG_n}\right); \quad \forall n \in \left[k\left(\sum_{i=1}^{j} n_i\right), k\left(\sum_{i=1}^{j+1} n_i\right) - 1\right),
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\left(e^{AG_n}\left(1 + \frac{1}{\sum_{j=n-k+1}^{n} \nu_{j1}}\right) > M_n \geq e^{AG_n}\right) \\
&\quad \wedge \left(\sum_{j=n-k+1}^{n} \nu_{j0} \geq 1\right) \wedge \left(\nu_{ji} \geq 0; \forall j \in \mathbf{N}_0, \forall i \in \overline{m}\right); \\
&\quad \forall n \in \left[k\left(\sum_{i=1}^{j+1} n_i\right), k\left(\sum_{i=1}^{j+2} n_i\right) - 1\right)
\end{aligned} \tag{3.19}$$

with $M_n > e^{AG_n}$ if $x_n \neq 0$ in (3.19), *for any given finite $j \in \mathbf{N}_0$ for any set of finite numbers $n_i \in \mathbf{N}$ for $i \geq 1$, $i \in \mathbf{N}_0$ with $n_0 = 0$, which satisfy $x_{k(\sum_{i=1}^{j+2} n_i)-1} \leq x_{k(\sum_{i=1}^{j+1} n_i)-1}$. The solution is alternately strictly decreasing and strictly increasing for each two consecutive such intervals.*

(ii) *Any nontrivial nonnegative solution $\{x_n\}$ of* (1.1) *for initial conditions* $\min(1, \min_{n \in \mathbf{N}_{0+}}(\frac{1-\sum_{j=n-k+1}^{n} \mu_{j0}}{\sum_{j=n-k+1}^{n} \sum_{i=1}^{m} \mu_{ji}})) \geq x_{-p+1} \geq \cdots \geq x_{-1} \geq x_0$ *is uniformly bounded and oscillatory if it satisfies the following constraints for each two consecutive intervals on nonnegative integers*:

$$\begin{aligned}
&\left(\left[\left(\sum_{j=n-k+1}^{n} \mu_{j0} < 1\right) \wedge \left(\sum_{j=n-k+1}^{n} \sum_{i=1}^{m} \mu_{ji} \geq 0\right)\right]\right. \\
&\quad \left. \vee \left[\left(\sum_{j=n-k+1}^{n} \mu_{j0} > 1\right) \wedge \left(\sum_{j=n-k+1}^{n} \sum_{i=1}^{m} \mu_{ji} \leq 0\right)\right]\right)
\end{aligned}$$

$$\wedge \left(M_n \geq e^{AG_n}\right); \quad \forall n \in \left[k\left(\sum_{i=1}^{j} n_i\right), k\left(\sum_{i=1}^{j+1} n_i\right) - 1\right), \tag{3.20}$$

$$\left(M_n \leq e^{AG_n}\right) \wedge \left(\sum_{j=n-k+1}^{n} \mu_{j0} \leq 1\right) \wedge (\mu_{ji} \leq 0; \forall j \in \mathbf{N}_{0+}, \forall i \in \overline{m});$$

$$\forall n \in \left[k\left(\sum_{i=1}^{j+1} n_i\right), k\left(\sum_{i=1}^{j+2} n_i\right) - 1\right) \tag{3.21}$$

with $M_n < e^{AG_n}$ *if* $x_n \neq 0$ *in* (3.21), *for any given finite* $j \in \mathbf{N}_0$ *and for any set of finite numbers* $n_i \in \mathbf{N}$ *for* $i \geq 1$, $i \in \mathbf{N}_{0+}$ *with* $n_0 = 0$, *which satisfy* $x_{k(\sum_{i=1}^{j+2} n_i)-1} \leq x_{k(\sum_{i=1}^{j+1} n_i)-1}$. *The solution is alternately strictly decreasing and strictly increasing for each two consecutive such intervals.*

(iii) *Any nontrivial nonnegative solution* $\{x_n\}$ *of* (1.1) *for initial conditions* $\min(1, \min_{n\in\mathbf{N}_{0+}}(\frac{\sum_{j=n-k+1}^{n} \nu_{j0}-1}{|\sum_{j=n-k+1}^{n}\sum_{i=1}^{m} \nu_{ji}|})) > x_{-p+1} \geq \cdots \geq x_{-1} \geq x_0$ *is uniformly bounded and oscillatory if it satisfies the following constraints for each two consecutive intervals on nonnegative integers*:

$$\left[\left(\sum_{j=n-k+1}^{n} \nu_{j0} < 1\right) \wedge (\nu_{ji} \leq 0; \forall j \in \mathbf{N}_0, \forall i \in \overline{m})\right]$$

$$\wedge \left(M_n \leq e^{AG_n}\right); \quad \forall n \in \left[k\left(\sum_{i=1}^{j} n_i\right), k\left(\sum_{i=1}^{j+1} n_i\right) - 1\right), \tag{3.22}$$

$$\left(e^{AG_n}\left(1 + \frac{1}{\sum_{j=n-k+1}^{n} \nu_{j1}}\right) > M_n \geq e^{AG_n}\right)$$

$$\wedge \left(\sum_{j=n-k+1}^{n} \nu_{j0} \geq 1\right) \wedge (\nu_{ji} \geq 0; \forall j \in \mathbf{N}_0, \forall i \in \overline{m});$$

$$\forall n \in \left[k\left(\sum_{i=1}^{j+1} n_i\right), k\left(\sum_{i=1}^{j+2} n_i\right) - 1\right) \tag{3.23}$$

with $M_n > e^{AG_n}$ *if* $x_n \neq 0$ *in* (3.23), *for any given finite* $j \in \mathbf{N}_0$ *for any set of finite numbers* $n_i \in \mathbf{N}$ *for* $i \geq 1$, $i \in \mathbf{N}_0$ *with* $n_0 = 0$, *which satisfy* $x_{k(\sum_{i=1}^{j+2} n_i)-1} \leq x_{k(\sum_{i=1}^{j+1} n_i)-1}$. *The solution is alternately strictly decreasing and strictly increasing for each two consecutive such intervals.*

(iv) *Any nontrivial nonnegative solution* $\{x_n\}$ *of* (1.1) *for initial conditions* $\min(1, \min_{n\in\mathbf{N}_{0+}}(\frac{\sum_{j=n-k+1}^{n} \nu_{j0}-1}{|\sum_{j=n-k+1}^{n}\sum_{i=1}^{m} \nu_{ji}|})) > x_{-p+1} \geq \cdots \geq x_{-1} \geq x_0$ *is uniformly bounded and oscillatory if it satisfies the following constraints for each two consecutive intervals on nonnegative integers*:

$$\left[\left(\sum_{j=n-k+1}^{n} \nu_{j0} < 1\right) \wedge (\nu_{ji} \leq 0; \forall j \in \mathbf{N}_{0+}, \forall i \in \overline{m})\right] \wedge \left(M_n \leq e^{AG_n}\right);$$

$$\forall n \in \left[k\left(\sum_{i=1}^{j} n_i\right), k\left(\sum_{i=1}^{j+1} n_i\right) - 1\right), \tag{3.24}$$

$$\left(M_n \le e^{AG_n}\right) \wedge \left(\sum_{j=n-k+1}^{n} \mu_{j0} \le 1\right) \wedge \left(\mu_{ji} \le 0; \forall j \in \mathbf{N}_{0+}, \forall i \in \overline{m}\right);$$

$$\forall n \in \left[k\left(\sum_{i=1}^{j+1} n_i\right), k\left(\sum_{i=1}^{j+2} n_i\right) - 1\right) \tag{3.25}$$

with $M_n < e^{AG_n}$ *if* $x_n \neq 0$ *in* (3.25), *for any given finite* $j \in \mathbf{N}_0$ *for any set of finite numbers* $n_i \in \mathbf{N}$ *for* $i \ge 1$, $i \in \mathbf{N}_0$, *with* $n_0 = 0$, *which satisfy* $x_{k(\sum_{i=1}^{j+2} n_i)-1} \le x_{k(\sum_{i=1}^{j+1} n_i)-1}$; $\forall j \in \mathbf{N}_0$. *The solution is alternately strictly decreasing and strictly increasing for each two consecutive such intervals.*

(v) *Any nontrivial nonnegative solution* $\{x_n\}$ *of* (1.1) *for initial conditions* $x_{-p+1} \le \cdots \le x_{-1} \le x_0 > 0$ *is uniformly bounded and oscillatory if it satisfies the following constraints for each two consecutive intervals on nonnegative integers*:

$$\left(e^{AG_n}\left(1 + \frac{1}{\sum_{j=n-k+1}^{n} \nu_{j1}}\right) > M_n \ge e^{AG_n}\right)$$

$$\wedge \left(\sum_{j=n-k+1}^{n} \nu_{j0} \ge 1\right) \wedge \left(\nu_{ji} \ge 0; \forall j \in \mathbf{N}_0, \forall i \in \overline{m}\right);$$

$$\forall n \in \left[k\left(\sum_{i=1}^{j} n_i\right), k\left(\sum_{i=1}^{j+1} n_i\right) - 1\right), \tag{3.26}$$

$$\left(\left[\left(\sum_{j=n-k+1}^{n} \mu_{j0} < 1\right) \wedge \left(\sum_{j=n-k+1}^{n} \sum_{i=1}^{m} \mu_{ji} \ge 0\right)\right]\right.$$

$$\left.\vee \left[\left(\sum_{j=n-k+1}^{n} \mu_{j0} > 1\right) \wedge \left(\sum_{j=n-k+1}^{n} \sum_{i=1}^{m} \mu_{ji} \le 0\right)\right]\right);$$

$$\forall n \in \left[k\left(\sum_{i=1}^{j+1} n_i\right), k\left(\sum_{i=1}^{j+2} n_i\right) - 1\right) \tag{3.27}$$

with $M_n > e^{AG_n}$ *if* $x_n \neq 0$ *in* (3.26), *for any given finite* $j \in \mathbf{N}_0$ *for any set of finite numbers* $n_i \in \mathbf{N}$ *for* $i \ge 1$, $i \in \mathbf{N}_0$ *with* $n_0 = 0$, *which satisfy* $x_{k(\sum_{i=1}^{j+2} n_i)-1} \le x_{k(\sum_{i=1}^{j+1} n_i)-1}$. *The solution is alternately strictly increasing and strictly decreasing for each two consecutive such intervals. If the above inequality is strict, then the solution converges asymptotically to the zero equilibrium point.*

(vi) *Any nontrivial nonnegative solution* $\{x_n\}$ *of* (1.1) *for initial conditions* $x_{-p+1} \le \cdots \le x_{-1} \le x_0 > 0$ *is uniformly bounded and oscillatory if it satisfies the following constraints for each two consecutive intervals on nonnegative integers*:

$$\left(M_n \le e^{AG_n}\right) \wedge \left(\sum_{j=n-k+1}^{n} \mu_{j0} \le 1\right) \wedge \left(\mu_{ji} \le 0; \forall j \in \mathbf{N}_{0+}, \forall i \in \overline{m}\right);$$

$$\forall n \in \left[k\left(\sum_{i=1}^{j} n_i\right), k\left(\sum_{i=1}^{j+1} n_i\right) - 1\right), \tag{3.28}$$

$$\left(\left[\left(\sum_{j=n-k+1}^{n} \mu_{j0} < 1\right) \wedge \left(\sum_{j=n-k+1}^{n} \sum_{i=1}^{m} \mu_{ji} \ge 0\right)\right]\right.$$

$$\vee\left[\left(\sum_{j=n-k+1}^{n}\mu_{j0}>1\right)\wedge\left(\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\mu_{ji}\leq 0\right)\right]\Bigg);$$

$$\forall n\in\left[k\left(\sum_{i=1}^{j+1}n_i\right),k\left(\sum_{i=1}^{j+2}n_i\right)-1\right) \tag{3.29}$$

with $M_n<e^{AG_n}$ *if* $x_n\neq 0$ *in* (3.28), *for any given finite* $j\in\mathbf{N}_0$ *for any set of finite numbers* $n_i\in\mathbf{N}$ *for* $i\geq 1$, $i\in\mathbf{N}_0$ *with* $n_0=0$, *which satisfy* $x_{k(\sum_{i=1}^{j+2}n_i)-1}\leq x_{k(\sum_{i=1}^{j+1}n_i)-1}$; $\forall j\in\mathbf{N}_0$. *The solution is alternately strictly increasing and strictly decreasing for each two consecutive such intervals. If the above inequality is strict, then the solution converges asymptotically to the zero equilibrium point.*

(vii) *Any nontrivial nonnegative solution* $\{x_n\}$ *of* (1.1) *for initial conditions* $x_{-p+1}\leq\cdots\leq x_{-1}\leq x_0>0$ *is uniformly bounded and oscillatory if it satisfies the following constraints for each two consecutive intervals on nonnegative integers*:

$$\left(e^{AG_n}\left(1+\frac{1}{\sum_{j=n-k+1}^{n}\nu_{j1}}\right)>M_n\geq e^{AG_n}\right)$$

$$\wedge\left(\sum_{j=n-k+1}^{n}\nu_{j0}\geq 1\right)\wedge(\nu_{ji}\geq 0;\forall j\in\mathbf{N}_{0+},\forall i\in\overline{m});$$

$$\forall n\in\left[k\left(\sum_{i=1}^{j}n_i\right),k\left(\sum_{i=1}^{j+1}n_i\right)-1\right), \tag{3.30}$$

$$\left[\left(\sum_{j=n-k+1}^{n}\nu_{j0}<1\right)\wedge(\nu_{ji}\leq 0;\forall j\in\mathbf{N}_{0+},\forall i\in\overline{m})\right]\wedge\left(M_n\leq e^{AG_n}\right);$$

$$\forall n\in\left[k\left(\sum_{i=1}^{j+1}n_i\right),k\left(\sum_{i=1}^{j+2}n_i\right)-1\right) \tag{3.31}$$

with $M_n>e^{AG_n}$ *if* $x_n\neq 0$ *in* (3.30) *for any given finite* $j\in\mathbf{N}_{0+}$ *for any set of finite numbers* $n_i\in\mathbf{N}$ *for* $i\geq 1$, $i\in\mathbf{N}_0$, *with* $n_0=0$, *which satisfy* $x_{k(\sum_{i=1}^{j+2}n_i)-1}\leq x_{k(\sum_{i=1}^{j+1}n_i)-1}$; $\forall j\in\mathbf{N}_0$. *The solution is alternately strictly increasing and strictly decreasing for each two consecutive such intervals. If the above inequality is strict, then the solution converges asymptotically to the zero equilibrium point.*

(viii) *Any nontrivial nonnegative solution* $\{x_n\}$ *of* (1.1) *for initial conditions* $x_{-p+1}\leq\cdots\leq x_{-1}\leq x_0>0$ *is uniformly bounded and oscillatory if it satisfies the following constraints for each two consecutive intervals on nonnegative integers*:

$$\left(M_n\leq e^{AG_n}\right)\wedge\left(\sum_{j=n-k+1}^{n}\mu_{j0}\leq 1\right)\wedge(\mu_{ji}\leq 0;\forall j\in\mathbf{N}_{0+},\forall i\in\overline{m});$$

$$\forall n\in\left[k\left(\sum_{i=1}^{j+1}n_i\right),k\left(\sum_{i=1}^{j+2}n_i\right)-1\right), \tag{3.32}$$

$$\left[\left(\sum_{j=n-k+1}^{n}\nu_{j0}<1\right)\wedge(\nu_{ji}\leq 0;\forall j\in\mathbf{N}_{0+},\forall i\in\overline{m})\right]\wedge\left(M_n\leq e^{AG_n}\right);$$

$$\forall n\in\left[k\left(\sum_{i=1}^{j}n_i\right),k\left(\sum_{i=1}^{j+1}n_i\right)-1\right) \tag{3.33}$$

with $M_n > e^{AG_n}$ if $x_n \neq 0$ in (3.32) *for any given finite $j \in \mathbf{N}_0$ for any set of finite numbers $n_i \in \mathbf{N}$ for $i \geq 1$, $i \in \mathbf{N}_0$, with $n_0 = 0$, which satisfy $x_{k(\sum_{i=1}^{j+2} n_i)-1} \leq x_{k(\sum_{i=1}^{j+1} n_i)-1}$; $\forall j \in \mathbf{N}_0$. The solution is alternately strictly increasing and strictly decreasing for each two consecutive such intervals. If the above inequality is strict, then the solution converges asymptotically to the zero equilibrium point.*

Proof Property (i) is proven directly as follows. For any given finite $j \in \mathbf{N}_0$, the solution subsequence $\{x_{k(\sum_{i=1}^{j+\ell} n_i)-1}\}$ satisfies the chain of inequalities $x_{k(\sum_{i=1}^{j+\ell} n_i)-1} \leq x_{k(\sum_{i=1}^{j+\ell-1} n_i)-1} \leq \cdots \leq x_{k(\sum_{i=1}^{j-1} n_i)-1} < +\infty$; $\forall \ell \in \mathbf{N}$ by construction for a set of finite $n_i \in \mathbf{N}$. Thus, the whole solution cannot possess any other unbounded subsequence since this would be a contradiction to the above chain of finitely upper-bounded inequalities by taking account of the fact that all the numbers $n_i \in \mathbf{N}$ are finite. The solution is alternately strictly increasing and decreasing from Theorem 3.4(i) and Theorem 3.3(i). The first part of Property (ii) is a dual one to Property (i) and it is proven also from Theorems 3.3-3.4 and the existence of a bounded chain of finitely bounded non-strict inequalities as above. The second part is proven as follows. The conditions (3.18) and (3.19) lead to strictly increasing and strictly decreasing sequences of the solution of finite sizes $n_i \in \mathbf{N}$, $i \in \mathbf{N}_0$, respectively. The strict inequality $x_{k(\sum_{i=1}^{j+2} n_i)-1} < x_{k(\sum_{i=1}^{j+1} n_i)-1}$; $\forall j \in \mathbf{N}_0$, together with the fact that the above sequences of pair order are strictly decreasing from (3.9) and Property (ii), that is, $x_{k(\sum_{i=1}^{2j} n_i)} > x_{k(\sum_{i=1}^{2j} n_i+1)} > \cdots > x_{k(\sum_{i=1}^{2j} n_i+n_{i,2j+1}+1)}$, lead to the contradiction $0 = \lim_{j\to\infty} x_{k(\sum_{i=1}^{2j} n_i)} > \max_{1\leq \ell \leq n_{i+2j-1}}(\liminf_{j\to\infty} x_{k(\sum_{i=1}^{2j+\ell} n_i)}) > 0$ if $\liminf_{j\to\infty} x_{k(\sum_{i=1}^{j} n_i)} > 0$. Hence, Property (ii). The remaining properties are proven 'mutatis-mutandis'. □

4 Global stability and instability

Consider, for the sake of a more complete discussion, the following generalisation of (1.1):

$$x_{n+1} = \theta_n \max\left(0, \left(1 - \sum_{j=0}^{k-1} F(x_{n-j})\right)\left(1 - M(\bar{x}_n)e^{-AG(\bar{x}_n)}\right)\right); \quad \forall n \in \mathbf{N}_0 \tag{4.1}$$

for initial conditions $x_i \geq 0$; $i = 1-p, 2-p, \ldots, 0$, where the general term of the real sequence $\{\theta_n\}$ satisfies $\theta_n = \theta_n(x_n) \in [0,1]$. This term can be interpreted as a total or partial 'culling' action on the infection in the sense that all of a part of the infected individuals are removed from the habitat by using, for instance, quarantine or simply removal. Physically, we can consider the first stage given by (1.1) producing the solution sequence x_{n+1}^-, which replaces the current solution value x_{n+1} in (1.1), and then the second stage involving an impulsive action leading to the value $x_{n+1} \equiv x_{n+1}^+$ given by (4.1). If $\theta_n \equiv 1$, then (4.1) is identical to (1.1). The global stability of (4.1) is discussed from Lyapunov stability theory as follows. Let us define a Lyapunov sequence candidate $\{V_n\}$ for (4.1) of the general term $V_n = V_n(x_n) = p_n x_n$, where the general term of the sequence $\{p_n\}$ satisfies $0 < \underline{p} \leq p_n \leq \overline{p} < +\infty$ for some $\underline{p}, \overline{p}\ (\geq \underline{p}) \in \mathbf{R}_+$. The following result follows.

Theorem 4.1 *Assume that the sequence $\{F_n\}$ satisfies the constraint $\sum_{i=0}^{m} \nu_{ni} x_n^i \leq F_n \leq \sum_{i=0}^{m} \mu_{ni} x_n^i$. Then the following properties hold.*

(i) *Assume that $M_n \leq e^{AG_n}$; $\forall n \in \mathbf{N}_0$. Any solution sequence of* (4.1) *is uniformly bounded for any bounded set of initial conditions $x_i \geq 0$; $i = 1-p, 2-p, \ldots, 0$, if for any $n \in \mathbf{N}_0$, one* *of the following constraints holds*:

$$\left(\sum_{j=n-k+1}^{n} \nu_{j0} \geq 1\right) \wedge (\nu_{ji} \geq 0; \forall j \in \mathbf{N}_0, \forall i \in \overline{m}) \\ \wedge \left(\theta_n \in \left[0, \max\left(\frac{1}{(1-M_n e^{-AG_n})\nu_{n1}}, 1\right)\right]\right) \tag{4.2}$$

and, furthermore, either $\theta_n < \frac{1}{(1-M_n e^{-AG_n})\nu_{n1}}$ *or* $\sum_{j=n-k+1}^{n} \nu_{j0} > 1$, *or at least one inequality* $\nu_{ji} \geq 0$ *is strict for each pair of integers* $j \in [n-k+1, n)$, $i \in \overline{m}$, *or at least one* ν_{ni} *is positive for* $i \in \overline{m}$. *If, furthermore, there is only an equilibrium point, then the solution sequence converges to such a point.*

(ii) *Assume that* $M_n > e^{AG_n}$; $\forall n \in \mathbf{N}_0$. *Any solution sequence of* (4.1) *is uniformly bounded for any bounded set of initial conditions* $x_i \geq 0$; $i = 1-p, 2-p, \ldots, 0$, *if for any* $n \in \mathbf{N}_0$, *one of the following constraints holds*:

$$\left(\sum_{j=n-k+1}^{n} \mu_{j0} \leq 1\right) \wedge (\mu_{ji} \leq 0; \forall j \in \mathbf{N}_{0+}, \forall i \in \overline{m}) \\ \wedge \left(\left(\theta_n \in \left[0, \max\left(\frac{1}{(1-M_n e^{-AG_n})|\mu_{n1}|}, 1\right)\right]\right)\right) \tag{4.3}$$

and, furthermore, either $\theta_n < \frac{1}{(M_n e^{-AG_n}-1)|\mu_{n1}|}$ *or* $\sum_{j=n-k+1}^{n} \mu_{j0} < 1$, *or at least one inequality* $\nu_{ji} \geq 0$ *is strict for each pair of integers* $j \in [n-k+1, n)$, $i \in \overline{m}$, *or at least one* μ_{ni} *is negative for* $i = 2, 3, \ldots, m$. *If, furthermore, there is only an equilibrium point, then the solution sequence converges to such a point.*

(iii) *Assume that the sequence* $\{1 - M_n e^{-AG_n}\}$ *is neither nonnegative nor nonpositive. Then, any solution sequence of* (4.1) *is uniformly bounded for any bounded set of initial conditions* $x_i \geq 0$; $i = 1-p, 2-p, \ldots, 0$, *if* (4.2) *holds for any* $n \in \mathbf{N}_{0+}$ *such that* $M_n < e^{AG_n}$ *and* (4.3) *holds for any* $n \in \mathbf{N}_0$ *such that* $M_n > e^{AG_n}$. *If, furthermore, there is only an equilibrium point, then the solution sequence converges to such a point.*

Proof Since $V_n = V_n(x_n) = p_n x_n$, it follows from (4.1) for $\lambda_n = \lambda_n(\bar{x}_n) = 1 - M_n e^{-AG_n} \geq 0$, equivalently, $M_n \leq e^{AG_n}$, that if $\sum_{j=n-k+1}^{n} F_j > 1$, one gets

$$\begin{aligned}
\Delta V_n &= V_{n+1} - V_n = p_{n+1} x_{n+1} - p_n x_n \\
&\leq -p_n x_n + p_{n+1}\theta_n\lambda_n - p_{n+1}\theta_n\lambda_n\left(\sum_{j=n-k+1}^{n} \nu_{j0} + \sum_{j=n-k+1}^{n}\sum_{i=1}^{m} \nu_{ji} x_j^i\right) \\
&\leq -(p_n + p_{n+1}\theta_n\lambda_n\nu_{n1})x_n - p_{n+1}\theta_n\lambda_n\left(\sum_{j=n-k+1}^{n} \nu_{j0} - 1\right) \\
&\quad - p_{n+1}\theta_n\lambda_n\left(\left(\sum_{j=n-k+1}^{n-1}\sum_{i=1}^{m} \nu_{ji}\right)\min_{n-k+1\leq j\leq n-1}\min\left(x_j, x_j^m\right) + \left(\sum_{i=2}^{m} \nu_{ni}\right)\min\left(x_n^2, x_n^m\right)\right) \\
&\leq -p_n x_n - p_{n+1}\theta_n\lambda_n\left(\sum_{j=n-k+1}^{n} \nu_{j0} - 1\right)
\end{aligned}$$

$$-p_{n+1}\theta_n\lambda_n\left(\sum_{j=n-k+1}^{n}\sum_{i=1}^{m}\nu_{ji}\right)\min_{n-k+1\le j\le n}\min\left(x_j,x_j^m\right)<0;$$

$$\forall n\in\mathbf{N}_0 \tag{4.4}$$

provided that $x_n \neq 0$ under the constraints (4.2) so that $\Delta V_n = V_{n+1} - V_n < 0$. If $\sum_{j=n-k+1}^{n} F_j \le 1$, then one also has that $\Delta V_n = V_{n+1} - V_n = -p_n x_n < 0$ for any nonzero x_n. Thus, the candidate sequence is strictly decreasing and then has a zero limit while being a Lyapunov sequence:

$$\limsup_{n\to\infty}\left(x_{n+1}-\frac{\overline{p}}{\underline{p}}x_n\right)\le\lim_{n\to\infty}\left(x_{n+1}-\frac{p_n}{p_{n+1}}x_n\right)=0$$

so that the sequence $\{x_n\}$ fulfils $x_{n+1}\to\frac{p_n}{p_{n+1}}x_n$ as $n\to\infty$. Since the stability conditions (4.2) are not dependent on the $\{p_n\}$ sequence, this one may be a chosen constant $p_n = p = \overline{p} = \underline{p} > 0$, then the solution sequence $\{x_n\}$ converges to a finite limit which can depend on the initial conditions. It turns out that if there is only an equilibrium point (some related conditions are given in Theorem 2.4), then such a point is globally asymptotically stable and all the solutions converge to it. Otherwise, either some solution tends to infinity or oscillates contradicting that it converges to a finite limit. Hence, Property (i).

The proof of Property (ii) follows in the same way from (4.4) by noting that $\lambda_n = -|\lambda_n| < 0$, $\mu_{n1} = -|\mu_{n1}| < 0$ and that if $x_n \neq 0$ then one gets, under the constraints (4.3),

$$\begin{aligned}\Delta V_n &\le -\left(p_n - p_{n+1}\theta_n|\lambda_n||\mu_{n1}|\right)x_n + p_{n+1}\theta_n|\lambda_n|\left(\sum_{j=n-k+1}^{n}\mu_{j0}-1\right)\\ &\quad + p_{n+1}\theta_n|\lambda_n|\left(\left(\sum_{j=n-k+1}^{n-1}\sum_{i=1}^{m}\mu_{ji}\right)\max_{n-k+1\le j\le n-1}\max\left(x_j,x_j^m\right)\right.\\ &\quad\left. + \left(\sum_{i=2}^{m}\mu_{ni}\right)\max\left(x_n^2,x_n^m\right)\right)<0;\end{aligned} \tag{4.5}$$

$\forall n \in \mathbf{N}_0$ if (4.3) holds by taking $0 < 1 = \underline{p} \le p_n = p_{n+1} = p \le \overline{p} = 1 < +\infty$ provided that $\sum_{j=n-k+1}^{n} F_j \le 1$. If $\sum_{j=n-k+1}^{n} F_j \le 1$, then one also has that $\Delta V_n = V_{n+1} - V_n = -p_n x_n < 0$ for any nonzero x_n. Hence, Property (ii). Property (iii) is a set of mixed conditions of Properties (i)-(ii). □

An instability theorem being dual to the stability Theorem 4.2 follows without proof since it is close to that of the following theorem.

Theorem 4.2 *Assume that the sequence* $\{F_n\}$ *satisfies the constraint* $\sum_{i=0}^{m}\nu_{ni}x_n^i \le F_n \le \sum_{i=0}^{m}\mu_{ni}x_n^i$. *Then the following properties hold.*

(i) *The solution sequence* $\{x_n\}$ *tends to infinity if the following conditions hold for any* $n \in \mathbf{N}_0$:

$$\begin{aligned}&\left(M_n > e^{AG_n}\right)\wedge\left(\sum_{j=n-k+1}^{n}\nu_{j0}\ge 1\right)\wedge\left(\nu_{ji}\ge 0;\forall j\in\mathbf{N}_0, i\in\overline{m}\right)\\ &\quad\wedge\left(\nu_{n1}>\frac{1}{|1-M_n e^{-AG_n}|}\right)\wedge\left(\theta_n\in\left[\frac{1}{|1-M_n e^{-AG_n}|\nu_{n1}},1\right]\right)\end{aligned} \tag{4.6}$$

and, furthermore, either $\theta_n > \frac{1}{|1-M_n e^{-AG_n}|\nu_{n1}}$ *or* $\sum_{j=n-k+1}^{n} \nu_{j0} > 1$, *or at least one inequality* $\nu_{ji} \geq 0$ *is strict for each pair of integers* $j \in [n-k+1, n)$, $i \in \overline{m}$, *or at least one* ν_{ni} *is positive for* $i = 2, 3, \ldots, m$.

(ii) *The solution sequence* $\{x_n\}$ *tends also to infinity if the following two conditions hold simultaneously for any* $n \in \mathbf{N}_0$:

$$\bigl(M_n < e^{AG_n}\bigr) \wedge \left(\sum_{j=n-k+1}^{n} \mu_{j0} \geq 1\right) \wedge \bigl(\mu_{ji} \geq 0; \forall j \in \mathbf{N}_0, i \in \overline{m}\bigr) \wedge \left(|\mu_{n1}| > \frac{1}{|1 - M_n e^{-AG_n}|}\right) \wedge \left(\theta_n \in \left[\frac{1}{|1 - M_n e^{-AG_n}||\mu_{n1}|}, 1\right]\right) \tag{4.7}$$

and, furthermore, either $\theta_n > \frac{1}{|1-M_n e^{-AG_n}||\mu_{n1}|}$ *or* $\sum_{j=n-k+1}^{n} \mu_{j0} < 1$, *or at least one inequality* $\mu_{ji} \leq 0$ *is strict for each pair of integers* $j \in [n-k+1, n)$, $i \in \overline{m}$, *or at least one* μ_{ni} *is negative for* $i = 2, 3, \ldots, m$.

Proof Note that if $\lambda_n = 1 - M_n e^{-AG_n} < 0$, equivalently $M_n > e^{AG_n}$, then if $x_n > 0$ and $p_n - p_{n+1}\theta_n|\lambda_n|\nu_{n1} \leq 0$, $\sum_{j=n-k+1}^{n} \nu_{j0} \geq 1$, $\nu_{ji} \geq 0$; $\forall j \in \mathbf{N}_0$, $\forall i \in \overline{m}$, with at least one of these inequalities being strict, one has

$$\begin{aligned} \Delta V_n \geq{}& -\bigl(p_n - p_{n+1}\theta_n|\lambda_n|\nu_{n1}\bigr)x_n + p_{n+1}\theta_n|\lambda_n|\left(\sum_{j=n-k+1}^{n} \nu_{j0} - 1\right) \\ &+ p_{n+1}\theta_n|\lambda_n|\left(\left(\sum_{j=n-k+1}^{n-1}\sum_{i=1}^{m} \nu_{ji}\right) \min_{n-k+1 \leq j \leq n-1} \min\bigl(x_j, x_j^m\bigr) \right. \\ &\left. + \left(\sum_{i=2}^{m} \nu_{ni}\right) \min\bigl(x_n^2, x_n^m\bigr)\right) > 0; \end{aligned} \tag{4.8}$$

$\forall n \in \mathbf{N}_0$. Then the solution sequence $\{x_n\}$ is strictly monotone so that it tends to infinity. Hence, Property (i), by choosing $p_n = p_{n+1} = 1$ with no loss in generality. If $\lambda_n = 1 - M_n e^{-AG_n} > 0$, equivalently, $M_n < e^{AG_n}$, then if $x_n > 0$ and $p_n - p_{n+1}\theta_n|\lambda_n||\mu_{n1}| \leq 0$, $\sum_{j=n-k+1}^{n} \mu_{j0} \leq 1$, $\mu_{ji} \leq 0$; $\forall j \in \mathbf{N}_0$, $\forall i \in \overline{m}$, with at least one of these inequalities being strict, one has

$$\begin{aligned} \Delta V_n \geq{}& -\bigl(p_n - p_{n+1}\theta_n\lambda_n|\mu_{n1}|\bigr)x_n - p_{n+1}\theta_n\lambda_n\left(\sum_{j=n-k+1}^{n} \mu_{j0} - 1\right) \\ &- p_{n+1}\theta_n\lambda_n\left(\left(\sum_{j=n-k+1}^{n-1}\sum_{i=1}^{m} \mu_{ji}\right) \max_{n-k+1 \leq j \leq n-1} \max\bigl(x_j, x_j^m\bigr) \right. \\ &\left. + \left(\sum_{i=2}^{m} \mu_{ni}\right) \max\bigl(x_n^2, x_n^m\bigr)\right) > 0; \end{aligned} \tag{4.9}$$

$\forall n \in \mathbf{N}_0$. Hence, Property (ii). □

5 Simulation example: a SIS epidemic model

A simulation example which adapts a background SIS epidemic model to the class of epidemic models studied in this paper is used to illustrate some of the theoretic results proved in the previous sections.

5.1 Epidemic model description

The SIS epidemic model describes the transmission of some infectious diseases within a host population. The whole population can be divided in two categories by taking into account the status with regard to the disease of the individuals within the host population. In this sense, there is a population susceptible to the infection (S) and an infectious population (I) which can transmit the infection to the susceptible population by contacts. The discrete-time SIS epidemic model is composed by the following difference equations [16]:

$$\begin{cases} S_{n+1} = S_n(1-p)^{\frac{\alpha I_n}{N_n}} + \nu N_n - \mu S_n + \beta(1-\mu)I_n, \\ I_{n+1} = S_n(1-(1-p)^{\frac{\alpha I_n}{N_n}}) - \mu I_n + (1-\beta(1-\mu))I_n, \end{cases} \tag{5.1}$$

where S_n, I_n and $N_n = S_n + I_n$, respectively, denote the susceptible, infectious and whole populations at the sampling time instant $t = nT$, for all $n \in N_0$, with T being the sampling period. The parameters μ and ν are, respectively, the mortality for natural causes and the birth probabilities within a time step. Then μS_n and μI_n are, respectively, the numbers of susceptible and infectious individuals dying within the time period $[nT,(n+1)T)$. Also, νN_n denotes the number of newborns in such a time period. In view of (5.1), the facts that all newborns are susceptible to the infection and that there is no mortality from causes related to the disease are assumed. Finally, the parameters $\alpha > 0$, $0 < p < 1$ and $0 \le \beta < 1$, respectively, denote the number of contacts between an individual and others within a time step, the probability of transmitting the infection from an infectious individual to a susceptible one after a contact between them, and the probability of transition from the infectious population to the susceptible one within a time step. In this sense, $\alpha I_n/N_n$ is the quantity of contacts of one individual with any infectious one within a time step. Then $S_n(1-p)^{\alpha \frac{I_n}{N_n}}$ is the number of individuals which remain in the susceptible category after the time interval $[nT,(n+1)T)$, and $S_n(1-(1-p)^{\alpha \frac{I_n}{N_n}})$ is the number of individuals which pass from the susceptible to the infectious category within such a time interval. Finally, $\beta(1-\mu)I_n$ and $[1-\beta(1-\mu)]I_n$, respectively, denote the number of infectious individuals which remain alive and pass from the infectious to the susceptible category and that of infectious individuals which remain alive and infectious after such a time interval. In summary, Eqs. (5.1) describe the transitions between such population categories within a time step. In the particular case, when $\beta = 0$, the SIS model becomes a SI model where there is no transition from the infectious population to the susceptible one. Then, once a susceptible individual catches the infection, he/she/it remains infectious for all his/her/its live.

From (5.1), the dynamics of the whole population is given by

$$N_{n+1} = (1+\nu-\mu)N_n. \tag{5.2}$$

By applying the variable changes $x_n = \frac{S_n}{N_n}$ and $y_n = \frac{I_n}{N_n}$, and the fact that $x_n + y_n = 1$, one directly obtains that

$$y_{n+1} = \frac{1-(1-p)^{\alpha y_n} + (\mu\beta - \mu - \beta + (1-p)^{\alpha y_n})y_n}{1+\nu-\mu}, \tag{5.3}$$

which describes the dynamics of the normalised infectious population which is the proportion of the infectious population with respect to the whole population. Such an equation can be rewritten as

$$\begin{aligned} z_{n+1} &= 1-(1-p)^{\alpha' z_n} + \frac{\mu\beta-\mu-\beta+(1-p)^{\alpha y_n}}{1+\nu-\mu} z_n \\ &= \frac{1+\nu-\mu+(\mu\beta-\mu-\beta)z_n+(1-p)^{\alpha' z_n}(z_n-1-\nu+\mu)}{1+\nu-\mu} \\ &= \left(1-\frac{\mu+\beta-\mu\beta}{1+\nu-\mu} z_n\right)\left(1-\frac{1+\nu-\mu-z_n}{1+\nu-\mu+(\mu\beta-\mu-\beta)z_n} e^{\alpha' \ln(1-p) z_n}\right), \end{aligned} \tag{5.4}$$

where $z_n = (1+\nu-\mu)y_n$ and $\alpha' = \alpha/(1+\nu-\mu)$. Such an equation possesses the same structure as (1.1) if one takes into account the fact that the infectious population has to be nonnegative for all time since a negative population is not reasonable. In this sense, it follows that

$$\begin{aligned} &k = 1; \quad F(z_n) = \frac{\mu+\beta-\mu\beta}{1+\nu-\mu} z_n, \\ &q = 1; z_n = \bar{z}_n; \quad M(\bar{z}_n) = \frac{1+\nu-\mu-z_n}{1+\nu-\mu+(\mu\beta-\mu-\beta)z_n}; \\ &G(\bar{z}_n) = z_n; \qquad A = -\alpha' \ln(1-p) > 0 \end{aligned} \tag{5.5}$$

by comparing (5.1) and (5.4).

5.2 Numerical results with a constant probability for the infection transmission

A numerical study based on simulations of an epidemic disease described by (5.4)-(5.5) is carried out. The time evolution of the proportion of the infectious population for different values of the parameters p, α and β is analysed. In this way, the dependence on the epidemics dynamics of such parameters is reflected. The values $\mu = 1/70$ years^{-1} = 3.9139×10^{-5} days^{-1}, $\nu = 4.9139 \times 10^{-5}$ days^{-1} and $T = 1$ day as the sampling period for the model (5.4) are considered. Also, the initial condition for such a model corresponds to an initial situation at which $S_0 = 950$ individuals (ind), $I_0 = 50$ ind so that $N_0 = 1{,}000$ ind and $y_0 = 0.05$ ($z_0 \approx y_0$).

Figure 1 displays the time evolution of the proportion of the infectious population for three different probabilities of the transmission of the infection after a contact between a susceptible individual with an infectious one. Concretely, the values $p_1 = 0.001$, $p_2 = 0.003$ and $p_3 = 0.005$ are considered. Moreover, the values $\alpha = 30$ and $\beta = 0.05$ are, respectively, chosen for the number of contacts between an individual and others and for the probability of transition from the infectious population to the susceptible one within a time step. One can see that the infection disappears from the host population in the stationary state if the disease transmission probability is smaller than 0.01. On the other hand, the proportion of the infectious population in the stationary state increases as the infection probability does.

Figure 2 displays the time evolution of the proportion of the infectious population for three different values for the number of contacts of an individual with others within a time step. Concretely, the values $\alpha_1 = 10$, $\alpha_2 = 30$ and $\alpha_3 = 50$ are considered. Moreover,

Figure 1 Time evolution of the normalized infectious population for different infection probability.

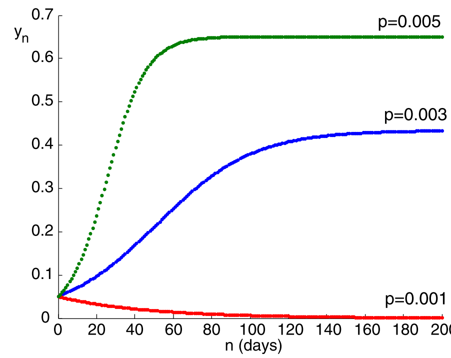


Figure 2 Time evolution of the normalized infectious population for different number of contacts in a time step.

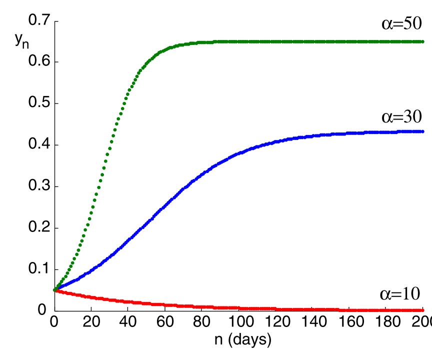
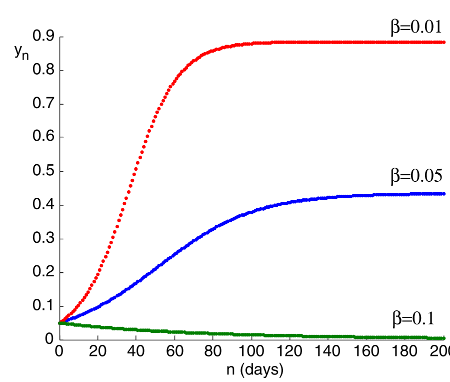


Figure 3 Time evolution of the normalized infectious population for different probabilities of removing from infection.



the values $p = 0.003$ and $\beta = 0.05$ are, respectively, chosen for the probability of infection transmission after a contact between a susceptible individual with an infectious one and for the probability of transition from the infectious population to the susceptible one within a time step. One can see that the infection disappears from the host population in the stationary state if the number of contacts is smaller than 10. Also, the proportion of the infectious population in the stationary state increases as such a number does.

Finally, Figure 3 displays the time evolution of the proportion of the infectious population for three different values for the probability of transition from the infectious population to the susceptible one. Concretely, the values $\beta_1 = 0.01$, $\beta_2 = 0.05$ and $\beta_3 = 0.1$ are considered. Moreover, the values $p = 0.003$ and $\alpha = 30$ are, respectively, chosen for the probability of infection transmission after a contact between a susceptible individual with an infectious one and for the number of contacts of an individual with others within a time step. One can see that the infection disappears from the host population in the stationary state if the probability of transition from the infectious population to the susceptible one is larger than 0.1. Also, the proportion of the infectious population in the stationary state increases as such a probability decreases.

5.3 Numerical results with a time-varying probability for the infection transmission

In a more realistic situation, the probability of transmitting the infection from an infectious individual to a susceptible one can depend on the time evolution of the proportion of the infectious population. This feature can be motivated by the fact that the whole population usually takes preventive measures for fighting against the propagation of the infection when the incidence of the infection is relevant. Such measures may imply an effective reduction in the probability of the infection transmission. This fact has been also modelled by means of a saturated transmission rate in other related research [17]. In such a situation, the dynamics of the infection can be also given by (5.4) but replacing the constant transmission probability p by $p(\bar{z}_n)$, where $\bar{z}_n = (z_n, z_{n-1}, \dots, z_{n-q+1})$ for some $q \in \mathbf{N}$, or, equivalently, by (1.1)-(5.5), by replacing x_n by z_n in (1.1) and $G(\bar{z}_n) = z_n$ and $A = -\alpha' \ln(1-p)$ by $G(\bar{z}_n) = -z_n \ln(1 - p(\bar{z}_n))$ and $A = \alpha'$, respectively.

An example based on an infectious disease where the current infection transmission depends on the time evolution of the proportion of infectious is analysed. The same initial condition and the same values for the parameters μ, ν and T as those used in the previous subsection are considered. Moreover, the values $\alpha = 30$ and $\beta = 0.05$ are used. Then, note that $y_n \approx z_n$ since $1 + \nu - \mu \approx 1$. The example considers that the transmission probability on the present day depends on the average of the registered values for the proportion of the infectious population within the previous seven days (a week). More specifically, at the beginning of the simulation, a constant probability $p_0 = 0.003$ is supposed in (5.4) and such a value is maintained until the value for $\bar{z}_n = \frac{1}{7}\sum_{j=0}^{6} z_{n-j}$ is larger than some upper bound $z_{\sup}$ (for instance, if $z_{\sup} = 0.25$ is used, then the transmission probability is maintained constant until the day on which the average proportion of infectious population within a week reaches 25% of the whole population). After such a day (namely, n_1), the probability of contracting the infection decreases from preventive behaviour. The following expression is supposed for the time evolution of such a probability:

$$p(n) = p_0 - k_1 \bar{z}_n (n - n_1); \quad \forall n \in [n_1 + 1, n_2], \tag{5.6}$$

where n_2 denotes the eventual day on which the proportion of the infectious population is smaller than a lower bound $z_{\inf}$ (for instance, if the value $z_{\inf} = 0.025$ is used, then the time-varying probability given by (5.6) is maintained until the day on which the average proportion of infectious population within a week is smaller than 2.5% of the whole population). Note from (5.6) that the probability of contracting the infection decays after the day n_1 with a decreasing step proportional to the average $\bar{z}_n = \frac{1}{7}\sum_{j=0}^{6} z_{n-j}$, where k_1 is a constant parameter appropriately chosen such that the time-varying probability $p(n) \in (0,1)$; $\forall n \in [n_1 + 1, n_2]$. Also, note that such a time-varying probability can be not decreasing after a certain day within the interval $[n_1 + 1, n_2]$ since $f_1(n) = k_1 \bar{z}_n (n - n_1)$ is not a monotone increasing function within such an interval.

After the day n_2, the probability of contracting the infection again can increase from the fact that the population could relax the preventive measures because the affectation of the disease within the host population has decreased to a small level. Then the following expression can be supposed for the time evolution of such a probability:

$$p(n) = p(n_2) + \frac{k_2(n - n_2)}{\bar{z}_n}; \quad \forall n \in [n_2 + 1, n_3], \tag{5.7}$$

where n_3 denotes the eventual day on which the proportion of the infectious population is again larger than the upper bound. Note from (5.7) that the probability of contracting the infection increases after the day n_2 with an increasing step inversely proportional to the average $\bar{z}_n = \frac{1}{7}\sum_{j=0}^{6} z_{n-j}$, where k_2 is a constant parameter appropriately chosen such that the time-varying probability $p(n) \in (0,1)$; $\forall n \in [n_2 + 1, n_3]$. Also, note that such a time-varying probability can be not increasing after a certain day within the interval $[n_2 + 1, n_3]$ since $f_2(n) = k_2(n - n_2)/\bar{z}_n$ is not a monotone increasing function within such an interval. After the day n_3, the probability of contracting the infection again can decrease as (5.6) from preventive measures because the affectation of the disease within the host population has reached a high level and so on. As a consequence, there is a cyclic behaviour in the transmission probability of the infection given by

$$p(n) = \begin{cases} p_0 & \text{for } n \le n_1, \\ p(n_i) - k_1\bar{z}_n(n - n_i) & \text{for } n \in [n_i + 1, n_{i+1}], \\ p(n_{i+1}) + \frac{k_2(n - n_{i+1})}{\bar{z}_n} & \text{for } n \in [n_{i+1} + 1, n_{i+2}] \end{cases} \tag{5.8}$$

for any odd $i \in N$. If the values for $z_{\sup} = 0.25$, $z_{\inf} = 0.025$, $k_1 = 3 \times 10^{-4}$ and $k_2 = 3 \times 10^{-7}$ are considered, then the time varying evolution of the proportion of infectious population and the transmission probability are, respectively, displayed in Figures 4 and 5.

One can see from Figure 4 that the proportion of the infectious population increases from the initial day until the n_1th day because the constant probability p_0 of transmitting the infection after contacts between susceptible and infectious individuals is large enough. On the n_1th day, the proportion of infectious population is large enough so that the whole population has to begin to take preventive measures in order to reduce the probability of transmission so that the impact of the disease be attenuated. Such preventive actions make such a probability begin to decrease via (5.6) within the time interval $[n_1 + 1, n^*]$, with n^*

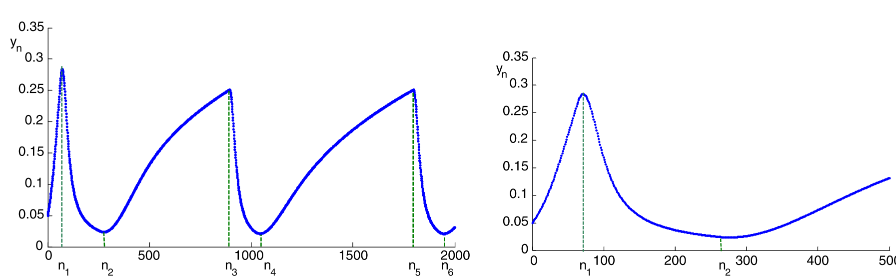


Figure 4 Time evolution of the normalized infectious population for a time-varying probability of infection transmission (the right-hand figure is a zoom for the first 500 days).

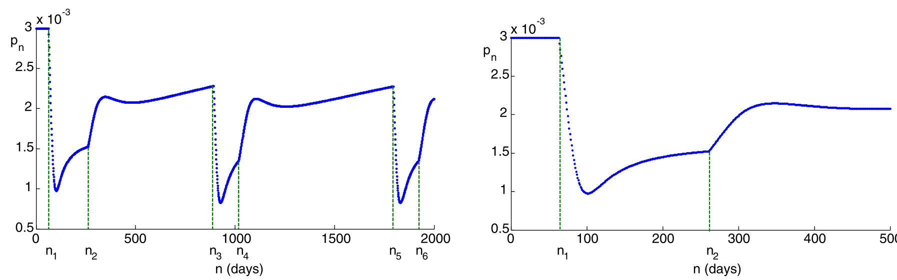


Figure 5 Time evolution of the probability of infection transmission (the right-hand figure is a zoom for the first 500 days).

denoting the day when the transmission probability reaches its minimum value within the time interval $[n_1 + 1, n_2]$, as it can be seen in Figure 5. As a consequence, the proportion of the infectious population decreases until the n_2th day when the population begins to relax the preventive measures since the individuals can feel that the infectious disease has disappeared because of the little proportion of the infectious population. Such behaviour makes the probability of transmitting the infection begin to increase after the n_2th day via (5.7). This fact implies an increment in the proportion of the infectious population until the n_3th day when such a proportion is large enough so that the whole population begins to take again preventive measures. In summary, a cyclic time evolution is obtained for the proportion of the infectious population as well as for the probability of transmission as it can be seen from Figures 4 and 5. Such a cyclic behaviour is a consequence of taking preventive actions for fighting against the disease transmission when the influence of the disease in the population is notable.

Competing interests
The authors declare they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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