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Boundedness of solutions for a class of second-order differential equation with singularity

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Abstract

In this paper, we study the following second-order periodic system:

$$x'' + V'(x) + p(t)|x|^\alpha = 0,$$

where $V(x)$ has a singularity. Under some assumptions on the $V(x)$ and $p(t)$ by Ortega's small twist theorem, we obtain the existence of quasi-periodic solutions and boundedness of all the solutions.

Keywords: boundedness of solutions; singularity; small twist theorem

1 Introduction and main result

In 1991, Levi [1] considered the following equation:

$$x'' + V'(x, t) = 0, \tag{1.1}$$

where $V(x, t)$ satisfies some growth conditions and $V(x, t) = V(x, t+1)$. The author reduced the system to a normal form and then applied Moser twist theorem to prove the existence of quasi-periodic solution and the boundedness of all solutions. This result relies on the fact that the nonlinearity $V(x, t)$ can guarantee the twist condition of KAM theorem. Later, several authors improved the Levi's result; we refer to [2–4] and the references therein.

Recently, Capietto, Dambrosio and Liu [5] studied the following equation:

$$x'' + V'(x) = F(x, t), \tag{1.2}$$

with $F(x, t) = p(t)$ is a π-periodic function and $V(x) = \frac{1}{2}x_+^2 + \frac{1}{(1-x^2)^\nu} - 1$, where $x_+ = \max\{x, 0\}$, $x_- = \max\{-x, 0\}$ and ν is a positive integer. Under the Lazer-Leach assumption that

$$1 + \frac{1}{2}\int_0^\pi p(t_0 + \theta)\sin\theta\, d\theta > 0, \quad \forall t_0 \in R, \tag{1.3}$$

they prove the boundedness of solutions and the existence of quasi-periodic solution by KAM theorem. It is the first time that the equation of the boundedness of all solution is treated in case of a singular potential.

We observe that $F(x, t) = p(t)$ in (1.2) is smooth and bounded, so a natural question is to find sufficient conditions on $F(x, t)$ such that all solutions of (1.2) are bounded when $F(x, t)$ is unbounded. The purpose of this paper is to deal with this problem.

Motivated by the papers [1, 5, 6], we consider the following equation:

$$x'' + V'(x) + p(t)|x|^\alpha = 0, \tag{1.4}$$

where $p(t)$ is a π-periodic function,

$$V(x) = \frac{1}{2}x_+^2 + \frac{1}{1 - x_-^2} - 1, \quad 0 < \alpha < 1, x > -1. \tag{1.5}$$

We suppose Lazer-Leach assumption hold:

$$\int_0^\pi p(t_0 + \theta)(\sin\theta)^{1+\alpha} \, d\theta > 0, \quad \forall t_0 \in R. \tag{1.6}$$

Our main result is the following theorem.

Theorem 1 *Under the assumptions* (1.5) *and* (1.6), *all the solutions of* (1.4) *are bounded.*

The main idea of our proof is acquired from [6]. The proof of Theorem 1 is based on a small twist theorem due to Ortega [7]. It mainly consists of two steps. The first one is to transform (1.4) into a perturbation of integrable Hamilton system. The second one is to show that Poincaré map of the equivalent system satisfies Ortega's twist theorem, then some desired result can be obtained.

Moreover, we have the following theorem on solutions of Aubry-Mather type.

Theorem 2 *Assume that* $p(t) \in C(R)$ *satisfies* (1.6); *then, there is an* $\epsilon_0 > 0$ *such that, for any* $\omega \in (\frac{1}{\pi}, \frac{1}{\pi} + \epsilon_0)$, *t equation* (1.4) *has a solution* $(x_\omega(t), x'_\omega(t))$ *of the Mather type with rotation number* ω. *More precisely*:

Case 1: $\omega = \frac{p}{q}$ *is rational. The solutions* $(x_\omega(t + 2i\pi), x'_\omega(t + 2i\pi))$, $1 \le i \le q - 1$ *are independent periodic solutions of periodic* $q\pi$; *moreover, in this case*,

$$\lim_{q \to \infty} \min_{t \in R}(|x_\omega(t)| + |x'_\omega(t)|) = +\infty.$$

Case 2: ω *is irrational. The solution* $(x_\omega(t), x'_\omega(t))$ *is either a usual quasi-periodic solution or a generalized one.*

We will apply Aubry-Mather theory, more precisely, the theorem in [8], to prove this theorem.

2 Proof of theorem
2.1 Action-angle variables and some estimates

Observe that (1.4) is equivalent to the following Hamiltonian system:

$$x' = \frac{\partial H}{\partial y}, \qquad y' = -\frac{\partial H}{\partial x} \tag{2.1}$$

with the Hamiltonian function

$$H(x, y, t) = \frac{1}{2}y^2 + V(x) + \frac{p(t)}{(\alpha + 1)}|x|^\alpha x.$$

In order to introduce action and angle variables, we first consider the auxiliary autonomous equation:

$$x' = y, \qquad y' = -V'(x), \tag{2.2}$$

which is an integrable Hamiltonian system with Hamiltonian function

$$H_1(x, y, t) = \frac{1}{2}y^2 + V(x).$$

The closed curves $H_1(x, y, t) = h > 0$ are just the integral curves of (2.2).

Denote by $T_0(h)$ the time period of the integral curve Γ_h of (2.2) defined by $H_1(x, y, t) = h$ and by I the area enclosed by the closed curve Γ_h for every $h > 0$. Let $-1 < -\alpha_h < 0 < \beta_h$ be such that $V(-\alpha_h) = V(\beta_h) = h$. It is easy to see that

$$I_0(h) = 2\int_{-\alpha_h}^{\beta_h} \sqrt{2(h - V(s))}\, ds, \quad \forall h > 0$$

and

$$T_0(h) = I_0'(h) = 2\int_{\alpha_h}^{\beta_h} \frac{1}{\sqrt{2(h - V(s))}}\, ds, \quad \forall h > 0.$$

By direct computation, we get

$$I_0(h) = 2\int_0^{\beta_h} \sqrt{2(h - V(s))}\, ds + 2\int_{-\alpha_h}^0 \sqrt{2(h - V(s))}\, ds$$

$$= \pi h + 2\int_0^{\alpha_h} \sqrt{2(h - V(-s))}\, ds,$$

so

$$T_0(h) = \pi + \int_0^{\alpha_h} \frac{1}{\sqrt{2(h - V(-s))}}\, ds.$$

We then have

$$I_0(h) = I_-(h) + I_+(h), \qquad T_0(h) = T_-(h) + T_+(h),$$

where

$$I_-(h) = 2\int_0^{-\alpha_h} \sqrt{2(h - V(s))}\, ds, \qquad I_+(h) = \pi h,$$

$$T_-(h) = 2\int_0^{-\alpha_h} \frac{1}{\sqrt{2(h - V(-s))}}\, ds, \qquad T_+(h) = \pi.$$

Similar in estimating in [5], we have the estimation of functions I_- and T_-.

Lemma 1 *We have*

$$h^n \left| \frac{d^n T_-(h)}{dh^n} \right| \le C h^{-\frac{1}{2}}$$

and

$$h^n \left| \frac{d^n I_-(h)}{dh^n} \right| \le C h^{\frac{1}{2}},$$

where $n = 0, 1, \ldots, 6$, $h \to +\infty$. Note that here and below we always use C, C_0 or C_0' to indicate some constants.

Remark 1 It follows from the definitions of $T_+(h)$, $T_-(h)$ and Lemma 1 that

$$\lim_{h \to +\infty} T_-(h) = 0, \qquad \lim_{h \to +\infty} T_+(h) = \pi.$$

Thus the time period $T_0(h)$ is dominated by $T_+(h)$ when h is sufficiently large. From the relation between $T_-(h)$ and $I_-(h)$, we know $I_0(h)$ is dominated by $I_+(h)$ when h is sufficiently large.

Remark 2 It also follow from the definition of $I(h)$, $I_-(h)$, $I_+(h)$ and Remark 1 that

$$\left| h^n \frac{d^n I_0(h)}{dh^n} \right| \le C_0 I_0(h) \quad \text{for } n \ge 1.$$

Remark 3 Note that $h = h_0(I_0)$ is the inverse function of I_0. By Remark 2, we have

$$\left| I^n \frac{d^n h(I)}{dI^n} \right| \le C_0 h(I) \quad \text{for } n \ge 1.$$

We now carry out the standard reduction to the action-angle variables. For this purpose, we define the generating function $S(x, I) = \int_C \sqrt{2(h - V(s))} \, ds$, where C is the part of the closed curve Γ_h connecting the point on the y-axis and point (x, y).

We define the well-know map $(\theta, I) \to (x, y)$ by

$$y = \frac{\partial S}{\partial x}(x, I), \qquad \theta = \frac{\partial S}{\partial I}(x, I),$$

which is symplectic since

$$dx \wedge dy = dx \wedge (S_{xx} \, dx + S_{xI} \, dI) = S_{xI} \, dx \wedge dI,$$

$$d\theta \wedge dI = (S_{Ix} \, dx + S_{II} \, dI) \wedge dI = S_{Ix} \, d \wedge dI.$$

From the above discussion, we can easily get

$$\theta = \begin{cases} \frac{\pi}{T_0(h(x,y))} \left(\frac{T_-(h(x,y))}{2} + \arcsin \frac{x}{\sqrt{2(h(x,y))}} \right), & \text{if } x > 0, y > 0, \\[2mm] \frac{\pi}{T_0(h(x,y))} \left(\frac{T_-(h(x,y))}{2} + \pi + \arcsin \frac{x}{\sqrt{2(h(x,y))}} \right), & \text{if } x > 0, y < 0, \\[2mm] \frac{\pi}{T_0(h(x,y))} \left(\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y) + 1 - (1 - s^2)^{-1})}} \, ds \right), & \text{if } x < 0, y > 0, \\[2mm] \frac{\pi}{T_0(h(x,y))} \left(T_0(h(x,y)) - \int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y) + 1 - (1 - s^2)^{-1})}} \, ds \right), & \text{if } x < 0, y < 0 \end{cases} \tag{2.3}$$

and

$$I(x,y) = I_0\big(h(x,y)\big) = 2 \int_{-\alpha_h}^{\beta_h} \sqrt{2\big(h(x,y) - V(s)\big)}\, ds. \tag{2.4}$$

In the new variables (θ, I), the system (2.1) is

$$\theta' = \frac{\partial H}{\partial I}, \qquad I' = -\frac{\partial H}{\partial \theta}, \tag{2.5}$$

where

$$H(\theta, I, t) = \pi h_0(I) + \pi \frac{p(t)}{(\alpha + 1)} \big|x(I,\theta)\big|^\alpha x(I,\theta). \tag{2.6}$$

In order to estimate $\pi \frac{p(t)}{(\alpha+1)}|x(I,\theta)|^\alpha x(I,\theta)$, we need the following lemma.

Lemma 2 [5, Lemma 2.2] *For I sufficient large and $-\alpha_h \leq x < 0$, the following estimates hold*:

$$\left| I^n \frac{\partial^n x(I,\theta)}{\partial I^n} \right| \leq c\sqrt{I}, \qquad \left| I^n \frac{\partial^n y(I,\theta)}{\partial I^n} \right| \leq c\sqrt{I} \quad \textit{for } 0 \leq n \leq 6.$$

2.2 New action and angle variables

Now we are concerned with the Hamiltonian system (2.5) with Hamiltonian function $H(\theta, I, t)$ given by (2.6). Note that

$$I\, d\theta - H\, dt = -(H\, dt - I\, d\theta).$$

This means that if one can solve I from (2.6) as a function of H (θ and t as parameters), then

$$\frac{dH}{d\theta} = -\frac{\partial I}{\partial t}(t, H, \theta), \qquad \frac{dt}{d\theta} = \frac{\partial I}{\partial H}(t, H, \theta), \tag{2.7}$$

is also a Hamiltonian system with Hamiltonian function I and now the action, angle and time variables are H, t and θ.

From (2.6) and Lemma 1, we have

$$\frac{\partial H}{\partial I} \to 1 \quad \text{as } I \to +\infty.$$

So, we assume that I can be written as

$$I = I_0\left(\frac{H}{\pi} + R(H, t, \theta)\right),$$

where R satisfies $|R| < \frac{H}{\pi}$. Recalling that h_0 is the inverse function of I_0, we have

$$\frac{H}{\pi} + R(H, t, \theta) = h_0(I),$$

which implies that

$$R(H,t,\theta) = \frac{p(t)}{(\alpha+1)}\left|x(I,\theta)\right|^{\alpha}x(I,\theta).$$

As a consequence, R is implicitly defined by

$$R(H,t,\theta) = \frac{p(t)}{(\alpha+1)}|x|^{\alpha}x\left(I_0\left(\frac{H}{\pi}+R(H,t,\theta)\right),\theta\right). \tag{2.8}$$

Lemma 3 *The function $R(H,t,\theta)$ satisfies the following estimates*:

$$\left|\frac{\partial^m R(H,t,\theta)}{\partial H^m}\right| \le H^{\frac{\alpha+1}{2}} \quad \text{for } m+l \le 6.$$

Proof Case $m = 0$. By (2.8), Lemma 2 and noticing that $\frac{H}{I} \to 1$ as $I \to +\infty$, we have

$$\begin{aligned}
\left|R(H,t,\theta)\right| &= \left|\pi\frac{p(t)}{(\alpha+1)}x^{\alpha+1}\left(I_0\left(\frac{H}{\pi}+R(H,t,\theta)\right),\theta\right)\right| \\
&\le \left|I_0\left(\frac{H}{\pi}+R(H,t,\theta)\right)\right|^{\frac{1+\alpha}{2}}p(t) \\
&= \left|I^{\frac{1+\alpha}{2}}\right|p(t) \le C\cdot H^{\frac{1+\alpha}{2}}.
\end{aligned}$$

Case $m \ge 1$. Derivative both sides of (2.8) with respect to H, we have

$$\frac{\partial R}{\partial H} = \frac{1}{\pi}\frac{1}{-1+\frac{1}{\frac{\partial x}{\partial I}I_0'(\frac{H}{\pi}+R)p(t)}}.$$

By Remark 2, Lemma 2 and the estimate of R, we have

$$\left|\frac{\partial x}{\partial I}I_0'\left(\frac{H}{\pi}+R\right)p(t)\right| \le C\cdot H^{\frac{-1+\alpha}{2}}.$$

Since

$$\left|\frac{1}{\frac{\partial x}{\partial I}I_0'(\frac{H}{\pi}+R)p(t)}\right| \gg 1 \quad \text{as } H \to +\infty,$$

we have

$$H\left|\frac{\partial R}{\partial H}\right| \le C\cdot H^{\frac{1+\alpha}{2}}.$$

We suppose that

$$\left|\frac{\partial^m R(H,t,\theta)}{\partial H^m}\right| \le H^{\frac{\alpha+1}{2}} \tag{2.9}$$

holds where $m = k-1$. We will prove (2.9) also holds where $m = k$, $k \le 6$.

By direct calculation, we have

$$\frac{\partial^k R}{\partial H^k} = \frac{\sum c_n j_1 \cdots j_n \frac{\partial^n x}{\partial I^n} \frac{\partial^{j_1}}{\partial H^{j_1}} (\frac{H}{\pi} + R) \cdots \frac{\partial^{j_n}}{\partial H^{j_n}} (\frac{H}{\pi} + R)}{1 - \frac{\partial x}{\partial I} I_0' (\frac{H}{\pi} + R) p(t)}, \tag{2.10}$$

where $1 \le n \le k$, $j_1 + \cdots + j_n = k$, $1 \le j_1, \ldots, j_n < k$.

Since

$$\frac{\partial^{j_n} I_0 (\frac{H}{\pi} + R)}{\partial H^{j_n}} = I_0^{(n)} \cdot \left[\left(\frac{H}{\pi} + R \right)' \right]^n + \cdots + I_0' \cdot \left(\frac{H}{\pi} + R \right)^{(n)},$$

by Lemma 1 and (2.9), when $j_n \ge 2$, we have

$$\left| \frac{\partial^{j_n} I_0 (\frac{H}{\pi} + R)}{\partial H^{j_n}} \right| \le C \cdot \left| I_0' \cdot \left(\frac{H}{\pi} + R \right)^{(n)} \right|$$

$$\le C \cdot H^{\frac{\alpha+1}{2}} \cdot H^{-j_n}$$

$$\le C \cdot H \cdot H^{-j_n}. \tag{2.11}$$

When $j_n = 1$, we have

$$\left| \frac{\partial^{j_n} I_0 (\frac{H}{\pi} + R)}{\partial H^{j_n}} \right| \le C. \tag{2.12}$$

By (2.11) and (2.12), we have

$$\left| \frac{\partial^{j_n} I_0 (\frac{H}{\pi} + R)}{\partial H^{j_n}} \right| \le C \cdot H^{1-j_n}, \tag{2.13}$$

where $1 \le j_n < k$.

By (2.13), we have

$$\left| \frac{\partial^{j_n} I_0 (\frac{H}{\pi} + R)}{\partial H^{j_n}} \right| \cdots \left| \frac{\partial^{j_n} I_0 (\frac{H}{\pi} + R)}{\partial H^{j_n}} \right| \le C \cdot H^{n-k}. \tag{2.14}$$

By (2.10), (2.14) and Lemma 2, we have (2.9) holds where $m = k$. Thus, we prove Lemma 3. □

Analogously, one may obtain, by a direct but cumbersome commutation, the following estimates.

Lemma 4 *The function $R(H, t, \theta)$ satisfies the following estimates*:

$$\left| \frac{\partial^{m+l} R(H, t, \theta)}{\partial H^m \partial t^l} \right| \le H^{\frac{\alpha+1}{2}} \quad \text{for } m + l \le 6.$$

Moreover, by the implicit function theorem, there exists a function $R_1 = R_1(t, H, \theta)$ such that

$$R(H, t, \theta) = \frac{p(t)}{(\alpha + 1)} \left| x(H, \theta) \right|^\alpha x(H, \theta) + R_1(H, t, \theta).$$

Since

$$R_1(H, t, \theta) = R(H, t, \theta) - \frac{p(t)}{(\alpha + 1)} |x(H, \theta)|^\alpha x(H, \theta)$$

for $x \geq 0$, we have

$$\left| R_1(H, t, \theta) \right| = \left| \frac{p(t)}{(\alpha + 1)} x^{\alpha+1} \left(I_0 \left(\frac{H}{\pi} + R(H, t, \theta) \right), \theta \right) - \frac{p(t)}{(\alpha + 1)} x^{\alpha+1}(H, \theta) \right|$$

$$= \left| \int_0^1 x^\alpha \left(H + s(\pi R + I_-) \right) \cdot \frac{\partial x}{\partial H} \left(H + s(\pi R + I_-) \right) \cdot (\pi R + I_-) p(t) \, ds \right|.$$

For $x < 0$, we have

$$\left| R_1(H, t, \theta) \right| = \left| \frac{p(t)}{(\alpha + 1)} (-x)^{\alpha+1} \left(I_0 \left(\frac{H}{\pi} + R(H, t, \theta) \right), \theta \right) - \frac{p(t)}{(\alpha + 1)} (-x)^{\alpha+1}(H, \theta) \right|$$

$$= \left| \int_0^1 (-x)^\alpha \left(H + s(\pi R + I_-) \right) \cdot \frac{\partial x}{\partial H} \left(H + s(\pi R + I_-) \right) \cdot (\pi R + I_-) p(t) \, ds \right|.$$

By Lemmas 1 and 4, we have the estimates on $R_1(H, t, \theta)$.

For concision, in the estimates and the calculation below, we only consider the case $x \geq 0$, since the case $x < 0$ have the similar result.

Lemma 5 $\left| \frac{\partial^{k+l} R_1(H, t, \theta)}{\partial^k H \, \partial^l t} \right| < H^{\frac{\alpha}{2}}$ *for* $k + l \leq 6$.

For the estimates of $I(\frac{H}{\pi} + R)$, we need the estimates on $I_-(\frac{H}{\pi} + R)$. By Lemmas 1 and 5, noticing that $|R| < \frac{H}{\pi}$, we have the following lemma.

Lemma 6 $\left| \frac{\partial^{k+l} I_-(\frac{H}{\pi} + R)}{\partial^k H \, \partial^l t} \right| < H^{\frac{1}{2}}$ *for* $k + l \leq 6$.

Now the new Hamiltonian function $I = I(t, H, \theta)$ is written in the form

$$I = I_0 \left(\frac{H}{\pi} + R \right) = I_+ \left(\frac{H}{\pi} + R \right) + I_- \left(\frac{H}{\pi} + R \right)$$

$$= H + \pi R(H, t, \theta) + I_- \left(\frac{H}{\pi} + R \right)$$

$$= H + \pi \frac{p(t)}{(\alpha + 1)} |x(H, \theta)|^\alpha x(H, \theta) + R_1(H, t, \theta) + I_- \left(\frac{H}{\pi} + R \right).$$

The system (2.7) is of the form

$$\begin{cases} \frac{dt}{d\theta} = \frac{\partial I}{\partial H} = 1 + \pi \frac{\partial x}{\partial H}(H, \theta) |x(H, \theta)|^\alpha p(t) + \frac{\partial R_1}{\partial H}(H, t, \theta) + \frac{\partial I_-}{\partial H}(H, t, \theta), \\ \frac{dH}{d\theta} = -\frac{\partial I}{\partial t} = -\pi \frac{p'(t)}{(\alpha+1)} |x(\theta, H)|^\alpha x(\theta, H) - \frac{\partial R_1}{\partial t}(t, H, \theta) - \frac{\partial I_-}{\partial t}(H, t, \theta). \end{cases} \tag{2.15}$$

Introduce a new action variable $\rho \in [1, 2]$ and a parameter $\epsilon > 0$ by $H = \epsilon^{-2} \rho$. Then $H \gg 1 \Leftrightarrow 0 < \epsilon \ll 1$. Under this transformation, the system (2.15) is changed into the form

$$\begin{cases} \frac{dt}{d\theta} = \frac{\partial I}{\partial H} = 1 + \pi \frac{\partial x}{\partial H}(H, \theta) |x|^\alpha (H, \theta) p(t) + \frac{\partial R_1}{\partial H}(H, t, \theta) + \frac{\partial I_-}{\partial H}(H, t, \theta), \\ \frac{d\rho}{d\theta} = -\frac{\partial I}{\partial t} = -\epsilon^2 [\pi \frac{p'(t)}{(\alpha+1)} |x(\theta, H)|^\alpha x(\theta, H) + \frac{\partial R_1}{\partial t}(t, H, \theta) + \frac{\partial I_-}{\partial t}(H, t, \theta)], \end{cases} \tag{2.16}$$

which is also Hamiltonian system with the new Hamiltonian function

$$\Gamma(t,\rho,\theta;\epsilon) = \rho + \pi\epsilon^{-2}\frac{p(t)}{\alpha+1}\big|x\big(\theta,\epsilon^{-2}\rho\big)\big|^{\alpha}x\big(\theta,\epsilon^{-2}\rho\big)$$
$$+ \epsilon^{-2}R_1\big(t,\epsilon^{-2}\rho,\theta\big) + \epsilon^{-2}I_-\big(t,\epsilon^{-2}\rho,\theta\big).$$

Obviously, if $\epsilon \ll 1$, the solution $(t(\theta,t_0,\rho_0),\rho(\theta,t_0,\rho_0))$ of (2.16) with the initial date $(t_0,\rho_0) \in R \times [1,2]$ is defined in the interval $\theta \in [0,2\pi]$ and $\rho(\theta,t_0,\rho_0) \in [\frac{1}{2},3]$. So the Poincaré map of (2.16) is well defined in the domain $R \times [1,2]$.

Lemma 7 [6, Lemma 5.1] *The Poincaré map of* (2.16) *has intersection property.*

The proof is similar to the corresponding one in [6].

For convenience, we introduce the notation $O_k(1)$ and $o_k(1)$. We say a function $f(t,\rho,\theta,\epsilon) \in O_k(1)$ if f is smooth in (t,ρ) and for $k_1 + k_2 \leq k$,

$$\left|\frac{\partial^{k_1+k_2}}{\partial t^{k_1}\,\partial\rho^{k_2}}f(t,\rho,\theta,\epsilon)\right| \leq C$$

for some constant $C > 0$ which is independent of the arguments t, ρ, θ, ϵ.

Similarly, we say $f(t,\rho,\theta,\epsilon) \in o_k(1)$ if f is smooth in (t,ρ) and for $k_1 + k_2 \leq k$,

$$\lim_{\epsilon\to 0}\left|\frac{\partial^{k_1+k_2}}{\partial t^{k_1}\,\partial\rho^{k_2}}f(t,\rho,\theta,\epsilon)\right| = 0,$$

uniformly in (t,ρ,θ).

2.3 Poincaré map and twist theorems

We will use Ortega's small twist theorem to prove that the Poincaré map P has an invariant closed curve, if ϵ is sufficiently small. Let us first recall the theorem in [7].

Lemma 8 (Ortega's theorem) *Let $A = \mathbb{S}^1 \times [a,b]$ be a finite cylinder with universal cover $\mathbb{A} = \mathbb{R} \times [a,b]$. The coordinate in $\mathbb{A}$ is denoted by (τ,v). Consider a map*

$$\bar{f} : A \to \mathbb{S} \times \mathbb{R}.$$

We assume that the map has the intersection property. Suppose that $f : A \to \mathbb{R} \times \mathbb{R}$, $(\tau_0,v_0) \to (\tau_1,v_1)$ is a lift of $\bar{f}$ and it has the form

$$\begin{cases} \tau_1 = \tau_0 + 2N\pi + \delta l_1(\tau_0,v_0) + \delta\tilde{g}_1(\tau_0,v_0), \\ v_1 = v_0 + \delta l_2(\tau_0,v_0) + \delta\tilde{g}_2(\tau_0,v_0), \end{cases} \tag{2.17}$$

where N is an integer, $\delta \in (0,1)$ is a parameter. The functions l_1, l_2, $\tilde{g}_1$ and $\tilde{g}_2$ satisfy

$$l_1 \in C^6(A), \qquad l_1(\tau_0,v_0) > 0, \qquad \frac{\partial l_1}{\partial v_0}(\tau_0,v_0) > 0, \quad \forall(\tau_0,v_0) \in A, \tag{2.18}$$

$$l_2(\cdot,\cdot),\tilde{g}_1(\cdot,\cdot,\epsilon),\tilde{g}_2(\cdot,\cdot,\epsilon) \in C^5(A).$$

In addition, we assume that there is a function $I : A \to R$ satisfying

$$I \in C^6(A), \qquad \frac{\partial I}{\partial v_0}(\tau_0, v_0) > 0, \quad \forall (\tau_0, v_0) \in A \tag{2.19}$$

and

$$l_1(\tau_0, v_0) \cdot \frac{\partial I}{\partial \tau_o}(\tau_0, v_0) + l_2(\tau_0, v_0) \cdot \frac{\partial I}{\partial v_0}(\tau_0, v_0) = 0, \quad \forall (\tau_0, v_0) \in A. \tag{2.20}$$

Moreover, suppose that there are two numbers $\tilde{a}$, and $\tilde{b}$ such that $a < \tilde{a} < \tilde{b} < b$ and

$$I_M(a) < I_m(\tilde{a}) \le I_M(\tilde{a}) < I_m(\tilde{b}) \le I_M(\tilde{b}) < I_m(b), \tag{2.21}$$

where

$$I_M(r) = \max_{\rho \in S^1} I(\rho_o, \tau_o), \qquad I_m(r) = \min_{\rho \in S^1} I(\rho_o, \tau_o).$$

Then there exist $\epsilon > 0$ and $\Delta > 0$ such that, if $\delta < \Delta$ and

$$\left\| \tilde{g}_1(\cdot, \cdot, \epsilon) \right\|_{C^5(A)} + \left\| \tilde{g}_2(\cdot, \cdot, \epsilon) \right\|_{C^5(A)} < \epsilon,$$

the mapping $\overline{f}$ has an invariant curve in Γ_A. The constant ϵ is independent of δ.

We make the ansatz that the solution of (2.16) with the initial condition $(t(0), \rho(0)) = (t_0, \rho_0)$ is of the form

$$t = t_0 + \theta + \epsilon^{1-\alpha} \Sigma_1(t_0, \rho_0, \theta; \epsilon), \qquad \rho = \rho_0 + \epsilon^{1-\alpha} \Sigma_2(t_0, \rho_0, \theta; \epsilon).$$

Then the Poincaré map of (2.16) is

$$P: t_1 = t_0 + 2\pi + \epsilon^{1-\alpha} \Sigma_1(t_0, \rho_0, 2\pi; \epsilon), \qquad \rho_1 = \rho_0 + \epsilon^{1-\alpha} \Sigma_2(t_0, \rho_0, 2\pi; \epsilon). \tag{2.22}$$

The functions Σ_1 and Σ_2 satisfy

$$\begin{cases} \Sigma_1 = \pi \epsilon^{\alpha-1} \int_0^\theta \frac{\partial x}{\partial H}(\theta, \epsilon^{-2}\rho) |x|^\alpha p(t) \, d\theta \\ \qquad + \epsilon^{\alpha-1} \int_0^\theta \left(\frac{\partial R_1}{\partial H}(H, t, \theta) + \frac{\partial I}{\partial H}(H, t, \theta) \right) d\theta, \\ \Sigma_2 = -\frac{\pi \epsilon^{\alpha+1}}{\alpha+1} \int_0^\theta |x(\theta, \epsilon^{-2}\rho)|^\alpha x(\theta, \epsilon^{-2}\rho) p'(t) \, d\theta \\ \qquad - \frac{\epsilon^{\alpha+1}}{\alpha+1} \int_0^\theta \left(\frac{\partial R_1}{\partial t}(H, t, \theta) - \frac{\partial I}{\partial t}(H, t, \theta) \right) d\theta, \end{cases} \tag{2.23}$$

where $t = t_0 + \theta + \epsilon^{1-\alpha} \Sigma_1$, $\rho = \rho_0 + \epsilon^{1-\alpha} \Sigma_2$. By Lemmas 4, 6 and 7, we know that

$$|\Sigma_1| + |\Sigma_2| \le C \quad \text{for } \theta \in [0, 2\pi]. \tag{2.24}$$

Hence, for $\rho_0 \in [1, 2]$, we may choose ϵ sufficiently small such that

$$\rho_0 + \epsilon \Sigma_2 \ge \frac{\rho_0}{2} \ge \frac{1}{2}. \tag{2.25}$$

Moreover, we can prove that

$$\Sigma_1, \Sigma_2 \in O_6(1). \tag{2.26}$$

Lemma 9 *The following estimates hold*:

$$x^{\alpha+1}\big(\theta, \epsilon^{-2}\rho\big) - x^{\alpha+1}\big(\theta, \epsilon^{-2}\rho_0\big) \in \epsilon^{-\alpha} O_6(1),$$

$$\frac{\partial x}{\partial H}\big(\theta, \epsilon^{-2}\rho\big) x^\alpha\big(\theta, \epsilon^{-2}\rho\big) - \frac{\partial x}{\partial H}\big(\theta, \epsilon^{-2}\rho_0\big) x^\alpha\big(\theta, \epsilon^{-2}\rho_0\big) \in \epsilon^{2-\alpha} O_6(1).$$

Proof Let

$$\Delta(t_0, \rho_0, \theta) = x^{\alpha+1}\big(\theta, \epsilon^{-2}\rho\big) - x^{\alpha+1}\big(\theta, \epsilon^{-2}\rho_0\big)$$

$$= \int_0^1 (\alpha+1) x^\alpha\big(\theta, \epsilon^{-2}\rho_0 + s\epsilon^{-1}\Sigma_2\big)$$

$$\cdot \frac{\partial x}{\partial H}\big(\theta, \epsilon^{-2}\rho_0 + s\epsilon^{-1}\Sigma_2\big) \epsilon^{-1}\Sigma_2 \, ds.$$

By Lemma 2 and (2.25), we have

$$\big|\Delta(t_0, \rho_0, \theta)\big| \le C \cdot \big(\epsilon^{-2}\rho_0 + s\epsilon^{-1}\Sigma_2\big)^{\frac{\alpha}{2}} \big(\epsilon^{-2}\rho_0 + s\epsilon^{-1}\Sigma_2\big)^{-\frac{1}{2}} \epsilon^{-1}\Sigma_2$$

$$\le C \cdot \big(\epsilon^{-2}\rho_0 + s\epsilon^{-1}\Sigma_2\big)^{\frac{\alpha-1}{2}} \epsilon^{\alpha-1} \epsilon^{-\alpha}$$

$$\le C \cdot \epsilon^{-\alpha}.$$

Take the derivative with respect to ρ_0 in the both sides of $\Delta(t_0, \rho_0, \theta)$, we have

$$\frac{\partial \Delta}{\partial \rho_0} = \int_0^1 \left[(\alpha+1)\alpha x^{\alpha-1} \frac{\partial x}{\partial H} \frac{1 + s\epsilon \frac{\partial \Sigma_2}{\partial \rho_0}}{\epsilon^2} \frac{\partial x}{\partial H} \frac{\Sigma_2}{\epsilon} + (\alpha+1) x^\alpha \frac{\partial^2 x}{\partial H^2} \frac{1 + s\epsilon \frac{\partial \Sigma_2}{\partial \rho_0}}{\epsilon^2} \frac{\Sigma_2}{\epsilon} \right.$$

$$\left. + (\alpha+1) x^\alpha \frac{\partial x}{\partial H} \frac{\partial \Sigma_2}{\partial \rho_0} \frac{1}{\epsilon} \right] ds.$$

Using Lemma 2 and noticing $|\Delta| \le C \cdot \epsilon^{-\alpha}$, we have

$$\left| \frac{\partial \Delta}{\partial \rho_0} \right| \le C \cdot \epsilon^{-\alpha}.$$

Analogously, one may obtain, by a direct but cumbersome commutation that

$$\left| \frac{\partial^{k+l} \Delta}{\partial \rho_0^k \, \partial t_0^l} \right| \le C \cdot \epsilon^{-\alpha},$$

which means that

$$x^{\alpha+1}\big(\theta, \epsilon^{-2}\rho\big) - x^{\alpha+1}\big(\theta, \epsilon^{-2}\rho_0\big) \in \epsilon^{-\alpha} O_6(1).$$

The estimates for $\frac{\partial x}{\partial H}(\theta, \epsilon^{-2}\rho) x^\alpha(\theta, \epsilon^{-2}\rho) - \frac{\partial x}{\partial H}(\theta, \epsilon^{-2}\rho_0) x^\alpha(\theta, \epsilon^{-2}\rho_0)$ follow from a similar argument, we omit it here. Thus, Lemma 9 is proved. □

Now we turn to give an asymptotic expression of Poincaré map of (2.15), that is, we study the behavior of the functions Σ_1 and Σ_2 at $\theta = \pi$ as $\epsilon \to 0$. In order to estimate Σ_1 and Σ_2, we need introduce the following definition and lemma. Let

$$\Theta_+(I) = \mathrm{meas}\{\theta \in [0,\pi], x(H_0,\theta) > 0\}, \qquad \Theta_-(I) = T_0 - \Theta_+(I),$$

where $H_0 = \epsilon^{-2}\rho_0$.

Lemma 10

$$\Theta_+(I) = \pi + \epsilon O_6(1), \qquad \Theta_-(I) = \epsilon O_6(1).$$

Proof This lemma was proved in [5], so we omit the details. □

For estimate Σ_1 and Σ_2, we need the estimates of x and x_H.

We recall that when $x < 0$, we have

$$\left|x(H_0,\theta)\right| = O_6(1), \qquad \left|x_H(H_0,\theta)\right| = \epsilon^2 O_5(1).$$

When $x > 0$, by the definition of θ, we have

$$\arcsin \frac{x(H_0,\theta)}{\sqrt{2h}} = \frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2} = \theta + \epsilon^2 O_5(1),$$

which yields that

$$x(H_0,\theta) = \sqrt{\frac{2H_0}{\pi}}\sin\theta + O_5(1), \qquad x_H(H_0,\theta) = \sqrt{\frac{1}{2H_0\pi}}\sin\theta + \epsilon^2 O_5(1).$$

Now we can give the estimates of Σ_1 and Σ_2.

Lemma 11 *The following estimates hold true*:

$$\Sigma_1(t_0,\rho_0,2\pi;\epsilon) = \left(\frac{\pi}{2\rho_0}\right)^{\frac{\alpha-1}{2}}\int_0^\pi (\sin\theta)^{1+\alpha}p(t_0+\theta)\,d\theta + o_6(1),$$

$$\Sigma_2(t_0,\rho_0,2\pi;\epsilon) = -\frac{1}{\alpha+1}\pi^{\frac{1-\alpha}{2}}(2\rho_0)^{\frac{\alpha+1}{2}}\int_0^\pi (\sin\theta)^{1+\alpha}p'(t_0+\theta)\,d\theta + o_6(1)$$

for $\epsilon \to 0$.

Proof Firstly, we consider Σ_1. By Lemmas 2, 6 and (2.23), we have

$$\Sigma_1(t_0,\rho_0,2\pi;\epsilon) = \pi\epsilon^{\alpha-1}\int_0^\pi \frac{\partial x}{\partial H}(\theta,\epsilon^{-2}\rho)\left|x(\theta,\epsilon^{-2}\rho)\right|^\alpha p(t)\,d\theta$$

$$+ \epsilon^{\alpha-1}\int_0^\pi \left(\frac{\partial R_1}{\partial H}(\epsilon^{-2}\rho,t,\theta) + \frac{\partial I_-}{\partial H}(\epsilon^{-2}\rho,t,\theta)\right)d\theta$$

$$= \pi\epsilon^{\alpha-1}\int_0^\pi \frac{\partial x}{\partial H}(\theta,\epsilon^{-2}\rho_0)\left|x(\theta,\epsilon^{-2}\rho_0)\right|^\alpha p(t_0+\theta)\,d\theta + \epsilon^\alpha O_6(1)$$

$$= \pi \epsilon^{\alpha-1} \left(\int_{\Theta_+} \frac{\partial x}{\partial H} \left(\theta, \epsilon^{-2} \rho \right) |x|^{\alpha} p(t_0 + \theta) \, d\theta \right.$$

$$\left. + \int_{\Theta_-} \frac{\partial x}{\partial H} \left(\theta, \epsilon^{-2} \rho \right) |x|^{\alpha} p(t_0 + \theta) \, d\theta \right) + \epsilon^{\alpha} O_6(1)$$

$$= \pi \epsilon^{\alpha-1} \int_{\Theta_+} \frac{\partial x}{\partial H} \left(\theta, \epsilon^{-2} \rho \right) |x|^{\alpha} p(t_0 + \theta) \, d\theta + \epsilon^{\alpha} O_6(1)$$

$$= \pi \epsilon^{\alpha-1} \int_0^{\pi} \frac{\partial x}{\partial H} \left(\theta, \epsilon^{-2} \rho \right) |x|^{\alpha} p(t_0 + \theta) \, d\theta + \epsilon^{\alpha} O_6(1)$$

$$= \left(\frac{\pi}{2\rho_0} \right)^{\frac{1-\alpha}{2}} \int_0^{\pi} (\sin \theta)^{\alpha+1} p(t_0 + \theta) \, d\theta + o_6(1).$$

Now we consider Σ_2.

$$\Sigma_2(t_0, \rho_0, 2\pi; \epsilon) = -\frac{\pi \epsilon^{\alpha+1}}{\alpha+1} \int_0^{\pi} \left| x\left(\theta, \epsilon^{-2} \rho \right) \right|^{\alpha} x\left(\theta, \epsilon^{-2} \rho \right) p'(t) \, d\theta$$

$$- \frac{\epsilon^{\alpha+1}}{\alpha+1} \int_0^{\pi} \left(\frac{\partial R_1}{\partial t} \left(\epsilon^{-2} \rho, t, \theta \right) + \frac{\partial I_-}{\partial t} \left(\epsilon^{-2} \rho, t, \theta \right) \right) d\theta$$

$$= -\frac{\pi \epsilon^{\alpha+1}}{\alpha+1} \int_0^{\pi} \left| x\left(\theta, \epsilon^{-2} \rho_0 \right) \right|^{\alpha} x\left(\theta, \epsilon^{-2} \rho_0 \right) p'(t_0 + \theta) \, d\theta + \epsilon^{\alpha} O_6(1)$$

$$= -\frac{\pi \epsilon^{\alpha+1}}{\alpha+1} \left(\int_{\Theta_+} \left| x\left(\theta, \epsilon^{-2} \rho \right) \right|^{\alpha} x\left(\theta, \epsilon^{-2} \rho_0 \right) p'(t_0 + \theta) \, d\theta \right.$$

$$\left. + \int_{\Theta_-} \left| x\left(\theta, \epsilon^{-2} \rho_0 \right) \right|^{\alpha} x\left(\theta, \epsilon^{-2} \rho \right) p'(t_0 + \theta) \, d\theta \right) + \epsilon^{\alpha} O_6(1)$$

$$= -\frac{\pi \epsilon^{\alpha+1}}{\alpha+1} \int_{\Theta_+} \left| x\left(\theta, \epsilon^{-2} \rho \right) \right|^{\alpha} x\left(\theta, \epsilon^{-2} \rho_0 \right) p'(t_0 + \theta) \, d\theta + \epsilon^{\alpha} O_6(1)$$

$$= -\frac{\pi \epsilon^{\alpha+1}}{\alpha+1} \int_0^{\pi} \left| x\left(\theta, \epsilon^{-2} \rho_0 \right) \right|^{\alpha} x\left(\theta, \epsilon^{-2} \rho \right) p'(t_0 + \theta) \, d\theta + \epsilon^{\alpha} O_6(1)$$

$$= -\frac{1}{\alpha+1} \pi^{\frac{1-\alpha}{2}} (2\rho_0)^{\frac{\alpha+1}{2}} \int_0^{\pi} (\sin \theta)^{1+\alpha} p'(t_0 + \theta) \, d\theta + o_6(1).$$

Thus, Lemma 11 is proved. □

2.4 Proof of Theorem 1

Let

$$\Psi_1(t_0, \rho_0) = \left(\frac{\pi}{2\rho_0} \right)^{\frac{1-\alpha}{2}} \int_0^{\pi} (\sin \theta)^{1+\alpha} p(t_0 + \theta) \, d\theta,$$

$$\Psi_2(t_0, \rho_0) = -\frac{1}{\alpha+1} \pi^{\frac{1-\alpha}{2}} (2\rho_0)^{\frac{\alpha+1}{2}} \int_0^{\pi} (\sin \theta)^{1+\alpha} p'(t_0 + \theta) \, d\theta.$$

Then there are two functions ϕ_1 and ϕ_2 such that the Poincaré map of (2.16), given by (2.22), is of the form

$$P: t_1 = t_0 + 2\pi + \epsilon^{1-\alpha} \Psi_1(t_0, \rho_0) + \epsilon^{1-\alpha} \phi_1, \qquad \rho_1 = \rho_0 + \epsilon^{1-\alpha} \Psi_2(t_0, \rho_0) + \epsilon^{1-\alpha} \phi_2,$$

where $\phi_1, \phi_2 \in o_6(1)$.

Since $\int_0^\pi p(t_0 + \theta)\sin\theta\, d\theta > 0$, $\forall t_0 \in R$, we have

$$\Psi_1 > 0, \qquad \frac{\partial \Psi_1}{\partial \rho_0} \neq 0.$$

Let

$$L = \frac{\rho_0^{-\frac{\alpha+1}{2}}}{\int_0^\pi (\sin\theta)^{1+\alpha} p(t_0 + \theta)\, d\theta}.$$

Then

$$\frac{\partial L}{\partial t_0}\Psi_1(t_0, \rho_0) + \frac{\partial L}{\partial \rho_0}\Psi_2(t_0, \rho_0) = 0.$$

The other assumptions of Ortega's theorem are easily verified. Hence, there is an invariant curve of P in the annulus $(t_0, \rho_0) \in S^1 \times [1, 2]$ which imply that the boundedness of our original equation (1.4). Then Theorem 1 is proved.

2.5 Proof of Theorem 2

We apply Aubry-Mather theory. By Theorem B in [8] and the monotone twist property of the Poincaré map P guaranteed by $\frac{\partial \Psi_1}{\partial \rho_0} < 0$. It is straightforward to check that Theorem 2 is correct.

Remark 4 In [9], the authors study the multiplicity of positive periodic solutions of singular Duffing equations

$$x'' + g(x) = p(t),$$

where $g(x)$ satisfies the semilinear condition at infinity and the time map satisfies an oscillation condition, and prove that the given equation possesses infinitely many positive 2π-periodic solutions by using the Poincaré-Birkhoff theorem. By the methods and techniques in [9], we can also prove the existence of 2π-periodic solutions of (1.4) where $V(x)$ satisfies the sublinear condition.

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