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On the solutions and conservation laws of the (1 + 1)-dimensional higher-order Broer-Kaup system

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Abstract

In this paper we obtain exact solutions of the (1 + 1)-dimensional higher-order Broer-Kaup system which was obtained from the Kadomtsev-Petviashvili equation by the symmetry constraints. The methods used to determine the exact solutions of the underlying system are the Lie group analysis and the simplest equation method. The solutions obtained are the solitary wave solutions. Moreover, we derive the conservation laws of the (1 + 1)-dimensional higher-order Broer-Kaup system by employing the multiplier approach and the new conservation theorem.

Keywords: the (1 + 1)-dimensional higher-order Broer-Kaup system; integrability; Lie group analysis; simplest equation method; solitary waves; conservation laws

1 Introduction

In this paper we study the $(1+1)$-dimensional higher-order Broer-Kaup system

$$u_t + 4\big(u_{xx} + u^3 - 3uu_x + 6uv\big)_x = 0, \tag{1.1a}$$

$$v_t + 4\big(v_{xx} + 3u^2v + 3uv_x + 3v^2\big)_x = 0, \tag{1.1b}$$

which was first introduced by Lou and Hu [1] by considering the symmetry constraints of the Kadomtsev-Petviashvili equation. The system (1.1a) and (1.1b) is in fact an extension of the well-known $(1+1)$-dimensional Broer-Kaup system [2–4]

$$u_t - u_{xx} + 2uu_x - 2v_x = 0, \tag{1.2a}$$

$$v_t + v_{xx} - 2(uv)_x = 0, \tag{1.2b}$$

which is used to model the bi-directional propagation of long waves in shallow water. In [5], Fan derived a unified Darboux transformation for the system (1.1a) and (1.1b) with the help of a gauge transformation of the spectral problem and as an application obtained some new explicit soliton-like solutions. Recently, Huang *et al.* [6] presented a new N-fold Darboux transformations of the $(1+1)$-dimensional higher-order Broer-Kaup system with the help of a gauge transformation of the spectral problem and found new explicit multi-soliton solutions of the system (1.1a) and (1.1b).

In the latter half of the nineteenth century, Sophus Lie (1842-1899) developed one of the most powerful methods to determine solutions of differential equations. This method, known as the Lie group analysis method, systematically unifies and extends well-known *ad hoc* techniques to construct explicit solutions of differential equations. It has proved to be a versatile tool for solving nonlinear problems described by the differential equations arising in mathematics, physics and in other scientific fields of study. For the theory and application of the Lie group analysis methods, see, *e.g.*, the Refs. [7–12].

Conservation laws play a vital role in the solution process of differential equations. Finding conservation laws of the system of differential equations is often the first step towards finding the solution [7]. Also, the conservation laws are useful in the numerical integration of partial differential equations [13], for example, to control numerical errors. The determination of conservation laws of the Korteweg de Vries equation, in fact, initiated the discovery of a number of methods to solve evolutionary equations [14]. Moreover, conservation laws play an important role in the theories of non-classical transformations [15, 16], normal forms and asymptotic integrability [17]. Recently, in [18] the conserved quantity was used to determine the unknown exponent in the similarity solution which cannot be obtained from the homogeneous boundary conditions.

In this paper, we use the Lie group analysis approach along with the simplest equation method to obtain exact solutions of the $(1+1)$-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b). Furthermore, conservation laws will be computed for (1.1a) and (1.1b) using the two approaches: the new conservation theorem due to Ibragimov [19] and the multiplier method [20, 21].

2 Symmetry reductions and exact solutions of (1.1a) and (1.1b)

The symmetry group of the $(1+1)$-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b) will be generated by the vector field of the form

$$X = \xi^1(t,x,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,u,v)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,u,v)\frac{\partial}{\partial v}.$$

Applying the third prolongation $\mathrm{pr}^{(3)}X$ [11] to (1.1a) and (1.1b) and solving the resultant overdetermined system of linear partial differential equations one obtains the following three Lie point symmetries:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x},\\
X_2 &= \frac{\partial}{\partial t},\\
X_3 &= -3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}.
\end{aligned}$$

2.1 One-dimensional optimal system of subalgebras

In this subsection we present an optimal system of one-dimensional subalgebras for the system (1.1a) and (1.1b) to obtain an optimal system of group-invariant solutions. The method which we use here for obtaining a one-dimensional optimal system of subalgebras is that given in [11]. The adjoint transformations are given by

$$\mathrm{Ad}\big(\exp(\epsilon X_i)\big)X_j = X_j - \epsilon[X_i,X_j] + \frac{1}{2}\epsilon^2\big[X_i,[X_i,X_j]\big] - \cdots.$$

Table 1 Commutator table of the Lie algebra of the system (1.1a) and (1.1b)

	X_1	X_2	X_3
X_1	0	0	$-X_1$
X_2	0	0	$-3X_2$
X_3	X_1	$3X_2$	0

Table 2 Adjoint table of the Lie algebra of the system (1.1a) and (1.1b)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 + \epsilon X_1$
X_2	X_1	X_2	$X_3 + 3\epsilon X_2$
X_3	$e^{-\epsilon} X_1$	$e^{-3\epsilon} X_2$	X_3

Here $[X_i, X_j]$ is the commutator given by

$$[X_i, X_j] = X_i X_j - X_j X_i.$$

The commutator table of the Lie point symmetries of the system (1.1a) and (1.1b) and the adjoint representations of the symmetry group of (1.1a) and (1.1b) on its Lie algebra are given in Table 1 and Table 2, respectively. Table 1 and Table 2 are used to construct an optimal system of one-dimensional subalgebras for the system (1.1a) and (1.1b).

From Tables 1 and 2 one can obtain an optimal system of one-dimensional subalgebras given by $\{\nu X_1 + X_2, X_2, X_3\}$.

2.2 Symmetry reductions of (1.1a) and (1.1b)

In this subsection we use the optimal system of one-dimensional subalgebras calculated above to obtain symmetry reductions that transform (1.1a) and (1.1b) into a system of ordinary differential equations (ODEs). Later, in the next subsection, we will look for exact solutions of (1.1a) and (1.1b).

Case 1. $\nu X_1 + X_2$

The symmetry $\nu X_1 + X_2$ gives rise to the group-invariant solution

$$u = F(z), \qquad v = G(z), \tag{2.1}$$

where $z = x - \nu t$ is an invariant of the symmetry $\nu X_1 + X_2$. Substitution of (2.1) into (1.1a) and (1.1b) results in the system of ODEs

$$\begin{aligned} &4F'''(z) - 12F(z)F''(z) + 24G(z)F'(z) - \nu F'(z) \\ &\quad + 12F(z)^2F'(z) - 12F'(z)^2 + 24F(z)G'(z) = 0, \end{aligned} \tag{2.2a}$$

$$\begin{aligned} &4G'''(z) + 12F'(z)G'(z) + 24F(z)G(z)F'(z) + 12F(z)G''(z) \\ &\quad + 12F(z)^2G'(z) - \nu G'(z) + 24G(z)G'(z) = 0. \end{aligned} \tag{2.2b}$$

Case 2. X_2

The symmetry X_2 gives rise to the group-invariant solution of the form

$$u = F(z), \qquad v = G(z), \tag{2.3}$$

where $z = x$ is an invariant of X_2 and the functions F and G satisfy the following system of ODEs:

$$
\begin{aligned}
&4F'''(z) - 12F(z)F''(z) + 24G(z)F'(z)\\
&\qquad + 12F(z)^2F'(z) - 12F'(z)^2 + 24F(z)G'(z) = 0,\\
&4G'''(z) + 12F'(z)G'(z) + 24F(z)G(z)F'(z)\\
&\qquad + 12F(z)G''(z) + 12F(z)^2G'(z) + 24G(z)G'(z) = 0.
\end{aligned}
$$

Case 3. X_3

By solving the corresponding Lagrange system for the symmetry X_3, one obtains an invariant $z = xt^{-1/3}$ and the group-invariant solution of the form

$$u = t^{-1/3}F(z), \qquad v = t^{-2/3}G(z), \tag{2.4}$$

where the functions F and G satisfy the following system of ODEs:

$$
\begin{aligned}
&F'(z)z - 72F'(z)G(z) - 12F'''(z) - 36F'(z)F(z)^2\\
&\qquad - 72G'(z)F(z) + F(z) + 36F''(z)F(z) + 36F'(z)^2 = 0,\\
&-36G'(z)F(z)^2 + 2G(z) - 36F'(z)G'(z) - 72G'(z)G(z)\\
&\qquad - 72F'(z)F(z)G(z) + G'(z)z - 12G'''(z) - 36G''(z)F(z) = 0.
\end{aligned}
$$

2.3 Exact solutions using the simplest equation method

In this subsection we use the simplest equation method, which was introduced by Kudryashov [22, 23] and modified by Vitanov [24] (see also [25]), to solve the ODE system (2.2a) and (2.2b), and as a result we will obtain the exact solutions of our $(1+1)$-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b). Bernoulli and Riccati equations will be used as the simplest equations.

Let us consider the solutions of the ODE system (2.2a) and (2.2b) in the form

$$F(z) = \sum_{i=0}^{M} A_i\big(H(z)\big)^i, \qquad G(z) = \sum_{i=0}^{M} B_i\big(H(z)\big)^i, \tag{2.5}$$

where $H(z)$ satisfies the Bernoulli and Riccati equations, M is a positive integer that can be determined by balancing procedure as in [24] and $A_0, \dots, A_M$, $B_0, \dots, B_M$ are parameters to be determined. It is well known that the Bernoulli and Riccati equations are nonlinear ODEs whose solutions can be written in terms of elementary functions.

We consider the Bernoulli equation

$$H'(z) = aH(z) + bH^2(z), \tag{2.6}$$

where a and b are constants. Its solution is given by

$$H(z) = a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\},$$

where C is a constant of integration.

For the Riccati equation

$$H'(z) = aH^2(z) + bH(z) + c, \tag{2.7}$$

where a, b and c are constants, we will use the solutions

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]$$

and

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})},$$

where $\theta^2 = b^2 - 4ac > 0$ and C is a constant of integration.

2.3.1 Solutions of (1.1a) *and* (1.1b) *using the Bernoulli equation as the simplest equation*

The balancing procedure [24] yields $M = 2$, so the solutions of (2.2a) and (2.2b) are of the form

$$F(z) = A_0 + A_1 H + A_2 H^2, \qquad G(z) = B_0 + B_1 H + B_2 H^2. \tag{2.8}$$

Substituting (2.8) into (2.2a) and (2.2b) and making use of (2.6) and then equating all coefficients of the functions H^i to zero, we obtain an algebraic system of equations in terms of A_0, A_1, A_2 and B_0, B_1, B_2. Solving the system of algebraic equations with the aid of Mathematica, we obtain the following cases.

Case 1

$$\begin{aligned}
A_0 &= \frac{1}{6}\left(\pm 3a \pm \sqrt{3}\sqrt{-a^2+\nu}\right),\\
A_1 &= \frac{bA_0(8a^2+\nu-12A_0^2)}{a(4a^2-\nu)},\\
A_2 &= 0,\\
B_0 &= 0,\\
B_1 &= \frac{-2ab^2+3abA_1-aA_1^2}{6b},\\
B_2 &= \frac{bB_1}{a}.
\end{aligned}$$

Thus, a solution of our (1 + 1)-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b) is

$$\begin{aligned}
u(t,x) ={}& A_0 + A_1 a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}\\
&+ A_2 a^2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2,
\end{aligned} \tag{2.9a}$$

$$
\begin{aligned}
v(t,x) = {} & B_0 + B_1 a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]} \right\} \\
& + B_2 a^2 \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]} \right\}^2,
\end{aligned}
\tag{2.9b}
$$

where $z = x - \nu t$ and C is a constant of integration.

Case 2

$$
\begin{aligned}
a &= \pm \frac{\sqrt{\nu}}{2}, \\
A_0 &= \pm \frac{\sqrt{\nu}}{2}, \\
A_1 &= \frac{6abA_0 \pm \sqrt{b^2\nu^2 + 36a^2b^2A_0^2 - 12b^2\nu A_0^2}}{\nu}, \\
A_2 &= 0, \\
B_0 &= 0, \\
B_1 &= \frac{-2ab^2 + 3abA_1 - aA_1^2}{6b}, \\
B_2 &= \frac{1}{6}\left(-2b^2 + 3bA_1 - A_1^2\right),
\end{aligned}
$$

and so a solution of the (1 + 1)-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b) is

$$
\begin{aligned}
u(t,x) = {} & A_0 + A_1 a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]} \right\} \\
& + A_2 a^2 \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]} \right\}^2,
\end{aligned}
\tag{2.10a}
$$

$$
\begin{aligned}
v(t,x) = {} & B_0 + B_1 a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]} \right\} \\
& + B_2 a^2 \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]} \right\}^2,
\end{aligned}
\tag{2.10b}
$$

where $z = x - \nu t$ and C is a constant of integration.

2.3.2 Solutions of (1.1a) *and* (1.1b) *using Riccati equation as the simplest equation*

The balancing procedure yields $M = 2$, so the solutions of the ODE system (2.2a) and (2.2b) are of the form

$$
F(z) = A_0 + A_1 H + A_2 H^2, \qquad G(z) = B_0 + B_1 H + B_2 H^2.
\tag{2.11}
$$

Substituting (2.11) into (2.2a) and (2.2b) and making use of (2.7), we obtain an algebraic system of equations in terms of A_0, A_1, A_2, B_0, B_1, B_2 by equating all coefficients of the functions H^i to zero. Solving the algebraic equations, one obtains the following cases.

Case 1

$$
\begin{aligned}
A_0 &= \frac{1}{6}\left(\pm 3b \pm \sqrt{3}\sqrt{-b^2+4ac+\nu}\right),\\
A_1 &= -\frac{a(-4b^2+4ac+\nu-12A_0^2)}{12bA_0},\\
A_2 &= 0,\\
B_0 &= \frac{1}{2}(-ac+cA_1),\\
B_1 &= \frac{1}{2}(-ab+bA_1),\\
B_2 &= \frac{aB_1}{b},
\end{aligned}
$$

and hence the solutions of the $(1+1)$-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b) are

$$
\begin{aligned}
u(t,x) =& A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}\\
&+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2, \qquad (2.12a)\\
v(t,x) =& B_0 + B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}\\
&+ B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2 \qquad (2.12b)
\end{aligned}
$$

and

$$
\begin{aligned}
u(t,x) =& A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}\\
&+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}^2, \qquad (2.13a)\\
v(t,x) =& B_0 + B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}\\
&+ B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}^2, \qquad (2.13b)
\end{aligned}
$$

where $z = x - \nu t$ and C is a constant of integration.

Case 2

$$
\begin{aligned}
b &= \pm\frac{\sqrt{\nu}}{6},\\
a &= -\frac{\nu}{18c},\\
A_0 &= \pm\frac{\sqrt{\nu}}{6},
\end{aligned}
$$

$$A_1 = \frac{aA_0 + a\sqrt{8b^2 + A_0^2}}{2b},$$

$$A_2 = 0,$$

$$B_0 = \frac{4a\nu - 3\nu A_1 + 36bA_0A_1}{108a},$$

$$B_1 = \frac{A_1(3ab + aA_0 - 2bA_1)}{6a},$$

$$B_2 = \frac{1}{6}\left(-2a^2 + 3aA_1 - A_1^2\right).$$

In this case the solutions of the $(1+1)$-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b) are given by

$$\begin{aligned} u(t,x) =& A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\ &+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2, \end{aligned} \tag{2.14a}$$

$$\begin{aligned} v(t,x) =& B_0 + B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\ &+ B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2 \end{aligned} \tag{2.14b}$$

and

$$\begin{aligned} u(t,x) =& A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\} \\ &+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}^2, \end{aligned} \tag{2.15a}$$

$$\begin{aligned} v(t,x) =& B_0 + B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\} \\ &+ B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}^2, \end{aligned} \tag{2.15b}$$

where $z = x - \nu t$ and C is a constant of integration.

Case 3

$$a = \frac{4b^2 - \nu}{16c},$$

$$A_0 = \pm b,$$

$$A_1 = \frac{2aA_0}{b},$$

$$A_2 = 0,$$

$$B_0 = \frac{-88ab^2 + 6a\nu + 12b^2A_1 - 3\nu A_1 + 32bA_0A_1}{96a},$$

$$B_1 = \frac{A_1(3ab + aA_0 - 2bA_1)}{6a},$$

$$B_2 = \frac{1}{6}\left(-2a^2 + 3aA_1 - A_1^2\right).$$

The solutions in this case are

$$\begin{aligned} u(t,x) =& A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\ &+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2, \end{aligned} \tag{2.16a}$$

$$\begin{aligned} v(t,x) =& B_0 + B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\ &+ B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2 \end{aligned} \tag{2.16b}$$

and

$$\begin{aligned} u(t,x) =& A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\} \\ &+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}^2, \end{aligned} \tag{2.17a}$$

$$\begin{aligned} v(t,x) =& B_0 + B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\} \\ &+ B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}^2, \end{aligned} \tag{2.17b}$$

where $z = x - \nu t$ and C is a constant of integration.

Case 4

$$a = \frac{4b^2 - \nu}{4c},$$

$$A_0 = 0,$$

$$A_1 = \pm a,$$

$$A_2 = 0,$$

$$B_0 = \frac{1}{8}\left(-4b^2 + \nu + 4cA_1\right),$$

$$B_1 = \frac{1}{2}(-ab + bA_1),$$

$$B_2 = \frac{aB_1}{b},$$

and so the solutions are

$$\begin{aligned}u(t,x) =& A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}\\ &+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2,\end{aligned} \tag{2.18a}$$

$$\begin{aligned}v(t,x) =& B_0 + B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}\\ &+ B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2\end{aligned} \tag{2.18b}$$

and

$$\begin{aligned}u(t,x) =& A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}\\ &+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}^2,\end{aligned} \tag{2.19a}$$

$$\begin{aligned}v(t,x) =& B_0 + B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}\\ &+ B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}^2,\end{aligned} \tag{2.19b}$$

where $z = x - \nu t$ and C is a constant of integration.

Case 5

$$\begin{aligned}
a &= \frac{4b^2 - \nu}{16c},\\
A_0 &= \frac{1}{4}(\pm 2b \pm \sqrt{\nu}),\\
A_1 &= \frac{a(4b^2 - \nu + 16A_0^2)}{16bA_0},\\
A_2 &= 0,\\
B_0 &= \frac{-8ab^2 + 2a\nu + 12b^2A_1 - 3\nu A_1 - 16cA_1^2}{96a},\\
B_1 &= bA_1,\\
B_2 &= \frac{aB_1}{b}.
\end{aligned}$$

The solutions are

$$\begin{aligned}u(t,x) =& A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}\\ &+ A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2,\end{aligned} \tag{2.20a}$$

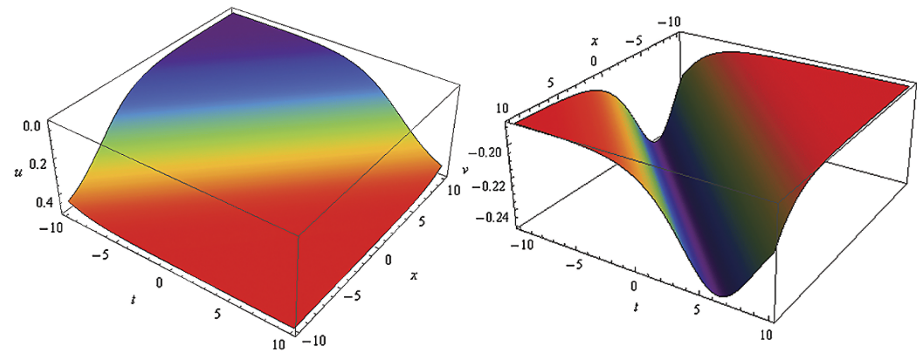

Figure 1 Profile of solitary waves (2.21a) and (2.21b).

$$v(t,x) = B_0 + B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} + B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2 \tag{2.20b}$$

and

$$u(t,x) = A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\} + A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}^2, \tag{2.21a}$$

$$v(t,x) = B_0 + B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\} + B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}(\frac{\theta z}{2})}{C\cosh(\frac{\theta z}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta z}{2})}\right\}^2, \tag{2.21b}$$

where $z = x - \nu t$ and C is a constant of integration.

A profile of the solution (2.21a) and (2.21b) is given in Figure 1.

3 Conservation laws of (1.1a) and (1.1b)

In this section, we derive conservation laws for the $(1+1)$-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b). Two different approaches will be used. Firstly, we use the new conservation method due to Ibragimov [19] and then employ the multiplier method [20, 21]. We now present some preliminaries that we will need later in this section.

3.1 Preliminaries

In this subsection we briefly present the notation and pertinent results which we utilize below. For details the reader is referred to [8–10, 19–21, 27].

3.1.1 Fundamental operators and their relationship

Consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{3.1}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second, ..., kth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$, respectively, with the *total derivative operator* with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \qquad i = 1, \ldots, n, \tag{3.2}$$

where the summation convention is used whenever appropriate.

The *Euler-Lagrange operator*, for each α, is given by [8–10]

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \cdots i_s}}, \quad \alpha = 1, \ldots, m, \tag{3.3}$$

and the *Lie-Bäcklund operator* is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \tag{3.4}$$

where $\mathcal{A}$ is the space of *differential functions*. The operator (3.4) is an abbreviated form of the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1 i_2 \cdots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \cdots i_s}}, \tag{3.5}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} &\zeta_i^\alpha = D_i\left(W^\alpha\right) + \xi^j u_{ij}^\alpha, \\ &\zeta^\alpha_{i_1 \cdots i_s} = D_{i_1} \cdots D_{i_s}\left(W^\alpha\right) + \xi^j u^\alpha_{j i_1 \cdots i_s}, \quad s > 1, \end{aligned} \tag{3.6}$$

in which W^α is the *Lie characteristic function* given by

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha. \tag{3.7}$$

The Lie-Bäcklund operator (3.5) can be written in a characteristic form as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1} \cdots D_{i_s}\left(W^\alpha\right) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \cdots i_s}}. \tag{3.8}$$

The *Noether operators* associated with the Lie-Bäcklund symmetry operator X are given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1} \cdots D_{i_s}\left(W^\alpha\right) \frac{\delta}{\delta u^\alpha_{i i_1 i_2 \cdots i_s}}, \quad i = 1, \ldots, n, \tag{3.9}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (3.3) by replacing u^α by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1} (-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial u^\alpha_{i j_1 j_2 \cdots j_s}}, \quad i = 1, \ldots, n, \alpha = 1, \ldots, m, \tag{3.10}$$

and the Euler-Lagrange , Lie-Bäcklund and Noether operators are connected by the operator identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{3.11}$$

The n-tuple vector $T = (T^1, T^2, \dots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \dots, n$, is a *conserved vector* of (3.1) if T^i satisfies

$$D_i T^i|_{(3.1)} = 0. \tag{3.12}$$

The equation (3.12) defines a *local conservation law* of the system (3.1).

3.1.2 Multiplier method

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \dots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{3.13}$$

hold identically. We consider multipliers of the third-order, that is,

$$\Lambda_\alpha = \Lambda_\alpha(t, x, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}).$$

The right-hand side of (3.13) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{3.14}$$

The conserved vectors are calculated via a homotopy formula [20, 21, 26] once the multipliers are obtained.

3.1.3 Variational method for a system and its adjoint

A system of *adjoint equations* for the system of kth-order differential equations (3.1) is defined by [27]

$$E^*_\alpha(x, u, v, \dots, u_{(k)}, v_{(k)}) = 0, \quad \alpha = 1, \dots, m, \tag{3.15}$$

where

$$E^*_\alpha(x, u, v, \dots, u_{(k)}, v_{(k)}) = \frac{\delta(v^\beta E_\beta)}{\delta u^\alpha}, \quad \alpha = 1, \dots, m, v = v(x) \tag{3.16}$$

and $v = (v^1, v^2, \dots, v^m)$ are new dependent variables.

The following results are given in Ibragimov [19] and recalled here.

Assume that the system of equations (3.1) admits the symmetry generator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \tag{3.17}$$

Then the system of adjoint equations (3.15) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_*^\alpha \frac{\partial}{\partial v^\alpha}, \quad \eta_*^\alpha = -\left[\lambda_\beta^\alpha v^\beta + v^\alpha D_i\left(\xi^i\right)\right], \tag{3.18}$$

where the operator (3.18) is an extension of (3.17) to the variable v^α and the λ_β^α are obtainable from

$$X(E_\alpha) = \lambda_\alpha^\beta E_\beta. \tag{3.19}$$

Theorem 1 [19] *Every Lie point, Lie-Bäcklund and nonlocal symmetry* (3.17) *admitted by the system of equations* (3.1) *gives rise to a conservation law for the system consisting of equation* (3.1) *and adjoint equation* (3.15), *where the components T^i of the conserved vector $T = (T^1, \dots, T^n)$ are determined by*

$$T^i = \xi^i L + W^\alpha \frac{\delta L}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1} \cdots D_{i_s}\left(W^\alpha\right) \frac{\delta L}{\delta u_{i i_1 i_2 \cdots i_s}^\alpha}, \quad i = 1, \dots, n, \tag{3.20}$$

with Lagrangian given by

$$L = v^\alpha E_\alpha(x, u, \dots, u_{(k)}). \tag{3.21}$$

3.2 Construction of conservation laws for (1.1a) and (1.1b)

We now construct conservation laws for the $(1+1)$-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b) using the two approaches.

3.2.1 Application of the multiplier method

For the $(1+1)$-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b), after some lengthy calculations, we obtain the third-order multipliers

$$\Lambda_1 = \Lambda_1(t, x, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx})$$

and

$$\Lambda_2 = \Lambda_2(t, x, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx})$$

that are given by

$$\begin{aligned}\Lambda_1 = {} & C_1(24tuv + 12tv_x) \\ & + C_2\left(6v_x u^2 + 4v_{xx}u + 2u_{xx}v + 6v_x v + 4u^3 v + 12uv^2 + 2u_x v_x + v_{xxx}\right) \\ & + C_3\left(3v_x u + 3u^2 v + 3v^2 + v_{xx}\right) + C_6(2uv + v_x) + C_5 v + C_4, \end{aligned} \tag{3.22}$$

$$\begin{aligned}\Lambda_2 = {} & C_1\left(12tu^2 + 24tv - 12tu_x - x\right) \\ & + C_2\left(-6u_x v - 6u_x u^2 + 4u_{xx}u + 12u^2 v + u^4 + 6v^2 + 3{u_x}^2 - u_{xxx} + 2v_{xx}\right) \\ & + C_3\left(-3u_x u + 6uv + u^3 + u_{xx}\right) + C_6\left(u^2 + 2v - u_x\right) + C_5 u + C_7, \end{aligned} \tag{3.23}$$

where C_i, $i = 1,2,3,4,5,6,7$ are arbitrary constants. Corresponding to the above multipliers, we obtain the following seven local conserved vectors of the (1 + 1)-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b):

$$\begin{aligned}
\Phi_1^t &= 6tv_xu - 6tu_xv + 12tu^2v + 12tv^2 - xv,\\
\Phi_1^x &= 2\{72tv_xu^3 - 144tu_xu^2v + 24tv_{xx}u^2 - 6xv_xu + 144tv_xuv\\
&\quad - 120tu_xv_xu + 48tu_{xx}uv - 3tv_tu - 24tu_x{}^2v + 3tu_tv + 48tv_{xx}v\\
&\quad + 72tu^4v + 288tu^2v^2 - 6xu^2v + 3uv + 96tv^3 - 6xv^2 + 24tu_{xx}v_x\\
&\quad - 24tu_xv_{xx} - 24tv_x{}^2 + 2v_x - 2xv_{xx}\},\\
\Phi_2^t &= \frac{1}{6}\{6vu^4 + 9v_xu^3 + 36v^2u^2 - 9vu_xu^2 + 8v_{xx}u^2 + 12vv_xu\\
&\quad + 4u_xv_xu + 12vu_{xx}u + 3v_{xxx}u + 12v^3 + 6vu_x{}^2 - 12v^2u_x\\
&\quad + 6vv_{xx} - 3vu_{xxx}\},\\
\Phi_2^x &= \frac{1}{6}\{72vu^6 + 72v_xu^5 + 792v^2u^4 - 288vu_xu^4 + 24v_{xx}u^4\\
&\quad + 864vv_xu^3 - 240u_xv_xu^3 + 96vu_{xx}u^3 - 9v_tu^3 + 1296v^3u^2\\
&\quad + 216vu_x{}^2u^2 + 216v_x{}^2u^2 - 864v^2u_xu^2 + 72v_xu_{xx}u^2 + 288vv_{xx}u^2\\
&\quad - 72u_xv_{xx}u^2 + 9vu_tu^2 - 8v_{tx}u^2 + 432v^2v_xu + 72u_x{}^2v_xu - 144vu_xv_xu\\
&\quad + 288v^2u_{xx}u - 144vu_xu_{xx}u + 144v_xv_{xx}u + 24u_{xx}v_{xx}u + 8v_xu_tu - 12vv_tu\\
&\quad + 12u_xv_tu - 12vu_{tx}u - 3v_{txx}u + 216v^4 + 24vu_{xx}{}^2 + 24v_{xx}{}^2 - 24u_xv_xu_{xx}\\
&\quad + 144v^2v_{xx} + 12v^2u_t + 3v_{xx}u_t + 6v_xv_t - 3u_{xx}v_t\\
&\quad - 3v_xu_{tx} - 6vv_{tx} + 3u_xv_{tx} + 3vu_{txx}\},\\
\Phi_3^t &= \frac{1}{2}\{2v_xu^2 - 2u_xuv + v_{xx}u + u_{xx}v + 2u^3v + 6uv^2\},\\
\Phi_3^x &= \frac{1}{2}\{24v_xu^4 - 72u_xu^3v + 8v_{xx}u^3 + 144v_xu^2v - 72u_xv_xu^2 + 24u_{xx}u^2v\\
&\quad - 2v_tu^2 - 72u_xuv^2 + 24u_{xx}v_xu + 48v_{xx}uv - 24u_xv_{xx}u + 2u_tuv - uv_{tx}\\
&\quad + 24u_{xx}v^2 - vu_{tx} + 24u^5v + 168u^3v^2 + 144uv^3 + u_tv_x + v_tu_x + 8u_{xx}v_{xx}\},\\
\Phi_4^t &= u,\\
\Phi_4^x &= 4\{-3u_xu + 6uv + u^3 + u_{xx}\},\\
\Phi_5^t &= uv,\\
\Phi_5^x &= 4\{3v_xu^2 - 3u_xuv + v_{xx}u + u_{xx}v + 3u^3v + 6uv^2 - u_xv_x\},\\
\Phi_6^t &= \frac{1}{2}\{v_xu - u_xv + 2u^2v + 2v^2\},\\
\Phi_6^x &= \frac{1}{2}\{24v_xu^3 - 48u_xu^2v + 8v_{xx}u^2 + 48v_xuv - 40u_xv_xu + 16u_{xx}uv\\
&\quad - v_tu - 8u_x{}^2v + u_tv + 16v_{xx}v + 24u^4v + 96u^2v^2 + 32v^3 + 8u_{xx}v_x - 8u_xv_{xx} - 8v_x{}^2\},
\end{aligned}$$

$$\Phi_7^t = v,$$

$$\Phi_7^x = 4\left\{3v_x u + 3u^2 v + 3v^2 + v_{xx}\right\}.$$

Remark 1 Higher-order conservation laws of (1.1a) and (1.1b) can be computed by increasing the order of multipliers.

3.2.2 Application of the new conservation theorem

The adjoint equations of (1.1a) and (1.1b), by invoking (3.16), are given by

$$-24\psi_x uv - 12\phi_x u^2 - 12\phi_{xx} u - 24\phi_x v - \phi_t - 12v_x\psi_x - 4\phi_{xxx} = 0, \tag{3.24a}$$

$$-12\psi_x u^2 + 12\psi_{xx} u - 24\phi_x u - 24\psi_x v - \psi_t + 12u_x\psi_x - 4\psi_{xxx} = 0, \tag{3.24b}$$

where $\phi = \phi(t,x)$ and $\psi = \psi(t,x)$ are the new dependent variables. By recalling (3.21), we get the following Lagrangian for the system of equations (1.1a) and (1.1b) and (3.24a) and (3.24b):

$$\begin{aligned} L =&\phi(t,x)\left\{u_t + 4\left(u_{xx} + u^3 - 3uu_x + 6uv\right)_x\right\} \\ &+ \psi(t,x)\left\{v_t + 4\left(v_{xx} + 3u^2 v + 3uv_x + 3v^2\right)_x\right\}. \end{aligned} \tag{3.25}$$

Because of the three Lie point symmetries of the $(1+1)$-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b), we have the following three cases to consider:

(i) We first consider the Lie point symmetry generator $X_1 = \partial_x$ of the $(1+1)$-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b). Corresponding to this symmetry, the Lie characteristic function is $W = -(u_x + v_x)$. Thus, by using (3.20), the components of the conserved vector are given by

$$T_1^t = -u_x\phi - v_x\psi,$$

$$T_1^x = 12v_x\psi_x u - 12u_x\phi_x u + u_t\phi + v_t\psi(t,x) + 4u_{xx}\phi_x - 4u_x\phi_{xx} + 4v_{xx}\psi_x - 4v_x\psi_{xx}.$$

(ii) The Lie point symmetry generator $X_2 = \partial_t$ has the Lie characteristic function $W = -(u_t + v_t)$. Hence using (3.20), one can obtain the conserved vector whose components are

$$\begin{aligned} T_2^t =&4\big(3v_x u^2\psi + 6u_x uv\psi + 3v_{xx} u\psi + 3u_x v_x\psi + 6v_x u\phi + 6u_x v\phi \\ &+ 3u_x u^2\phi - 3u_{xx} u\phi - 3{u_x}^2\phi + u_{xxx}\phi + 6v_x v\psi + v_{xxx}\psi\big), \\ T_2^x =& -4\big(3v_t u^2\psi + 6u_t uv\psi - 3v_t\psi_x u + 3u\psi v_{tx} + 3u_t v_x\psi + 6v_t u\phi \\ &+ 6u_t v\phi + 3u_t u^2\phi + 3u_t\phi_x u - 3u\phi u_{tx} - 3u_t u_x\phi + \phi u_{txx} + 6v_t v\psi \\ &+ \psi v_{txx} + u_t\phi_{xx} - \phi_x u_{tx} + v_t\psi_{xx} - \psi_x v_{tx}\big). \end{aligned}$$

(iii) Finally, we consider the symmetry generator $X_3 = -3t\partial_t - x\partial_x + u\partial_u + 2v\partial_v$. For this case, the Lie characteristic function $W = u + 2v + 3tu_t + 3tv_t + xu_x + xv_x$, and by using (3.20), the components of the conserved vector are given by

$$\begin{aligned}
T_3^t = {} & -36t\phi u_x u^2 - 36t\psi v_x u^2 + \phi u - 72tv\psi u_x u - 72t\phi v_x u + 36t\phi u_{xx} u \\
& - 36t\psi v_{xx} u + 36t\phi {u_x}^2 + 2v\psi + x\phi u_x - 72tv\phi u_x + x\psi v_x - 72tv\psi v_x \\
& - 36t\psi u_x v_x - 12t\phi u_{xxx} - 12t\psi v_{xxx}, \\
T_3^x = {} & 12\phi u^3 + 48v\psi u^2 + 12\phi_x u^2 + 36t\phi u_t u^2 + 36t\psi v_t u^2 + 72v\phi u - 36\phi u_x u \\
& + 48\psi v_x u + 12xu_x\phi_x u - 24v\psi_x u - 12xv_x\psi_x u + 4\phi_{xx} u + 72tv\psi u_t u + 36t\phi_x u_t u \\
& + 72t\phi v_t u - 36t\psi_x v_t u - 36t\phi u_{tx} u + 36t\psi v_{tx} u \\
& + 48v^2\psi - 8u_x\phi_x - 12v_x\psi_x + 12\phi u_{xx} \\
& - 4x\phi_x u_{xx} + 16\psi v_{xx} - 4x\psi_x v_{xx} + 4xu_x\phi_{xx} + 8v\psi_{xx} + 4xv_x\psi_{xx} - x\phi u_t + 72tv\phi u_t \\
& - 36t\phi u_x u_t + 36t\psi v_x u_t + 12t\phi_{xx} u_t - x\psi v_t + 72tv\psi v_t \\
& + 12t\psi_{xx} v_t - 12t\phi_x u_{tx} - 12t\psi_x v_{tx} \\
& + 12t\phi u_{txx} + 12t\psi v_{txx}.
\end{aligned}$$

Remark 2 The components of the conserved vectors contain the arbitrary solutions ϕ and ψ of adjoint equations (3.24a) and (3.24b), and hence one can obtain an infinite number of conservation laws.

4 Concluding remarks

In this paper we have studied the (1 + 1)-dimensional higher-order Broer-Kaup system (1.1a) and (1.1b). Similarity reductions and exact solutions, with the aid of the simplest equation method, were obtained based on optimal systems of one-dimensional subalgebras for the underlying system. We have verified the correctness of the solutions obtained here by substituting them back into the system (1.1a) and (1.1b). Furthermore, conservation laws for the system (1.1a) and (1.1b) were derived by using the multiplier method and the new conservation theorem.

Competing interests
The author declares that he has no competing interests.

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