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Strong convergence theorems for a generalized mixed equilibrium problem and variational inequality problems

Jae Ug Jeong*

*Correspondence:
jujeong@deu.ac.kr
Department of Mathematics,
Dongeui University, Busan, 614-714,
South Korea

Abstract
In this paper, a new iterative scheme based on the extragradient-like method for finding a common element of the set of common fixed points of a finite family of nonexpansive mappings, the set of solutions of variational inequalities for a strongly positive linear bounded operator and the set of solutions of a mixed equilibrium problem is proposed. A strong convergence theorem for this iterative scheme in Hilbert spaces is established. Our results extend recent results announced by many others.

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Keywords: fixed point; inverse strongly monotone mapping; variational inequality; equilibrium problem

1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T. Let P_C be the projection of H onto the convex subset C. Moreover, we also denote by $\mathbb{R}$ the set of all real numbers.

Peng and Yao [1] considered the generalized mixed equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Fx, y - x\rangle \ge 0, \quad \forall y \in C, \tag{1.1}$$

where $F : C \to H$ is a nonlinear mapping and $\varphi : C \to \mathbb{R}$ is a function and $\Theta : C \times C \to \mathbb{R}$ is a bifunction. The set of solutions of problem (1.1) is denoted by GMEP.

In the case of $F = 0$, problem (1.1) reduces to the mixed equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C,$$

which was considered by Ceng and Yao [2]. GMEP is denoted by MEP.

In the case of $\varphi = 0$, problem (1.1) reduces to the generalized equilibrium problem of finding $x \in C$ such that

$$\Theta(x,y) + \langle Fx, y - x\rangle \geq 0, \quad \forall y \in C,$$

which was studied by Takahashi and Takahashi [3] and many others, for example, [4–10].

In the case of $\varphi = 0$ and $F = 0$, problem (1.1) reduces to the equilibrium problem of finding $x \in C$ such that

$$\Theta(x,y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $\mathrm{EP}(\Theta)$.

In the case $\Theta = 0$, $\varphi = 0$ and $F = A$, problem (1.1) reduces to the classical variational inequality problem of finding $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of problem (1.3) is denoted by $\mathrm{VI}(A, C)$.

Problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, for instance, [2, 3, 11]. Peng and Yao [1] considered iterative methods for finding a common element of the set of solutions of problem (1.1), the set of solutions of problem (1.2) and the set of fixed points of a nonexpansive mapping.

Let $G_1, G_2 : C \times C \to \mathbb{R}$ be two bifunctions and let $B_1, B_2 : C \to H$ be two nonlinear mappings. We consider the generalized equilibrium problem $(\overline{x}, \overline{y}) \in C \times C$ such that

$$\begin{cases} G_1(\overline{x}, x) + \langle B_1\overline{y}, x - \overline{x}\rangle + \frac{1}{\mu_1}\langle \overline{x} - \overline{y}, x - \overline{x}\rangle \geq 0, & \forall x \in C, \\ G_2(\overline{y}, y) + \langle B_2\overline{x}, y - \overline{y}\rangle + \frac{1}{\mu_2}\langle \overline{y} - \overline{x}, y - \overline{y}\rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.4}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants.

In the case $G_1 = G_2 = 0$, problem (1.4) reduces to the general system of variational inequalities of finding $(\overline{x}, \overline{y}) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1\overline{y} + \overline{x} - \overline{y}, x - \overline{x}\rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2\overline{x} + \overline{y} - \overline{x}, y - \overline{y}\rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.5}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants, which was considered by Ceng, Wang and Yao [12]. In particular, if $B_1 = B_2 = A$, then problem (1.5) reduces to the system of variational inequalities of finding $(\overline{x}, \overline{y}) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 A\overline{y} + \overline{x} - \overline{y}, x - \overline{x}\rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 A\overline{x} + \overline{y} - \overline{x}, y - \overline{y}\rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.6}$$

which was studied by Verma [13].

If $\bar{x} = \bar{y}$ in (1.6), then (1.6) reduces to the classical variational inequality (1.3). Further, problem (1.6) is equivalent to the following projection formulas:

$$\begin{cases} \bar{x} = P_C(I - \mu_1 A)\bar{y}, \\ \bar{y} = P_C(I - \mu_2 A)\bar{x}. \end{cases}$$

Recently, Ceng *et al.* [12] introduced and studied a relaxed extragradient method for finding solutions of problem (1.5).

Let $\{T_i\}$ be an infinite family of nonexpansive mappings of C into itself and $\{\lambda_{n1}\}, \{\lambda_{n2}\}, \ldots, \{\lambda_{nN}\}$ be real sequences such that $\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nN} \in (0,1]$ for every $n \in N$. For any $n \geq 1$, we define a mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n0} &= I, \\ U_{n1} &= \lambda_{n1} T_1 U_{n0} + (1 - \lambda_{n1}) I, \\ U_{n2} &= \lambda_{n2} T_2 U_{n1} + (1 - \lambda_{n2}) I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1}) I, \\ W_n = U_{n,N} &= \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N}) I. \end{aligned}$$

Such a mapping W_n is called the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\{\lambda_{n1}\}, \{\lambda_{n2}\}, \ldots, \{\lambda_{nN}\}$. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. Moreover, in [1], it is shown that $F(W_n) = \bigcap_{i=1}^N F(T_i)$.

Throughout this article, let us assume that a bifunction $\Theta : C \times C \to \mathbb{R}$ and a convex function $\varphi : C \to \mathbb{R}$ satisfy the following conditions:

(H1) $\Theta(x,x) = 0$ for all $x \in C$;

(H2) Θ is monotone, *i.e.*, $\Theta(x,y) + \Theta(y,x) \leq 0$ for all $x, y \in C$;

(H3) for each $y \in C$, $x \mapsto \Theta(x,y)$ is weakly upper semicontinuous;

(H4) for each $x \in C$, $y \mapsto \Theta(x,y)$ is convex and lower semicontinuous;

(A1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(A2) C is a bounded set.

Recently, Qin *et al.* [8] studied the problem of finding a common element of the set of common fixed points of a finite family of nonexpansive mappings, the set of solutions of variational inequalities for a relaxed cocoercive mapping and the set of solutions of an equilibrium problem. More precisely, they proved the following theorem.

Theorem 1.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (H1)-(H4). Let $T_1, T_2, \ldots, T_N$ be a finite family of nonexpansive mappings of C into H and let B be a μ-Lipschitz, relaxed (u,v)-cocoercive mapping of C into H such that $F = \bigcap_{i=1}^N F(T_i) \cap \mathrm{EP}(\Theta) \cap \mathrm{VI}(A,C) \neq \phi$. Let f be a contraction* *of H into itself with a coefficient α $(0 < \alpha < 1)$ and let A be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} \Theta(y_n, \eta) + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \geq 0, & \forall \eta \in C, \\ x_{n+1} = \alpha_n \gamma f(W_n x_n) + (1 - \alpha_n A) W_n P_C (I - s_n B) y_n, & n \geq 1, \end{cases}$$

where $\alpha_n \in (0,1]$ and $\{r_n\}, \{s_n\} \subset [0,\infty)$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ *and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;
(iii) $\liminf_{n\to\infty} r_n > 0$;
(iv) $\{s_n\} \subset [a,b]$ *for some* a, b *with* $0 \leq a \leq b \leq \frac{2(\nu - u\mu^2)}{\mu^2}$, $\nu \geq u\mu^2$;
(v) $\sum_{n=0}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty$ *for all* $i = 1, 2, \ldots, N$.

Then, both $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in F$, where $q = P_F(\gamma f + (I - A))(q)$, which solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F.$$

In this paper, motivated by Takahashi and Takahashi [3], Ceng, Wang and Yao [12], Peng and Yao [1] and Qin, Shang and Su [8], we introduce the general iterative scheme for finding a common element of the set of common fixed points of a finite family of nonexpansive mappings, the set of solutions of the generalized mixed equilibrium problem (1.1) and the set of solutions of the generalized equilibrium problem (1.4), which solves the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathfrak{F},$$

where $\mathfrak{F} = \bigcap_{i=1}^{N} F(T_i) \cap \mathrm{GMEP} \cap \Omega$ and Ω is the set of solutions of the generalized equilibrium problem (1.4). The results obtained in this paper improve and extend the recent results announced by Qin *et al.* [8], Chen *et al.* [14], Combetters and Hirstoaga [15], Iiduka and Takahashi [16], Marino and Xu [17], Takahashi and Takahashi [18], Wittmann [19] and many others.

2 Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C, *i.e.*,

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Further, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

It is also known that H satisfies Opial's condition [20] if for each sequence $\{x_n\}_{n=1}^{\infty}$ in H which converges weakly to a point $x \in H$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in H, y \neq x.$$

Moreover, we assume that A is a bounded strongly positive operator on H with a constant $\overline{\gamma}$, that is, there exists $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

A mapping $B : C \to H$ is called β-inverse strongly monotone if there exists $\beta > 0$ such that

$$\langle x - y, Bx - By\rangle \geq \beta\|Bx - By\|^2, \quad \forall x, y \in C.$$

It is obvious that any inverse strongly monotone mapping is Lipschitz continuous.

In order to prove our main results in the next section, we need the following lemmas and proposition.

Lemma 2.1 [2] *Let C be a nonempty closed convex subset of H. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (H1)-(H4) *and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping*

$$T_r^{(\Theta,\varphi)}(x) = \left\{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. Assume that either (A1) *or* (A2) *holds. Then the following results hold*:

(i) $T_r^{(\Theta,\varphi)}(x) \neq \phi$ *for each $x \in H$ and $T_r^{(\Theta,\varphi)}$ is single-valued*;

(ii) $T_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\left\| T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\right\|^2 \leq \left\langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\right\rangle;$$

(iii) $F(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;

(iv) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex.*

Remark 2.1 If $\varphi = 0$, then $T_r^{(\Theta,\varphi)}$ is rewritten as T_r^{Θ}.

By a similar argument as that in the proof of Lemma 2.1 in [12], we have the following result.

Lemma 2.2 *Let C be a nonempty closed convex subset of H. Let $G_1, G_2 : C \times C \to \mathbb{R}$ be two bifunctions satisfying conditions* (H1)-(H4) *and let the mappings $B_1, B_2 : C \to H$ be β_1-inverse strongly monotone and β_2-inverse strongly monotone, respectively. Then, for given $\overline{x}, \overline{y} \in C$, $(\overline{x}, \overline{y})$ is a solution of* (1.4) *if and only if $\overline{x}$ is a fixed point of the mapping $\Gamma : C \to C$ defined by*

$$\Gamma(x) = T_{\mu_1}^{G_1}\left[T_{\mu_2}^{G_2}(x - \mu_2 B_2 x) - \mu_1 B_1 T_{\mu_2}^{G_2}(x - \mu_2 B_2 x)\right], \quad \forall x \in C,$$

where $\overline{y} = T_{\mu_2}^{G_2}(\overline{x} - \mu_2 B_2 \overline{x})$.

The set of fixed points of the mapping Γ is denoted by Ω.

Proposition 2.1 [3] *Let C, H, Θ, φ and $T_r^{(\Theta,\varphi)}$ be as in Lemma* 2.1. *Then the following holds*:

$$\left\| T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x \right\|^2 \le \frac{s-t}{s}\left\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\rho)}x, T_s^{(\Theta,\varphi)}x - x\right\rangle$$

for all $s, t > 0$ and $x \in H$.

Lemma 2.3 [21] *Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of H. If T has a fixed point, then $I - T$ is demi-closed*; *that is, when $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ and the sequence $\{(I - T)x_n\}$ converges strongly to some y, it follows that $(I - T)x = y$.*

Lemma 2.4 [22] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \quad \forall n \ge 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 [17] *Assume A is a strong positive linear bounded operator on a Hilbert space H with a coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.*

The following lemma is an immediate consequence of an inner product.

Lemma 2.6 *In a real Hilbert space H, the following inequality holds*:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$$

for all $x, y \in H$.

3 Main results

Now we state and prove our main results.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta, G_1, G_2 : C \times C \to \mathbb{R}$ be three bifunctions which satisfy assumptions* (H1)-(H4) *and $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function with assumption* (A1) *or* (A2). *Let the mappings $F, B_1, B_2 : C \to H$ be ζ-inverse strongly monotone, β_1-inverse strongly monotone and β_2-inverse strongly monotone, respectively. Let $T_1, T_2, \ldots, T_N$ be a finite family of nonexpansive mappings of C into H such that $\mathfrak{F} = \bigcap_{i=1}^N F(T_i) \cap \mathrm{GMEP} \cap \Omega \neq \phi$. Let f be a contraction of C into itself with a constant α $(0 < \alpha < 1)$ and let A be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$ such that $\|A\| \le 1$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} z_n = T_{\delta_n}^{(\Theta,\varphi)}(x_n - \delta_n F x_n), \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \alpha_n \gamma f(W_n x_n) + (1 - \alpha_n A) W_n y_n, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where $\alpha_n \in [0,1]$, $\mu_1 \in (0,2\beta_1)$, $\mu_2 \in (0,2\beta_2)$ *and* $\{\delta_n\} \subset [0,2\zeta]$ *satisfy the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 2\zeta$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;

(iii) $\lim_{n\to\infty} \lambda_{n,i} = 0$ *and* $\sum_{n=1}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty$ *for all* $i = 1,2,\dots,N$.

Then $\{x_n\}$ *converges strongly to* $x^* = P_{\mathfrak{F}}(\gamma f + (I - A))(x^*)$, *which solves the following variational inequality*:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathfrak{F},$$

and (x^*, y^*) *is a solution of problem* (1.4), *where* $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 B_2 x^*)$.

Proof We divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, take $p \in \mathfrak{F} = \bigcap_{i=1}^{N} F(T_i) \cap \mathrm{GMEP} \cap \Omega \neq \phi$ arbitrarily. Since $p = T_{\delta_n}^{(\Theta,\varphi)}(p - \delta_n Fp)$, F is ζ-inverse strongly monotone and $0 \leq \delta_n \leq 2\zeta$, we obtain that for any $n \geq 1$,

$$\begin{aligned}
\|z_n - p\|^2 &= \left\| T_{\delta_n}^{(\Theta,\varphi)}(x_n - \delta_n Fx_n) - T_{\delta_n}^{(\Theta,\varphi)}(p - \delta_n Fp) \right\|^2 \\
&\leq \left\| (x_n - p) - \delta_n(Fx_n - Fp) \right\|^2 \\
&= \|x_n - p\|^2 - 2\delta_n \langle x_n - p, Fx_n - Fp \rangle + \delta_n^2 \|Fx_n - Fp\|^2 \\
&\leq \|x_n - p\|^2 - 2\delta_n \zeta \|Fx_n - Fp\|^2 + \delta_n^2 \|Fx_n - Fp\|^2 \\
&= \|x_n - p\|^2 + \delta_n(\delta_n - 2\zeta) \|Fx_n - Fp\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned} \tag{3.2}$$

Putting $u_n = T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n)$ and $u = T_{\mu_2}^{G_2}(p - \mu_2 B_2 p)$, we have

$$\begin{aligned}
\|u_n - u\|^2 &= \left\| T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - T_{\mu_2}^{G_2}(p - \mu_2 B_2 p) \right\|^2 \\
&\leq \left\| (z_n - p) - \mu_2(B_2 z_n - B_2 p) \right\|^2 \\
&= \|z_n - p\|^2 - 2\mu_2 \langle z_n - p, B_2 z_n - B_2 p \rangle + \mu_2^2 \|B_2 z_n - B_2 p\|^2 \\
&\leq \|z_n - p\|^2 - 2\mu_2 \beta_2 \|B_2 z_n - B_2 p\|^2 + \mu_2^2 \|B_2 z_n - B_2 p\|^2 \\
&= \|z_n - p\|^2 + \mu_2(\mu_2 - 2\beta_2) \|B_2 z_n - B_2 p\|^2 \\
&\leq \|z_n - p\|^2.
\end{aligned} \tag{3.3}$$

And since $p = T_{\mu_1}^{G_1}(u - \mu_1 B_1 u)$, we know that for any $n \geq 1$,

$$\begin{aligned}
\|y_n - p\|^2 &= \left\| T_{\mu_1}^{G_1}(u_n - \mu_1 B_1 u_n) - T_{\mu_1}^{G_1}(u - \mu_1 B_1 u) \right\|^2 \\
&\leq \left\| (u_n - u) - \mu_1(B_1 u_n - B_1 u) \right\|^2 \\
&\leq \|u_n - u\|^2 - 2\mu_1 \langle u_n - u, B_1 u_n - B_1 u \rangle + \mu_1^2 \|B_1 u_n - B_1 u\|^2 \\
&\leq \|u_n - u\|^2 - 2\mu_1 \beta_1 \|B_1 u_n - B_1 u\| + \mu_1 \|B_1 u_n - b_1 u\|^2 \\
&\leq \|u_n - u\|^2 + \mu_1(\mu_1 - 2\beta_1) \|B_1 u_n - B_1 u\|
\end{aligned}$$

$$\begin{aligned}
&\le \|u_n - u\|^2 \\
&\le \|z_n - p\|^2. \qquad (3.4)
\end{aligned}$$

Furthermore, from (3.1), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n \gamma f(W_n x_n) + (1 - \alpha_n A) W_n y_n - p\| \\
&= \|\alpha_n (\gamma f(W_n x_n) - Ap) + (1 - \alpha_n A)(W_n y_n - p)\| \\
&\le \alpha_n \|\gamma f(W_n x_n) - Ap\| + \|I - \alpha_n A\| \|W_n y_n - p\| \\
&\le \alpha_n (\gamma \|f(W_n x_n) - f(p)\| + \|\gamma f(p) - Ap\|) + (1 - \alpha_n \overline{\gamma}) \|y_n - p\| \\
&\le \alpha_n (\gamma \alpha \|x_n - p\| + \|\gamma f(p) - Ap\|) + (1 - \alpha_n \overline{\gamma}) \|x_n - p\| \\
&\le [1 - (\overline{\gamma} - \gamma\alpha)\alpha_n] \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|.
\end{aligned}$$

By induction, we obtain that for all $n \ge 1$,

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{1}{\overline{\gamma} - \gamma\alpha} \|\gamma f(p) - Ap\|\right\}.$$

Hence $\{x_n\}$ is bounded. Consequently, we deduce immediately that $\{z_n\}$, $\{y_n\}$, $\{f(W_n x_n)\}$ and $\{W_n(y_n)\}$ are bounded.

Step 2. $\lim_{n\to\infty} \|W_{n+1} y_n - W_n y_n\| = 0$.

It follows from the definition of W_n that

$$\begin{aligned}
&\|W_{n+1} y_n - W_n y_n\| \\
&\quad = \|\lambda_{n+1,N} T_N U_{n+1,N-1} y_n - (1 - \lambda_{n+1,N}) y_n - \lambda_{n,N} T_N U_{n,N-1} y_n - (1 - \lambda_{n,N}) y_n\| \\
&\quad \le |\lambda_{n+1,N} - \lambda_{n,N}| \|y_n\| + \|\lambda_{n+1,N} T_N U_{n+1,N-1} y_n - \lambda_{n,N} T_N U_{n,N-1} y_n\| \\
&\quad \le |\lambda_{n+1,N} - \lambda_{n,N}| \|y_n\| + \|\lambda_{n+1,N} (T_N U_{n+1,N-1} y_n - T_N U_{n,N-1} y_n)\| \\
&\qquad + |\lambda_{n+1,N} - \lambda_{n,N}| \|T_N U_{n,N-1} y_n\| \\
&\quad \le |\lambda_{n+1,N} - \lambda_{n,N}| \|y_n\| + \lambda_{n+1,N} \|U_{n+1,N-1} y_n - U_{n,N-1} y_n\| \\
&\qquad + |\lambda_{n+1,N} - \lambda_{n,N}| \|T_N U_{n,N-1} y_n\|.
\end{aligned}$$

Since $\{y_n\}$ is bounded and T_k, $U_{n,k}$ are nonexpansive, $\lim_{n\to\infty} \|W_{n+1} y_n - W_n y_n\| = 0$.

Step 3. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

We estimate $\|y_{n+1} - y_n\|$, $\|W_{n+1} x_{n+1} - W_n x_n\|$ and $\|W_{n+1} y_{n+1} - W_n y_n\|$. From (3.1) we have

$$\begin{aligned}
\|y_{n+1} - y_n\|^2 &= \|T_{\mu_1}^{G_1}(u_{n+1} - \mu_1 B_1 u_{n+1}) - T_{\mu_1}^{G_1}(u_n - \mu_1 B_1 u_n)\|^2 \\
&\le \|(u_{n+1} - u_n) - \mu_1 (B_1 u_{n+1} - B_1 u_n)\|^2 \\
&\le \|u_{n+1} - u_n\|^2 + \mu_1(\mu_1 - 2\beta_1) \|B_1 u_{n+1} - B_1 u_n\|^2 \\
&\le \|u_{n+1} - u_n\|^2 \\
&= \|T_{\mu_2}^{G_2}(z_{n+1} - \mu_2 B_2 z_{n+1}) - T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n)\|^2 \\
&\le \|(z_{n+1} - z_n) - \mu_2 (B_2 z_{n+1} - B_2 z_n)\|^2
\end{aligned}$$

$$\begin{aligned}
&\le \|z_{n+1}-z_n\|^2+\mu_2(\mu_2-2\beta_2)\|B_2z_{n+1}-B_2z_n\|^2 \\
&\le \|z_{n+1}-z_n\|^2, \qquad (3.5)
\end{aligned}$$

$$\begin{aligned}
&\bigl\|(x_{n+1}-\delta_{n+1}Fx_{n+1})-(x_n-\delta_nFx_n)\bigr\| \\
&\quad\le \bigl\|x_{n+1}-x_n-\delta_{n+1}(Fx_{n+1}-Fx_n)\bigr\|+|\delta_n-\delta_{n+1}|\|Fx_n\| \\
&\quad\le \|x_{n+1}-x_n\|+|\delta_n-\delta_{n+1}|\|Fx_n\|
\end{aligned}$$

and

$$\begin{aligned}
&\|z_{n+1}-z_n\| \\
&\quad= \bigl\|T^{(\Theta,\varphi)}_{\delta_{n+1}}(x_{n+1}-\delta_{n+1}Fx_{n+1})-T^{(\Theta,\varphi)}_{\delta_n}(x_n-\delta_nFx_n)\bigr\| \\
&\quad\le \bigl\|T^{(\Theta,\varphi)}_{\delta_{n+1}}(x_{n+1}-\delta_{n+1}Fx_{n+1})-T^{(\Theta,\varphi)}_{\delta_{n+1}}(x_n-\delta_nFx_n)\bigr\| \\
&\qquad+\bigl\|T^{(\Theta,\varphi)}_{\delta_{n+1}}(x_n-\delta_nFx_n)-T^{(\Theta,\varphi)}_{\delta_n}(x_n-\delta_nFx_n)\bigr\| \\
&\quad\le \bigl\|(x_{n+1}-\delta_{n+1}Fx_{n+1})-(x_n-\delta_nFx_n)\bigr\| \\
&\qquad+\bigl\|T^{(\Theta,\varphi)}_{\delta_{n+1}}(x_n-\delta_nFx_n)-T^{(\Theta,\varphi)}_{\delta_n}(x_n-\delta_nFx_n)\bigr\| \\
&\quad\le \|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|\|Fx_n\| \\
&\qquad+\bigl\|T^{(\Theta,\varphi)}_{\delta_{n+1}}(x_n-\delta_nFx_n)-T^{(\Theta,\varphi)}_{\delta_n}(x_n-\delta_nFx_n)\bigr\|. \qquad (3.6)
\end{aligned}$$

It follows from (3.5) and (3.6) that

$$\begin{aligned}
\|y_{n+1}-y_n\| &\le \|z_{n+1}-z_n\| \\
&\le \|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|\|Fx_n\| \\
&\quad+\bigl\|T^{(\Theta,\varphi)}_{\delta_{n+1}}(x_n-\delta_nFx_n)-T^{(\Theta,\varphi)}_{\delta_n}(x_n-\delta_nFx_n)\bigr\|. \qquad (3.7)
\end{aligned}$$

Without loss of generality, let us assume that there exists a real number a such that $\delta_n > a > 0$ for all n. Utilizing Proposition 2.1, we have

$$\begin{aligned}
&\bigl\|T^{(\Theta,\varphi)}_{\delta_{n+1}}(x_n-\delta_nFx_n)-T^{(\Theta,\varphi)}_{\delta_n}(x_n-\delta_nFx_n)\bigr\| \\
&\quad\le \frac{|\delta_{n+1}-\delta_n|}{\delta_{n+1}}\bigl\|T^{(\Theta,\varphi)}_{\delta_{n+1}}(I-\delta_nF)x_n\bigr\| \\
&\quad\le \frac{|\delta_{n+1}-\delta_n|}{a}\bigl\|T^{(\Theta,\varphi)}_{\delta_{n+1}}(I-\delta_nF)x_n\bigr\|. \qquad (3.8)
\end{aligned}$$

It follows from the definition of W_n that

$$\begin{aligned}
&\|W_{n+1}y_n-W_ny_n\| \\
&\quad= \bigl\|\lambda_{n+1,N}T_NU_{n+1,N-1}y_n+(1-\lambda_{n+1,N})y_n-\lambda_{n,N}T_NU_{n,N-1}y_n-(1-\lambda_{n,N})y_n\bigr\| \\
&\quad\le |\lambda_{n+1,N}-\lambda_{n,N}|\|y_n\|+\|\lambda_{n+1,N}T_NU_{n+1,N-1}y_n-\lambda_{n,N}T_NU_{n,N-1}y_n\| \\
&\quad\le |\lambda_{n+1,N}-\lambda_{n,N}|\|y_n\|+\bigl\|\lambda_{n+1,N}(T_NU_{n+1,N-1}y_n-T_NU_{n,N-1}y_n)\bigr\| \\
&\qquad+|\lambda_{n+1,N}-\lambda_{n,N}|\|T_NU_{n,N-1}y_n\|
\end{aligned}$$

$$\begin{aligned}
&\le |\lambda_{n+1,N} - \lambda_{n,N}| \|y_n\| + \lambda_{n+1,N} \|U_{n+1,N-1}y_n - U_{n,N-1}y_n\| \\
&\quad + |\lambda_{n+1,N} - \lambda_{n,N}| \|T_N U_{n,N-1}y_n\| \\
&\le M_1 |\lambda_{n+1,N} - \lambda_{n,N}| + \lambda_{n+1,N} \|U_{n+1,N-1}y_n - U_{n,N-1}y_n\|,
\end{aligned} \tag{3.9}$$

where M_1 is a constant such that $M_1 \ge 2\max\{\sup_{n\ge1}\|y_n\|, \sup_{n\ge1}\|T_N U_{n,N-1}y_n\|\}$. Next, we consider

$$\begin{aligned}
&\|U_{n+1,N-1}y_n - U_{n,N-1}y_n\| \\
&\quad = \big\|\lambda_{n+1,N-1} T_{N-1} U_{n+1,N-2}y_n + (1 - \lambda_{n+1,N-1})y_n \\
&\qquad - \lambda_{n,N-1} T_{N-1} U_{n,N-2}y_n - (1 - \lambda_{n,N-1})y_n\big\| \\
&\quad \le |\lambda_{n+1,N-1} - \lambda_{n,N-1}| \|y_n\| + \|\lambda_{n+1,N-1} T_{N-1} U_{n+1,N-2}y_n - \lambda_{n,N-1} T_{N-1} U_{n,N-2}y_n\| \\
&\quad \le |\lambda_{n+1,N-1} - \lambda_{n,N-1}| \|y_n\| + \lambda_{n+1,N-1} \|T_{N-1} U_{n+1,N-2}y_n - T_{N-1} U_{n,N-2}y_n\| \\
&\qquad + |\lambda_{n+1,N-1} - \lambda_{n,N-1}| \|T_{N-1} U_{n,N-2}y_n\| \\
&\quad \le M_2 |\lambda_{n+1,N-1} - \lambda_{n,N-1}| + \|U_{n+1,N-2}y_n - U_{n,N-2}y_n\|,
\end{aligned}$$

where M_2 is a constant that $M_2 \ge 2\max\{\sup_{n\ge1}\|y_n\|, \sup_{n\ge1}\|T_{N-1} U_{n,N-2}y_n\|\}$. In a similar way, we obtain

$$\|U_{n+1,N-1}y_n - U_{n,N-1}y_n\| \le M_3 \sum_{i=1}^{N-1} |\lambda_{n+1,i} - \lambda_{n,i}|, \tag{3.10}$$

where M_3 is an appropriate constant. Substituting (3.10) into (3.9), we have that

$$\begin{aligned}
\|W_{n+1}y_n - W_n y_n\| &\le M_1 |\lambda_{n+1,N} - \lambda_{n,N}| + \lambda_{n+1,N} M_3 \sum_{i=1}^{N-1} |\lambda_{n+1,i} - \lambda_{n,i}| \\
&\le M_4 \sum_{i=1}^{N} |\lambda_{n+1,i} - \lambda_{n,i}|,
\end{aligned} \tag{3.11}$$

where M_4 is a constant such that $M_4 \ge \max\{M_1, M_3\}$. Similarly, we have

$$\|W_{n+1}x_n - W_n x_n\| \le M_5 \sum_{i=1}^{N} |\lambda_{n+1,i} - \lambda_{n,i}|, \tag{3.12}$$

where M_5 is an appropriate constant. Hence it follows from (3.1), (3.7), (3.8), (3.11) and (3.12) that

$$\begin{aligned}
&\|x_{n+2} - x_{n+1}\| \\
&\quad = \big\|(I - \alpha_{n+1}A)(W_{n+1}y_{n+1} - W_n y_n) - (\alpha_{n+1} - \alpha_n)AW_n y_n \\
&\qquad + \gamma\big[\alpha_{n+1}\big(f(W_{n+1}x_{n+1}) - f(W_n x_n)\big) + f(W_n x_n)(\alpha_{n+1} - \alpha_n)\big]\big\| \\
&\quad \le (1 - \alpha_{n+1}\overline{\gamma})\big(\|W_{n+1}y_{n+1} - W_{n+1}y_n\| + \|W_{n+1}y_n - W_n y_n\|\big) \\
&\qquad + |\alpha_{n+1} - \alpha_n| \|AW_n y_n\|
\end{aligned}$$

$$
\begin{aligned}
&\quad + \gamma\big[\alpha_{n+1}\big\|f(W_{n+1}x_{n+1}) - f(W_nx_n)\big\| + f(W_nx_n)(\alpha_{n+1} - \alpha_n)\big] \\
&\le (1 - \alpha_{n+1}\overline{\gamma})\big(\|y_{n+1} - y_n\| + \|W_{n+1}y_n - W_ny_n\|\big) \\
&\quad + |\alpha_{n+1} - \alpha_n|\|AW_ny_n\| \\
&\quad + \gamma\big[\alpha_{n+1}\alpha\big(\|x_{n+1} - x_n\| + \|W_{n+1}x_n - W_nx_n\|\big) + |\alpha_{n+1} - \alpha_n|\big\|f(W_nx_n)\big\|\big] \\
&\le (1 - \alpha_{n+1}\overline{\gamma})\Bigg(\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|Fx_n\| \\
&\quad + \big\|T_{\delta_{n+1}}^{(\Theta,\varphi)}(x_n - \delta_nFx_n) - T_{\delta_n}^{(\Theta,\varphi)}(x_n - \delta_nFx_n)\big\| + M_4\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|\Bigg) \\
&\quad + |\alpha_{n+1} - \alpha_n|\|AW_ny_n\| \\
&\quad + \gamma\alpha\alpha_{n+1}\|x_{n+1} - x_n\| + \gamma\alpha\alpha_{n+1}\|W_{n+1}x_n - W_nx_n\| \\
&\quad + \gamma|\alpha_{n+1} - \alpha_n|\big\|f(W_nx_n)\big\| \\
&\le \big[1 - \alpha_{n+1}(\overline{\gamma} - \gamma\alpha)\big]\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|Fx_n\| \\
&\quad + \frac{|\delta_{n+1} - \delta_n|}{a}\big\|T_{\delta_{n+1}}^{(\Theta,\varphi)}(I - \delta_nF)x_n\big\| + M_4\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}| \\
&\quad + |\alpha_{n+1} - \alpha_n|\|AW_ny_n\| + \gamma\alpha\alpha_{n+1}M_5\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}| \\
&\quad + \gamma|\alpha_{n+1} - \alpha_n|\big\|f(W_nx_n)\big\| \\
&\le \big[1 - \alpha_{n+1}(\overline{\gamma} - \gamma\alpha)\big]\|x_{n+1} - x_n\| \\
&\quad + M_6\Bigg[2|\delta_{n+1} - \delta_n| + \sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}| + (1 + \gamma)|\alpha_{n+1} - \alpha_n|\Bigg],
\end{aligned}
$$

where M_6 is a constant such that $M_6 \ge \max\{\sup_{n\ge1}\|Fx_n\|, \frac{1}{a}\sup_{n\ge1}\|T_{\delta_{n+1}}^{(\Theta,\varphi)}(I - \delta_nF)x_n\|, M_4 + \gamma M_5, \sup_{n\ge1}\|AW_ny_n\|, \sup_{n\ge1}\|f(W_nx_n)\|\}$. By Lemma 2.4, we get $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 4. $\lim_{n\to\infty}\|Fx_n - Fp\| = 0$, $\lim_{n\to\infty}\|B_1u_n - B_1u\| = 0$ and $\lim_{n\to\infty}\|B_2z_n - B_2p\| = 0$.

Indeed, from (3.1)-(3.4) we get

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \big\|\alpha_n\big(\gamma f(W_nx_n) - Ap\big) + (I - \alpha_nA)(W_ny_n - p)\big\|^2 \\
&\quad \le \big(\alpha_n\big\|\gamma f(W_nx_n) - Ap\big\| + (1 - \alpha_n\overline{\gamma})\|y_n - p\|\big)^2 \\
&\quad \le \alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|^2 + (1 - \alpha_n\overline{\gamma})\|y_n - p\|^2 \\
&\qquad + 2\alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|\|y_n - p\| \\
&\quad \le \alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|^2 + (1 - \alpha_n\overline{\gamma})\big[\|u_n - u\|^2 + \mu_1(\mu_1 - 2\beta_1)\|B_1u_n - B_1u\|^2\big] \\
&\qquad + 2\alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|\|y_n - p\| \\
&\quad \le \alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|^2 + (1 - \alpha_n\overline{\gamma})\big[\|z_n - p\|^2 + \mu_2(\mu_2 - 2\beta_2)\|B_2z_n - B_2p\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \mu_1(\mu_1 - 2\beta_1)\|B_1u_n - B_1u\|^2\big] + 2\alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|\|y_n - p\| \\
&\le \alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|^2 + \|x_n - p\|^2 + \delta_n(\delta_n - 2\zeta)\|F_n - Fp\|^2 \\
&\quad + \mu_2(\mu_2 - 2\beta_2)\|B_2z_n - B_2p\|^2 + \mu_1(\mu_1 - 2\beta_1)\|B_1u_n - B_1u\|^2 \\
&\quad + 2\alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|\|y_n - p\|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\delta_n(2\zeta - \delta_n)\|Fx_n - Fp\|^2 + \mu_2(2\beta_2 - \mu_2)\|B_2z_n - B_2p\|^2 + \mu_1(2\beta_1 - \mu_1)\|B_1u_n - B_1u\|^2 \\
&\quad \le \alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|^2 + \|x_n - p\| - \|x_{n+1} - p\|^2 + 2\alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|\|y_n - p\| \\
&\quad \le \alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|^2 + \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\| \\
&\qquad + 2\alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|\|y_n - p\|.
\end{aligned}
$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty}\|Fx_n - Fp\| = 0$, $\lim_{n\to\infty}\|B_1u_n - B_1u\| = 0$ and $\lim_{n\to\infty}\|B_2z_n - B_2p\| = 0$.

Step 5. $\lim_{n\to\infty}\|x_n - z_n\| = 0$, $\lim_{n\to\infty}\|z_n - y_n\| = 0$ and $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Indeed, from (3.2), (3.3) and Lemma 2.1, we have

$$
\begin{aligned}
&\|u_n - u\|^2 \\
&\quad = \big\|T_{\mu_2}^{G_2}(z_n - \mu_2B_2z_n) - T_{\mu_2}^{G_2}(p - \mu_2B_2p)\big\|^2 \\
&\quad \le \big\langle (z_n - \mu_2B_2z_n) - (p - \mu_2B_2p), u_n - u\big\rangle \\
&\quad = \frac{1}{2}\big[\big\|(z_n - \mu_2B_2z_n) - (p - \mu_2B_2p)\big\|^2 + \|u_n - u\|^2 \\
&\qquad - \big\|(z_n - \mu_2B_2z_n) - (p - \mu_2B_2p) - (u_n - u)\big\|^2\big] \\
&\quad \le \frac{1}{2}\big[\|z_n - p\|^2 + \|u_n - u\|^2 - \big\|(z_n - u_n) - \mu_2(B_2z_n - B_2p) - (p - u)\big\|^2\big] \\
&\quad \le \frac{1}{2}\big[\|x_n - p\|^2 + \|u_n - u\|^2 - \big\|(z_n - u_n) - (p - u)\big\|^2 \\
&\qquad + 2\mu_2\big\langle (z_n - u_n) - (p - u), B_2z_n - B_2p\big\rangle - \mu_2^2\|B_2z_n - B_2p\|^2\big]
\end{aligned}
$$

and

$$
\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \big\|T_{\mu_1}^{G_1}(u_n - \mu_1B_1u_n) - T_{\mu_1}^{G_1}(u - \mu_1B_1u)\big\|^2 \\
&\quad \le \big\langle (u_n - \mu_1B_1u_n) - (u - \mu_1B_1u), y_n - p\big\rangle \\
&\quad = \frac{1}{2}\big[\big\|(u_n - \mu_1B_1u_n) - (u - \mu_1B_1u)\big\|^2 + \|y_n - p\|^2 \\
&\qquad - \big\|(u_n - \mu_1B_1u_n) - (u - \mu_1B_1u) - (y_n - p)\big\|^2\big] \\
&\quad \le \frac{1}{2}\big[\|u_n - u\|^2 + \|y_n - p\|^2 - \big\|(u_n - y_n) + (p - u)\big\|^2 \\
&\qquad + 2\mu_1\big\langle B_1u_n - B_1u, (u_n - y_n) + (p - u)\big\rangle - \mu_1^2\|B_1u_n - B_1u\|^2\big]
\end{aligned}
$$

$$\begin{aligned}&\leq \frac{1}{2}\big[\|x_n - p\|^2 + \|y_n - p\|^2 - \big\|(u_n - y_n) + (p - u)\big\|^2 \\ &\quad + 2\mu_1\big\langle B_1u_n - B_1u, (u_n - y_n) + (p - u)\big\rangle\big],\end{aligned}$$

which imply that

$$\begin{aligned}\|u_n - u\|^2 &\leq \|x_n - p\|^2 - \big\|(z_n - u_n) - (p - u)\big\|^2 \\ &\quad + 2\mu_2\big\langle (z_n - u_n) - (p - u), B_2z_n - B_2p\big\rangle - \mu_2^2\|B_2z_n - B_2p\|^2 \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\|y_n - p\|^2 &\leq \|x_n - p\|^2 - \big\|(u_n - y_n) + (p - u)\big\|^2 \\ &\quad + 2\mu_1\|B_1u_n - B_1u\|\big\|(u_n - y_n) + (p - u)\big\|.\end{aligned} \tag{3.14}$$

It follows from (3.14) that

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\quad = \big\|\alpha_n\big(\gamma f(W_nx_n) - Ap\big) + (I - \alpha_nA)(W_ny_n - p)\big\|^2 \\ &\quad \leq \alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|^2 + (1 - \alpha_n\overline{\gamma})\|y_n - p\|^2 \\ &\qquad + 2\alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|\|y_n - p\| \\ &\quad \leq \alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|^2 + (1 - \alpha_n\overline{\gamma})\big[\|x_n - p\|^2 - \big\|(u_n - y_n) + (p - u)\big\|^2 \\ &\qquad + 2\mu_1\|B_1u_n - B_1u\|\big\|(u_n - y_n) + (p - u)\big\|\big] + 2\alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|\|y_n - p\|,\end{aligned}$$

which gives that

$$\begin{aligned}&(1 - \alpha_n\overline{\gamma})\big\|(u_n - y_n) + (p - u)\big\|^2 \\ &\quad \leq \alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\qquad + 2\mu_1(1 - \alpha_n\overline{\gamma})\|B_1u_n - B_1u\|\big\|(u_n - y_n) + (p - u)\big\| \\ &\qquad + 2\alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|\|y_n - p\| \\ &\quad \leq \alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|^2 + \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\| \\ &\qquad + 2\mu_1(1 - \alpha_n\overline{\gamma})\|B_1u_n - B_1u\|\big\|(u_n - y_n) + (p - u)\big\| \\ &\qquad + 2\alpha_n\big\|\gamma f(W_nx_n) - Ap\big\|\|y_n - p\|.\end{aligned}$$

Since $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|B_1u_n - B_1u\| \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\big\|(u_n - y_n) + (p - u)\big\|^2 = 0. \tag{3.15}$$

Also, from (3.4) and (3.13), we have

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\quad = \big\|\alpha_n\big(\gamma f(W_nx_n) - Ap\big) + (I - \alpha_nA)(W_ny_n - p)\big\|^2\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n \big\| \gamma f(W_n x_n) - Ap \big\|^2 + (1 - \alpha_n \overline{\gamma}) \|y_n - p\|^2 \\
&\quad + 2\alpha_n \big\| \gamma f(W_n x_n) - Ap \big\| \|y_n - p\| \\
&\le \alpha_n \big\| \gamma f(W_n x_n) - Ap \big\|^2 + (1 - \alpha_n \overline{\gamma}) \|u_n - u\|^2 \\
&\quad + 2\alpha_n \big\| \gamma f(W_n x_n) - Ap \big\| \|y_n - p\| \\
&\le \alpha_n \big\| \gamma f(W_n x_n) - Ap \big\|^2 + (1 - \alpha_n \overline{\gamma}) \big[\|x_n - p\|^2 - \big\| (z_n - u_n) - (p - u) \big\|^2 \\
&\quad + 2\mu_2 \big\langle (z_n - u_n) - (p - u), B_2 z_n - B_2 p \big\rangle \big] \\
&\quad + 2\alpha_n \big\| \gamma f(W_n x_n) - Ap \big\| \|y_n - p\|.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
&(1 - \alpha_n \overline{\gamma}) \big\| (z_n - u_n) - (p - u) \big\|^2 \\
&\quad \le \alpha_n \big\| \gamma f(W_n x_n) - Ap \big\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad + 2\mu_2 \big\| (z_n - u_n) - (p - u) \big\| \|B_2 z_n - B_2 p\| \\
&\qquad + 2\alpha_n \big\| \gamma f(W_n x_n) - Ap) \big\| \|y_n - p\| \\
&\quad \le \alpha_n \big\| \gamma f(W_n x_n) - Ap \big\|^2 + \big(\|x_n - p\| + \|x_{n+1} - p\| \big) \|x_n - x_{n+1}\| \\
&\qquad + 2\mu_2 \big\| (z_n - u_n) - (p - u) \big\| \|B_2 z_n - B_2 p\| \\
&\qquad + 2\alpha_n \big\| \gamma f(W_n x_n) - Ap \big\| \|y_n - p\|.
\end{aligned}
$$

Note that $\|B_2 z_n - B_2 p\| \to 0$ as $n \to \infty$. Then we have

$$
\lim_{n\to\infty} \big\| (z_n - u_n) - (p - u) \big\| = 0. \tag{3.16}
$$

In addition, from the firm nonexpansivity of $T_{\delta_n}^{(\Theta,\varphi)}$, we have

$$
\begin{aligned}
&\|z_n - p\|^2 \\
&\quad = \big\| T_{\delta_n}^{(\Theta,\varphi)}(x_n - \delta_n F x_n) - T_{\delta_n}^{(\Theta,\varphi)}(p - \delta_n F p) \big\|^2 \\
&\quad \le \big\langle (x_n - \delta_n F x_n) - (p - \delta_n F p), z_n - p \big\rangle \\
&\quad = \frac{1}{2} \Big[\big\| (x_n - \delta_n F x_n) - (p - \delta_n F p) \big\|^2 + \|z_n - p\|^2 \\
&\qquad - \big\| (x_n - \delta_n F x_n) - (p - \delta_n F p) - (z_n - p) \big\|^2 \Big] \\
&\quad \le \frac{1}{2} \Big[\|x_n - p\|^2 + \|z_n - p\|^2 - \big\| x_n - z_n - \delta_n (F x_n - F p) \big\|^2 \Big] \\
&\quad = \frac{1}{2} \Big[\|x_n - p\|^2 + \|z_n - p\|^2 - \|x_n - z_n\|^2 + 2\delta_n \langle F x_n - F p, x_n - z_n \rangle \\
&\qquad - \delta_n^2 \|F x_n - F p\|^2 \Big],
\end{aligned}
$$

which implies that

$$
\|z_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\delta_n \|F x_n - F p\| \|x_n - z_n\|. \tag{3.17}
$$

From (3.1), (3.4) and (3.17), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad = \bigl\|\alpha_n\bigl(\gamma f(W_nx_n)-Ap\bigr)+(I-\alpha_nA)(W_ny_n-p)\bigr\|^2 \\
&\quad \le \alpha_n\bigl\|\gamma f(W_nx_n)-Ap\bigr\|^2+(1-\alpha_n\overline{\gamma})\|y_n-p\|^2 \\
&\qquad + 2\alpha_n\bigl\|\gamma f(W_nx_n)-Ap\bigr\|\|y_n-p\| \\
&\quad \le \alpha_n\bigl\|\gamma f(W_nx_n)-Ap\bigr\|^2+(1-\alpha_n\overline{\gamma})\|z_n-p\|^2 \\
&\qquad + 2\alpha_n\bigl\|\gamma f(W_nx_n)-Ap\bigr\|\|y_n-p\| \\
&\quad \le \alpha_n\bigl\|\gamma f(W_nx_n)-Ap\bigr\|^2+(1-\alpha_n\overline{\gamma})\bigl[\|x_n-p\|^2-\|x_n-z_n\|^2 \\
&\qquad + 2\delta_n\|Fx_n-Fp\|\|x_n-z_n\|\bigr]+2\alpha_n\bigl\|\gamma f(W_nx_n)-Ap\bigr\|\|y_n-p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1-\alpha_n\overline{\gamma})\|x_n-z_n\|^2 \\
&\quad \le \alpha_n\bigl\|\gamma f(W_nx_n)-Ap\bigr\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\qquad + 2(1-\alpha_n\overline{\gamma})\delta_n\|Fx_n-Fp\|\|x_n-z_n\|+2\alpha_n\bigl\|\gamma f(W_nx_n)-Ap\bigr\|\|y_n-p\| \\
&\quad \le \alpha_n\bigl\|\gamma f(W_nx_n)-Ap\bigr\|^2+\bigl(\|x_n-p\|+\|x_{n+1}-p\|\bigr)\|x_n-x_{n+1}\| \\
&\qquad + 2(1-\alpha_n\overline{\gamma})\delta_n\|Fx_n-Fp\|\|x_n-z_n\|+2\alpha_n\bigl\|\gamma f(W_nx_n)-Ap\bigr\|\|y_n-p\|.
\end{aligned}$$

Since $\|Fx_n-Fp\|\to 0$ as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.18}$$

Thus, from (3.15), (3.16) and (3.18), we obtain that

$$\begin{aligned}
\lim_{n\to\infty}\|z_n-y_n\| &= \lim_{n\to\infty}\bigl\|(z_n-u_n)-(p-u)+(u_n-y_n)+(p-u)\bigr\| \\
&\le \lim_{n\to\infty}\bigl\|z_n-u_n-(p-u)\bigr\|+\lim_{n\to\infty}\bigl\|u_n-y_n+(p-u)\bigr\| \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\|x_n-y_n\| &\le \lim_{n\to\infty}\|x_n-z_n\|+\lim_{n\to\infty}\|z_n-y_n\| \\
&= 0.
\end{aligned}$$

Step 6. $\lim_{n\to\infty}\|y_n-W_ny_n\|=0$.

Indeed, observe that

$$\begin{aligned}
\|x_n-W_ny_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-W_ny_n\| \\
&= \|x_n-x_{n+1}\|+\bigl\|\alpha_n\gamma f(W_nx_n)+(I-\alpha_nA)W_ny_n-W_ny_n\bigr\| \\
&\le \|x_n-x_{n+1}\|+\alpha_n\bigl[\gamma\bigl\|f(W_nx_n)\bigr\|+\|A\|\|W_ny_n\|\bigr].
\end{aligned}$$

From Step 3 and $\alpha_n \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty} \|x_n - W_n y_n\| = 0$. Consequently,

$$\begin{aligned}\lim_{n\to\infty} \|y_n - W_n y_n\| &\le \lim_{n\to\infty} \big(\|y_n - x_n\| + \|x_n - W_n y_n\|\big)\\ &= 0.\end{aligned}$$

Step 7. $\limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle \le 0$, where $x^* = P_{\mathfrak{F}}(\gamma f + (I - A))(x^*)$.

Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle = \lim_{i\to\infty} \langle \gamma f(x^*) - Ax^*, x_{n_i} - x^*\rangle.$$

Correspondingly, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$. Since $\{y_{n_i}\}$ is bounded, there exists a subsequence of y_{n_i} which converges weakly to w. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup w$. Next we show $w \in \mathfrak{F}$. First, we prove that $w \in \Omega$. Utilizing Lemma 2.1, we have for all $x, y \in C$

$$\begin{aligned}&\big\|\Gamma(x) - \Gamma(y)\big\|^2\\ &\quad= \big\| T_{\mu_1}^{G_1}\big[T_{\mu_2}^{G_2}(x - \mu_2 B_2 x) - \mu_1 B_1 T_{\mu_2}^{G_2}(x - \mu_2 B_2 x)\big]\\ &\qquad - T_{\mu_1}^{G_1}\big[T_{\mu_2}^{G_2}(y - \mu_2 B_2 y) - \mu_1 B_1 T_{\mu_2}^{G_2}(y - \mu_2 B_2 y)\big]\big\|^2\\ &\quad\le \big\| T_{\mu_2}^{G_2}(x - \mu_2 B_2 x) - T_{\mu_2}^{G_2}(y - \mu_2 B_2 y)\\ &\qquad - \mu_1\big[B_1 T_{\mu_2}^{G_2}(x - \mu_2 B_2 x) - B_1 T_{\mu_2}^{G_2}(y - \mu_2 B_2 y)\big]\big\|^2\\ &\quad\le \big\| T_{\mu_2}^{G_2}(x - \mu_2 B_2 x) - T_{\mu_2}^{G_2}(y - \mu_2 B_2 y)\big\|^2\\ &\qquad + \mu_1(\mu_1 - 2\beta_1)\big\|B_1 T_{\mu_2}^{G_2}(x - \mu_2 B_2 x) - B_1 T_{\mu_2}^{G_2}(y - \mu_2 B_2 y)\big\|^2\\ &\quad\le \big\| T_{\mu_2}^{G_2}(x - \mu_2 B_2 x) - T_{\mu_2}^{G_2}(y - \mu_2 B_2 y)\big\|^2\\ &\quad\le \big\|x - y - \mu_2(B_2 x - B_2 y)\big\|^2\\ &\quad\le \|x - y\|^2 + \mu_2(\mu_2 - 2\beta_2)\|B_2 x - B_2 y\|^2\\ &\quad\le \|x - y\|^2.\end{aligned}$$

This shows that $\Gamma : C \to C$ is nonexpansive. Note that

$$\begin{aligned}\big\|y_n - \Gamma(y_n)\big\| &= \big\|\Gamma(z_n) - \Gamma(y_n)\big\|\\ &\le \|z_n - y_n\|\\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

According to Lemma 2.2 and Lemma 2.3, we obtain $w \in \Omega$.

Next, let us show that $w \in \mathrm{GMEP}$. From $z_n = T_{\delta_n}^{(\Theta,\varphi)}(x_n - \delta_n F x_n)$, we obtain

$$\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{\delta_n}\big\langle y - z_n, z_n - (x_n - \delta_n F x_n)\big\rangle \ge 0, \quad \forall y \in C.$$

It follows from (H2) that

$$\varphi(y) - \varphi(z_n) + \langle y - z_n, F x_n\rangle + \frac{1}{\delta_n}\langle y - z_n, z_n - x_n\rangle \ge \Theta(y, z_n), \quad \forall y \in C. \tag{3.19}$$

Replacing n by n_i, we have

$$\varphi(y) - \varphi(z_{n_i}) + \langle y - z_{n_i}, Fx_{n_i}\rangle + \left\langle y - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\delta_{n_i}}\right\rangle \geq \Theta(y, z_{n_i}), \quad \forall y \in C.$$

Let $z_t = ty + (1-t)w$ for all $t \in [0,1]$ and $y \in C$. Then we have $z_t \in C$. It follows from (3.19) that

$$\begin{aligned}\langle z_t - z_{n_i}, Fz_t\rangle &\geq \langle z_t - z_{n_i}, Fz_t\rangle - \varphi(z_t) + \varphi(z_{n_i}) - \langle z_t - z_{n_i}, Fx_{n_i}\rangle \\ &\quad - \left\langle z_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\delta_{n_i}}\right\rangle + \Theta(z_t, z_{n_i}) \\ &= \langle z_t - z_{n_i}, Fz_t - Fz_{n_i}\rangle + \langle z_t - z_{n_i}, Fz_{n_i} - Fx_{n_i}\rangle \\ &\quad - \varphi(z_t) + \varphi(z_{n_i}) - \left\langle z_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\delta_{n_i}}\right\rangle + \Theta(z_t, z_{n_i}).\end{aligned}$$

Since $\|z_{n_i} - x_{n_i}\| \to 0$, we have $\|Fz_{n_i} - Fx_{n_i}\| \to 0$. From the monotonicity of F, we have

$$\langle Fz_t - Fz_{n_i}, z_t - z_{n_i}\rangle \geq 0.$$

From (H4), the weakly lower semicontinuity of φ, $\frac{z_{n_i} - x_{n_i}}{\delta_{n_i}} \to 0$ and $z_{n_i} \rightharpoonup w$, we have

$$\langle z_t - w, Fz_t\rangle \geq -\varphi(z_t) + \varphi(w) + \Theta(z_t, w) \tag{3.20}$$

as $i \to \infty$. By (H1), (H4) and (3.20), we obtain

$$\begin{aligned}0 &= \Theta(z_t, z_t) + \varphi(z_t) - \varphi(z_t) \\ &= \Theta\bigl(z_t, ty + (1-t)w\bigr) + \varphi\bigl(ty + (1-t)w\bigr) - \varphi(z_t) \\ &\leq t\Theta(z_t, y) + (1-t)\Theta(z_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(z_t) \\ &= t\bigl[\Theta(z_t, y) + \varphi(y) - \varphi(z_t)\bigr] + (1-t)\bigl[\Theta(z_t, w) + \varphi(w) - \varphi(z_t)\bigr] \\ &\leq t\bigl[\Theta(z_t, y) + \varphi(y) - \varphi(z_t)\bigr] + (1-t)\langle z_t - w, Fz_t\rangle \\ &= t\bigl[\Theta(z_t, y) + \varphi(y) - \varphi(z_t)\bigr] + (1-t)t\langle y - w, Fz_t\rangle.\end{aligned}$$

Hence we obtain

$$0 \leq \Theta(z_t, y) + \varphi(y) - \varphi(z_t) + (1-t)\langle y - w, Fz_t\rangle.$$

Putting $t \to 0$, we have

$$0 \leq \Theta(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Fw\rangle, \quad \forall y \in C.$$

This implies that $w \in \mathrm{GMEP}$.

Since Hilbert spaces satisfy Opial's condition, it follows from Step 5 that

$$\begin{aligned}\liminf_{i\to\infty} \|y_{n_i} - w\| &< \liminf_{i\to\infty} \|y_{n_i} - W_n w\| \\ &\leq \liminf_{i\to\infty} \bigl[\|y_{n_i} - W_n y_{n_i}\| + \|W_n y_{n_i} - W_n w\|\bigr]\end{aligned}$$

$$\begin{aligned}
&\le \limsup_{i\to\infty} \|y_{n_i} - W_n y_{n_i}\| + \liminf_{i\to\infty} \|W_n y_{n_i} - W_n w\| \\
&= \liminf_{i\to\infty} \|W_n y_{n_i} - W_n w\| \\
&\le \liminf_{i\to\infty} \|y_{n_i} - w\|,
\end{aligned}$$

which derives a contraction. This implies that $w \in F(W_n)$. It follows from $F(W_n) = \bigcap_{i=1}^N F(T_i)$ that $w \in \bigcap_{i=1}^N F(T_i)$.

Since $x^* = P_{\mathfrak{F}}(\gamma f + (I - A))(x^*)$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle &= \lim_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_{n_i} - x^* \rangle \\
&= \langle \gamma f(x^*) - Ax^*, w - x^* \rangle \\
&\le 0.
\end{aligned}$$

Step 8. $x_n \to x^*$ as $n \to \infty$.

Indeed, from Lemma 2.6 and (3.4), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad= \|\alpha_n \gamma f(W_n x_n) + (I - \alpha_n A) W_n y_n - x^*\|^2 \\
&\quad= \|(I - \alpha_n A)(W_n y_n - x^*) + \alpha_n(\gamma f(W_n x_n) - Ax^*)\|^2 \\
&\quad\le \|(I - \alpha_n A)(W_n y_n - x^*)\|^2 + 2\alpha_n \langle \gamma f(W_n x_n) - Ax^*, x_{n+1} - x^* \rangle \\
&\quad\le (1 - \alpha_n \overline{\gamma})^2 \|y_n - x^*\|^2 + 2\alpha_n \langle \gamma f(W_n x_n) - Ax^*, x_{n+1} - x^* \rangle \\
&\quad\le (1 - \alpha_n \overline{\gamma})^2 \|x_n - x^*\|^2 + 2\alpha_n \gamma \langle f(W_n x_n) - f(x^*), x_{n+1} - x^* \rangle \\
&\qquad + 2\alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle \\
&\quad\le (1 - \alpha_n \overline{\gamma})^2 \|x_n - x^*\|^2 + 2\alpha_n \gamma \alpha \|x_n - x^*\| \|x_{n+1} - x^*\| \\
&\qquad + 2\alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle \\
&\quad\le (1 - \alpha_n \overline{\gamma})^2 \|x_n - x^*\|^2 + \alpha_n \gamma \alpha (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\qquad + 2\alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad\le \frac{(1 - \alpha_n \overline{\gamma})^2 + \alpha_n \gamma \alpha}{1 - \alpha_n \gamma \alpha} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n \gamma \alpha} \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle \\
&\quad= \left(1 - \frac{2\alpha_n(\overline{\gamma} - \alpha\gamma)}{1 - \alpha_n \gamma \alpha}\right) \|x_n - x^*\|^2 \\
&\qquad + \frac{2\alpha_n(\overline{\gamma} - \alpha\gamma)}{1 - \alpha_n \gamma \alpha} \left[\frac{\alpha_n \overline{\gamma}^2}{2(\overline{\gamma} - \alpha\gamma)} \|x_n - x^*\|^2 + \frac{1}{\overline{\gamma} - \alpha\gamma} \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle \right].
\end{aligned}$$

Put

$$\gamma_n = \frac{2\alpha_n(\overline{\gamma} - \alpha\gamma)}{1 - \alpha_n\gamma\alpha}$$

and

$$\xi_n = \frac{2\alpha_n(\overline{\gamma} - \alpha\gamma)}{1 - \alpha_n\gamma\alpha}\left[\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma} - \alpha\gamma)}\|x_n - x^*\|^2 + \frac{1}{\overline{\gamma} - \alpha\gamma}\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle\right].$$

Then we can write the last inequality as

$$a_{n+1} \leq (1 - \gamma_n)a_n + \xi_n.$$

It follows from condition (i) and Step 6 that

$$\sum_{n=1}^{\infty}\gamma_n = +\infty$$

and

$$\begin{aligned}\limsup_{n\to\infty}\frac{\xi_n}{\gamma_n} &= \limsup_{n\to\infty}\left\{\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma} - \alpha\gamma)}\|x_n - x^*\|^2 \right.\\ &\quad \left. + \frac{1}{\overline{\gamma} - \alpha\gamma}\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle\right\}\\ &\leq 0.\end{aligned}$$

Hence, applying Lemma 2.4, we immediately obtain that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

As corollaries of Theorem 3.1, we have the following results.

Corollary 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta, G_1, G_2 : C \times C \to \mathbb{R}$ be three bifunctions which satisfy assumptions* (H1)-(H4) *and $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function satisfying* (A1) *or* (A2). *Let the mappings $B_1, B_2 : C \to H$ be β_1-inverse strongly monotone and β_2-inverse strongly monotone, respectively. Let $T_1, T_2, \ldots, T_N$ be a finite family of nonexpansive mappings of C into H such that $\mathfrak{F} = \bigcap_{i=1}^N F(T_i) \cap \mathrm{MEP} \cap \Omega \neq \phi$. Let f be a contraction of H into itself with a constant α $(0 < \alpha < 1)$ and let A be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence defined by*

$$\begin{cases}\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{\delta_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C,\\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n)],\\ x_{n+1} = \alpha_n\gamma f(W_n x_n) + (1 - \alpha_n A)W_n y_n, \quad n \geq 1,\end{cases}$$

where $\alpha_n \in [0,1]$, $\mu_1 \in (0, 2\beta_1)$, $\mu_2 \in (0, 2\beta_2)$ and $\{\delta_n\} \subset (0, \infty)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $\lim_{n\to\infty} \lambda_{n,i} = 0$, $\sum_{n=1}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty$ *for all* $i = 1, 2, \ldots, N$,
(iii) $0 < \liminf_{n\to\infty} \delta_n \le \limsup_{n\to\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{N} |\delta_{n+1} - \delta_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $x^* = P_{\bigcap_{i=1}^N F(T_i)\cap \mathrm{MEP}\cap\Omega}(\overline{\gamma} f + (I - A))(x^*)$ *and* (x^*, y^*) *is a solution of problem* (1.4), *where* $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 B_2 x^*)$, *which solves the following variational inequality*:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \ge 0, \quad \forall x \in \bigcap_{i=1}^{N} F(T_i) \cap \mathrm{MEP} \cap \Omega.$$

Proof In Theorem 3.1, for all $n \ge 0$, $z_n = T_{\delta_n}^{(\Theta,\varphi)}(x_n - \delta_n F x_n)$ is equivalent to

$$\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle F x_n, y - z_n \rangle + \frac{1}{\delta_n} \langle y - z_n, z_n - x_n \rangle \ge 0, \quad \forall y \in C. \tag{3.21}$$

Putting $F \equiv 0$, we obtain

$$\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{\delta_n} \langle y - z_n, z_n - x_n \rangle \ge 0, \quad \forall y \in C.$$ □

Corollary 3.2 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $G_1, G_2 : C \times C \to \mathbb{R}$ *be two bifunctions with satisfy assumptions* (H1)-(H4). *Let the mappings* $F, B_1, B_2 : C \to H$ *be* ζ*-inverse strongly monotone,* β_1*-inverse strongly monotone and* β_2*-inverse strongly monotone, respectively. Let* $T_1, T_2, \ldots, T_N$ *be a finite family of nonexpansive mappings of* C *into* H *such that* $\bigcap_{i=1}^N F(T_i) \cap \mathrm{VI}(A, C) \cap \Omega \ne \phi$. *Let* f *be a contraction of* H *into itself with a constant* α $(0 < \alpha < 1)$ *and let* A *be a strongly positive linear bounded operator with a coefficient* $\overline{\gamma} > 0$ *such that* $\|A\| \le 1$. *Assume that* $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. *Let* $x_1 \in C$ *and let* $\{x_n\}$ *be a sequence defined by*

$$\begin{cases} z_n = P_C(x_n - \delta F x_n), \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \alpha_n \gamma f(W_n x_n) + (1 - \alpha_n A) W_n y_n, \quad n \ge 1, \end{cases}$$

where $\alpha_n \in [0, 1]$, $\mu_1 \in (0, 2\beta_1)$, $\mu_2 \in (0, 2\beta_2)$ *and* $\{\delta_n\} \subset [0, 2\zeta]$ *satisfy the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $\lim_{n\to\infty} \lambda_{n,i} = 0$ *and* $\sum_{n=1}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty$ *for all* $i = 1, 2, \ldots, N$;
(iii) $\liminf_{n\to\infty} \delta_n > 0$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $x^* = P_{\bigcap_{i=1}^N F(T_i)\cap \mathrm{VI}(A,C)\cap\Omega}(\gamma f + (I - A))(x^*)$ *and* (x^*, y^*) *is a solution of problem* (1.4), *where* $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 B_2 x^*)$, *which solves the following variational inequality*:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \ge 0, \quad \forall x \in \bigcap_{i=1}^{N} F(T_i) \cap \mathrm{VI}(A, C) \cap \Omega.$$

Proof Put $\Theta = 0$ and $\varphi = 0$ in Theorem 3.1. Then we have from (3.21) that

$$\langle F x_n, y - z_n \rangle + \frac{1}{\delta_n} \langle y - z_n, z_n - x_n \rangle \ge 0, \quad \forall y \in C, n \ge 1.$$

That is,

$$\langle y - z_n, x_n - \delta_n F x_n - z_n \rangle \le 0, \quad \forall y \in C.$$

It follows that $P_C(x_n - \delta_n F x_n) = z_n$ for all $n \ge 1$. We can obtain the desired conclusion easily. □

Remark 3.1 We can see easily that Takahashi and Takahashi [18], Peng and Yao's [1] results are special cases of Theorem 3.1.

Competing interests
The author declares that they have no competing interests.

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