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On the property of T-distributivity

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Abstract

In this paper, we introduce the notion of T-distributivity for any t-norm on a bounded lattice. We determine a relation between the t-norms T and T', where T' is a T-distributive t-norm. Also, for an arbitrary t-norm T, we give a necessary and sufficient condition for T_D to be T-distributive and for T to be $T_\wedge$-distributive. Moreover, we investigate the relation between the T-distributivity and the concepts of the T-partial order, the divisibility of t-norms. We also determine that the T-distributivity is preserved under the isomorphism. Finally, we construct a family of t-norms which are not distributive over each other with the help of incomparable elements in a bounded lattice.

MSC: 03B52; 03E72

Keywords: triangular norm; bounded lattice; T-partial order; divisibility; distributivity

1 Introduction

Triangular norms based on a notion used by Menger [1] were introduced by Schweizer and Sklar [2] in the framework of probabilistic metric spaces, and they play a fundamental role in several branches of mathematics like in fuzzy logics and their applications [3, 4], the games theory [5], the non-additive measures and integral theory [6–8].

A triangular norm (t-norm for short) $T : [0,1]^2 \to [0,1]$ is a commutative, associative, non-decreasing operation on $[0,1]$ with a neutral element 1. The four basic t-norms on $[0,1]$ are the minimum T_M, the product T_P, the Łukasiewicz t-norm T_L and the drastic product T_D given by, respectively, $T_M(x,y) = \min(x,y)$, $T_P(x,y) = xy$, $T_L(x,y) = \max(0, x + y - 1)$ and

$$T_D(x,y) = \begin{cases} x, & \text{if } y = 1, \\ y, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that for any t-norms T_1 and T_2, T_1 is called weaker than T_2 if for every $(x,y) \in [0,1]^2$, $T_1(x,y) \le T_2(x,y)$.

T-norms are defined on a bounded lattice $(L, \le, 0, 1)$ in a similar way, and then extremal t-norms T_D as well as $T_\wedge$ on L are defined similarly T_D and T_M on $[0,1]$. For more details on t-norms on bounded lattices, we refer to [9–17]. Also, the order between t-norms on a bounded lattice is defined similarly.

In the present paper, we introduce the notion of T-distributivity for any t-norms on a bounded lattice $(L, \le, 0, 1)$. The aim of this study is to discuss the properties of

T-distributivity. The paper is organized as follows. Firstly, we recall some basic notions in Section 2. In Section 3, we define the T-distributivity for any t-norm on a bounded lattice. For any two t-norms T_1 and T_2, where T_1 is T_2-distributive, we show that T_1 is weaker than T_2 and give an example illustrating the converse of this need not be true. Also, we prove that the only t-norm T, where every t-norm is T-distributive, is the infimum t-norm $T_\wedge$ when the lattice L is especially a chain. If L is not a chain, we give an example illustrating any t-norm need not be $T_\wedge$. Also, we show that for any t-norm T on a bounded lattice, T_D is T-distributive. Moreover, we show that the T-distributivity is preserved under the isomorphism. For any two t-norms T_1 and T_2 such that T_1 is T_2-distributive, we prove that the divisibility of t-norm T_1 requires the divisibility of t-norm T_2. Also, we obtain that for any two t-norms T_1 and T_2, where T_1 is T_2-distributive, the T_1-partial order implies T_2-partial order. Finally, we construct a family of t-norms which are not distributive over each other with the help of incomparable elements in a bounded lattice.

2 Notations, definitions and a review of previous results

Definition 1 [14] Let $(L, \leq, 0, 1)$ be a bounded lattice. A triangular norm T (t-norm for short) is a binary operation on L which is commutative, associative, monotone and has a neutral element 1.

Let

$$T_D(x,y) = \begin{cases} x, & \text{if } y = 1, \\ y, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then T_D is a t-norm on L. Since it holds that $T_D \leq T$ for any t-norm T on L, T_D is the smallest t-norm on L.

The largest t-norm on a bounded lattice $(L, \leq, 0, 1)$ is given by $T_\wedge(x,y) = x \wedge y$.

Definition 2 [18] A t-norm T on L is divisible if the following condition holds:

$$\forall x, y \in L \text{ with } x \leq y, \text{ there is a } z \in L \text{ such that } \quad x = T(y,z).$$

A basic example of a non-divisible t-norm on any bounded lattice (*i.e.*, $\operatorname{card} L > 2$) is the weakest t-norm T_D. Trivially, the infimum $T_\wedge$ is divisible: $x \leq y$ is equivalent to $x \wedge y = x$.

Definition 3 [12] Let L be a bounded lattice, T be a t-norm on L. The order defined as follows is called a T-partial order (triangular order) for a t-norm T.

$$x \preceq_T y : \quad \Leftrightarrow \quad T(\ell, y) = x \quad \text{for some } \ell \in L.$$

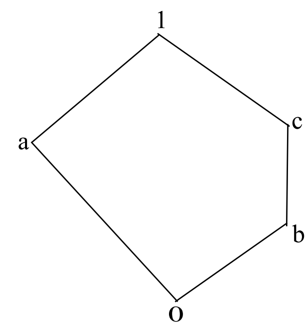
Definition 4 [19]

(i) A t-norm T on a lattice L is called $\wedge$-distributive if

$$T(a, b_1 \wedge b_2) = T(a, b_1) \wedge T(a, b_2)$$

for every $a, b_1, b_2 \in L$.

Figure 1 ($L = \{0, a, b, c, 1\}, \leq, 0, 1$).



(ii) A t-norm T on a complete lattice $(L, \leq, 0, 1)$ is called *infinitely* $\wedge$*-distributive* if

$$T(a, \wedge_I b_\tau) = \wedge_I T(a, b_\tau)$$

for every subset $\{a, b_\tau \in L, \tau \in I\}$ of L.

3 *T*-distributivity

Definition 5 Let $(L, \leq, 0, 1)$ be a bounded lattice and T_1 and T_2 be two t-norms on L. For every $x, y, z \in L$ such that at least one of the elements y, z is not 1, if the condition

$$T_1\big(x, T_2(y, z)\big) = T_2\big(T_1(x, y), T_1(x, z)\big)$$

is satisfied, then T_1 is called T_2-distributive or we say that T_1 is distributive over T_2.

Example 1 Let $(L = \{0, a, b, c, 1\}, \leq, 0, 1)$ be a bounded lattice whose lattice diagram is displayed in Figure 1.

The functions T_1 and T_2 on the lattice L defined by

$$T_1(x, y) = \begin{cases} 0, & \text{if } x = a, y = a, \\ b, & \text{if } x = c, y = c, \\ x \wedge y, & \text{otherwise} \end{cases}$$

and

$$T_2(x, y) = \begin{cases} b, & \text{if } x = c, y = c, \\ x \wedge y, & \text{otherwise} \end{cases}$$

are obviously t-norms on L such that T_1 is T_2-distributive.

Proposition 1 *Let $(L, \leq, 0, 1)$ be a bounded lattice and T_1 and T_2 be two t-norms on L. If T_1 is T_2-distributive, then T_1 is weaker than T_2.*

Proof Since all t-norms coincide on the boundary of L^2, it is sufficient to show that $T_1 \leq T_2$ for all $x, y, z \in L \backslash \{0, 1\}$. By the T_2-distributivity of T_1, it is obtained that

$$T_1(x, y) = T_1\big(T_2(x, 1), y\big) = T_2\big(T_1(x, y), T_1(1, y)\big) = T_2\big(T_1(x, y), y\big) \leq T_2(x, y).$$

Thus, $T_1 \leq T_2$, *i.e.*, T_1 is weaker than T_2. □

Remark 1 The converse of Proposition 1 need not be true. Namely, for any two t-norms T_1 and T_2, even if T_1 is weaker than T_2, T_1 may not be T_2-distributive. Now, let us investigate the following example.

Example 2 Consider the product T_P and the Łukasiewicz t-norm T_L. It is clear that $T_L < T_P$. Since

$$T_L\left(\frac{3}{4}, T_P\left(\frac{5}{8}, \frac{1}{2}\right)\right) = T_L\left(\frac{3}{4}, \frac{5}{16}\right) = \frac{1}{16}$$

and

$$T_P\left(T_L\left(\frac{3}{4}, \frac{5}{8}\right), T_L\left(\frac{3}{4}, \frac{1}{2}\right)\right) = T_P\left(\frac{3}{8}, \frac{1}{4}\right) = \frac{3}{32}$$

T_L is not T_P-distributive.

Corollary 1 *Let L be a bounded lattice and T_1 and T_2 be any two t-norms on L. If both T_1 is T_2-distributive and T_2 is T_1-distributive, then $T_1 = T_2$.*

Proposition 2 *Let L be a bounded chain and T' be a t-norm on L. For every t-norm T, T is T'-distributive if and only if $T' = T_\wedge$.*

Proof :$\Rightarrow$ Let T be an arbitrary t-norm on L such that T'-distributive. By Proposition 1, it is obvious that $T \leq T'$ for any t-norm T. Thus, $T' = T_\wedge$.

$\Leftarrow$: Since L is a chain, for any $y, z \in L$, either $y \leq z$ or $z \leq y$. Suppose that $y \leq z$. By using the monotonicity of any t-norm T, it is obtained that for any $x \in L$, $T(x,y) \leq T(x,z)$. Then

$$T(x,y) = T(x,y) \wedge T(x,z)$$

holds. Thus, for any $x, y, z \in L$,

$$\begin{aligned} T\bigl(x, T_\wedge(y,z)\bigr) &= T(x,y) \\ &= T(x,y) \wedge T(x,z) \\ &= T_\wedge\bigl(T(x,y), T(x,z)\bigr) \end{aligned}$$

is satisfied, which shows that any t-norm T is $T_\wedge$-distributive. □

Remark 2 In Proposition 2, if L is not a chain, then the left-hand side of Proposition 2 may not be satisfied. Namely, if L is not a chain, then any t-norm T need not be $T_\wedge$-distributive. Moreover, even if L is a distributive lattice, any t-norm on L may not be $T_\wedge$-distributive. Now, let us investigate the following example.

Example 3 Consider the lattice ($L = \{0, x, y, z, a, 1\}, \leq$) as displayed in Figure 2.

Obviously, L is a distributive lattice. Define the function T on L as shown in Table 1.

One can easily check that T is a t-norm. Since

$$T\bigl(a, T_\wedge(y,z)\bigr) = T(a,x) = 0$$

Figure 2 ($L = \{0, x, y, z, a, 1\}, \leq$).

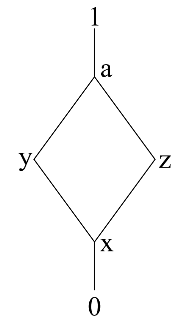


Table 1 ***T*-norm on the lattice ($L = \{0, x, y, z, a, 1\}, \leq$)**

T	0	x	y	z	a	1
0	0	0	0	0	0	0
x	0	0	0	0	0	x
y	0	0	y	0	y	y
z	0	0	0	z	z	z
a	0	0	y	z	a	a
1	0	x	y	z	a	1

and

$$T_\wedge\big(T(a,y), T(a,z)\big) = T_\wedge(y,z) = x,$$

T is not $T_\wedge$-distributive.

Remark 3 The fact that any t-norm T is $T_\wedge$-distributive means that T is $\wedge$-distributive.

Theorem 1 *Let* $(L, \leq, 0, 1)$ *be a bounded lattice. For any t-norm* T *on* L, T_D *is* T-*distributive.*

Proof Let T be an arbitrary t-norm on L. We must show that the equality

$$T_D\big(x, T(y,z)\big) = T\big(T_D(x,y), T_D(x,z)\big)$$

holds for every element x, y, z of L with $y \neq 1$ or $z \neq 1$. Suppose that $z \neq 1$. If $x = 1$, the desired equality holds since $T_D(x, T(y,z)) = T(y,z)$ and $T(T_D(x,y), T_D(x,z)) = T(y,z)$. Let $x \neq 1$. Then $y = 1$ or $y \neq 1$. If $y = 1$, since $T_D(x, T(y,z)) = T_D(x,z) = 0$ and $T(T_D(x,y), T_D(x,z)) = T(x,0) = 0$, the equality holds again. Now, let $y \neq 1$. Since $T(y,z) \leq y \leq 1$ and $y \neq 1$, $T(y,z) \neq 1$. Then $T_D(x, T(y,z)) = 0$ and $T(T_D(x,y), T_D(x,z)) = T(0,0) = 0$, whence the equality holds. Thus, T_D is T-distributive for any t-norm T on L. □

Proposition 3 [20] *If* T *is a t-norm and* $\varphi : [0,1] \to [0,1]$ *is a strictly increasing bijection, then the operation* $T_\varphi : [0,1]^2 \to [0,1]$ *given by*

$$T_\varphi(x,y) = \varphi^{-1}\big(T\big(\varphi(x), \varphi(y)\big)\big)$$

is a t-norm which is isomorphic to T. *This t-norm is called* φ-*transform of* T.

Let T_1 and T_2 be any two t-norms on $[0,1]$ and let φ be a strictly increasing bijection from $[0,1]$ to $[0,1]$. Denote the φ-transforms of the t-norms T_1 and T_2 by T_φ^1 and T_φ^2, respectively.

Theorem 2 *Let T_1 and T_2 be any t-norms on $[0,1]$ and let φ be a strictly increasing bijection from $[0,1]$ to $[0,1]$. T_1 is T_2-distributive if and only if T_φ^1 is T_φ^2-distributive.*

Proof Let T_1 be T_2-distributive. We must show that for every $x,y,z \in [0,1]$ with $y \neq 1$ or $z \neq 1$,

$$T_\varphi^1\big(x, T_\varphi^2(y,z)\big) = T_\varphi^2\big(T_\varphi^1(x,y), T_\varphi^1(x,z)\big).$$

Since $\varphi : [0,1] \to [0,1]$ is a strictly increasing bijection, for every element $y,z \in [0,1]$ with $y \neq 1$ or $z \neq 1$, it must be $\varphi(y) \neq 1$ or $\varphi(z) \neq 1$. By using T_2-distributivity of T_1, we obtain that the equality

$$\begin{aligned}
T_\varphi^1\big(x, T_\varphi^2(y,z)\big) &= \varphi^{-1}\big(T_1\big(\varphi(x), \varphi\big(T_\varphi^2(y,z)\big)\big)\big) \\
&= \varphi^{-1}\big(T_1\big(\varphi(x), \varphi\big(\varphi^{-1}\big(T_2\big(\varphi(y), \varphi(z)\big)\big)\big)\big)\big) \\
&= \varphi^{-1}\big(T_1\big(\varphi(x), T_2\big(\varphi(y), \varphi(z)\big)\big)\big) \\
&= \varphi^{-1}\big(T_2\big(T_1\big(\varphi(x), \varphi(y)\big), T_1\big(\varphi(x), \varphi(z)\big)\big)\big) \\
&= \varphi^{-1}\big(T_2\big(\big(\varphi \circ \varphi^{-1}\big)T_1\big(\varphi(x), \varphi(y)\big), \big(\varphi \circ \varphi^{-1}\big)T_1\big(\varphi(x), \varphi(z)\big)\big)\big) \\
&= \varphi^{-1}\big(T_2\big(\varphi\big(\varphi^{-1}\big(T_1\big(\varphi(x), \varphi(y)\big)\big)\big), \varphi\big(\varphi^{-1}\big(T_1\big(\varphi(x), \varphi(z)\big)\big)\big)\big)\big) \\
&= \varphi^{-1}\big(T_2\big(\varphi\big(T_\varphi^1(x,y)\big), \varphi\big(T_\varphi^1(x,z)\big)\big)\big) \\
&= T_\varphi^2\big(T_\varphi^1(x,y), T_\varphi^1(x,z)\big)
\end{aligned}$$

holds. Thus, T_φ^1 is T_φ^2-distributive.

Conversely, let T_φ^1 be T_φ^2-distributive. We will show that $T_1(x, T_2(y,z)) = T_2(T_1(x,y), T_1(x,z))$ for every element $x,y,z \in [0,1]$ with $y \neq 1$ or $z \neq 1$. Since T_φ^1 is the φ-transform of the t-norm T_1, for every $x,y \in [0,1]$, $T_\varphi^1(x,y) = \varphi^{-1}(T_1(\varphi(x), \varphi(y)))$. Since φ is a bijection, it is clear that

$$T_1\big(\varphi(x), \varphi(y)\big) = \varphi\big(T_\varphi^1(x,y)\big) \tag{1}$$

holds. Also, by using (1), it is obtained that

$$T_1(x,y) = T_1\big(\varphi\big(\varphi^{-1}(x)\big), \varphi\big(\varphi^{-1}(y)\big)\big) = \varphi\big(T_\varphi^1\big(\varphi^{-1}(x), \varphi^{-1}(y)\big)\big) \tag{2}$$

From (2), it follows

$$T_\varphi^1\big(\varphi^{-1}(x), \varphi^{-1}(y)\big) = \varphi^{-1}\big(T_1(x,y)\big). \tag{3}$$

Also, the similar equalities for t-norm T_2 can be written. Since $\varphi^{-1}(y) \neq 1$ or $\varphi^{-1}(z) \neq 1$ for every $y,z \in [0,1]$ with $y \neq 1$ or $z \neq 1$, by using T_φ^2-distributivity of T_φ^1, it is obtained that the following equalities:

$$\begin{aligned} T_1\bigl(x, T_2(y,z)\bigr) &\overset{(2)}{=} T_1\bigl(x, \varphi\bigl(T_\varphi^2\bigl(\varphi^{-1}(y), \varphi^{-1}(z)\bigr)\bigr)\bigr) \\ &\overset{(2)}{=} \varphi\bigl(T_\varphi^1\bigl(\varphi^{-1}(x), \varphi^{-1}\bigl(\varphi\bigl(T_\varphi^2\bigl(\varphi^{-1}(y), \varphi^{-1}(z)\bigr)\bigr)\bigr)\bigr)\bigr) \\ &= \varphi\bigl(T_\varphi^1\bigl(\varphi^{-1}(x), T_\varphi^2\bigl(\varphi^{-1}(y), \varphi^{-1}(z)\bigr)\bigr)\bigr) \\ &= \varphi\bigl(T_\varphi^2\bigl(T_\varphi^1\bigl(\varphi^{-1}(x), \varphi^{-1}(y)\bigr), T_\varphi^1\bigl(\varphi^{-1}(x), \varphi^{-1}(z)\bigr)\bigr)\bigr) \\ &\overset{(3)}{=} \varphi\bigl(T_\varphi^2\bigl(\varphi^{-1}\bigl(T_1(x,y)\bigr), \varphi^{-1}\bigl(T_1(x,z)\bigr)\bigr)\bigr) \\ &\overset{(2)}{=} \varphi\bigl(\varphi^{-1}\bigl(T_2\bigl(T_1(x,y), T_1(x,z)\bigr)\bigr)\bigr) \\ &= T_2\bigl(T_1(x,y), T_1(x,z)\bigr) \end{aligned}$$

hold. Thus, T_1 is T_2-distributive. □

Proposition 4 *Let $(L, \leq, 0, 1)$ be a bounded lattice and T_1 and T_2 be two t-norms on L such that T_1 is T_2-distributive. If T_1 is divisible, then T_2 is also divisible.*

Proof Consider two elements x, y of L with $x \leq y$. If $x = y$, then T_2 would be always a divisible t-norm since $T_2(y,1) = y = x$. Let $x \neq y$. Since T_1 is divisible, there exists an element $1 \neq z$ of L such that $T_1(y,z) = x$. Then, by using T_2-distributivity of T_1, it is obtained that

$$\begin{aligned} x &= T_1(y,z) = T_1\bigl(y, T_2(z,1)\bigr) \\ &= T_2\bigl(T_1(y,z), T_1(y,1)\bigr) \\ &= T_2\bigl(T_1(y,z), y\bigr). \end{aligned}$$

Thus, for any elements x, y of L with $x \leq y$ and $x \neq y$, since there exists an element $T_1(y,z) \in L$ such that $x = T_2(T_1(y,z), y)$, T_2 is a divisible t-norm. □

Corollary 2 *Let $(L, \leq, 0, 1)$ be a bounded lattice and T_1 and T_2 be two t-norms on L. If T_1 is T_2-distributive, then the T_1-partial order implies the T_2-partial order.*

Proof Let $a \preceq_{T_1} b$ for any $a, b \in L$. If $a = b$, then it would be $a \preceq_{T_2} b$ since $T_2(b,1) = b = a$ for the element $1 \in L$. Now, suppose that $a \preceq_{T_1} b$ but $a \neq b$. Then there exists an element $\ell \in L$ such that $T_1(b,\ell) = a$. Since $a \neq b$, it must be $\ell \neq 1$. Then $T_1(b, T_2(\ell,1)) = T_1(b,\ell) = a$. Since T_1 is T_2-distributive, it is obtained that

$$\begin{aligned} a &= T_1\bigl(b, T_2(\ell,1)\bigr) = T_2\bigl(T_1(b,\ell), T_1(b,1)\bigr) \\ &= T_2(a,b). \end{aligned}$$

for elements $b, \ell, 1 \in L$ with $\ell \neq 1$, whence $a \preceq_{T_2} b$. So, we obtain that $\preceq_{T_1} \subseteq \preceq_{T_2}$. □

Remark 4 For any t-norms T_1 and T_2, if T_1 is T_2-distributive, then we show that T_1 is weaker than T_2 in Proposition 1 and the T_1-partial order implies the T_2-partial order in Proposition 2. Although T_1 is weaker than T_2, that does not require the T_1-partial order to imply the T_2-partial order. Let us investigate the following example illustrating this case.

Example 4 Consider the drastic product T_P and the function defined as follows:

$$T^*(x,y) = \begin{cases} xy, & \text{if } (x,y) \in [0,\frac{1}{2}]^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

It is clear that the function T^* is a t-norm such that $T_P \leq T^*$, but $\preceq_{T_P} \nsubseteq \preceq_{T^*}$. Indeed.

First, let us show that $\frac{3}{8} \npreceq_{T^*} \frac{1}{2}$. Suppose that $\frac{3}{8} \preceq_{T^*} \frac{1}{2}$. Then, for some $\ell \in [0,1]$,

$$T^*\left(\ell, \frac{1}{2}\right) = \frac{3}{8}.$$

For $\ell \in [0,1]$, either $\ell \leq \frac{1}{2}$ or $\ell > \frac{1}{2}$. Let $\ell \leq \frac{1}{2}$. Since $\frac{3}{8} = T^*(\ell, \frac{1}{2}) = \frac{1}{2}\ell$, it is obtained that $\ell = \frac{3}{4}$, which contradicts $\ell \leq \frac{1}{2}$. Then it must be $\ell > \frac{1}{2}$. Since $\frac{3}{8} = T^*(\ell, \frac{1}{2}) = \min(\ell, \frac{1}{2}) = \frac{1}{2}$, which is a contradiction. Thus, it is obtained that $\frac{3}{8} \npreceq_{T^*} \frac{1}{2}$. On the other hand, since $x \preceq_{T_P} y$ means that there exists an element ℓ of L such that $T_p(\ell, y) = \ell y = x$ and $T_P(\frac{1}{2}, \frac{3}{4}) = \frac{3}{8}$, we have that $\frac{3}{8} \preceq_{T_P} \frac{1}{2}$. So, it is obtained that $\preceq_{T_P} \nsubseteq \preceq_{T^*}$.

Now, let us construct a family of t-norms which are not distributive over each other with the help of incomparable elements in a bounded lattice.

Theorem 3 *Let L be a complete lattice and $\{S_\alpha | \alpha \in I\}$ be a nonempty family of nonempty sets consisting of the elements in L which are all incomparable to each other with respect to the order on L. If for any element $u \in S_\alpha$, $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$ is comparable to every element in L, then the family $(T_u)_{u \in S_\alpha}$ defined by*

$$T_u(x,y) = \begin{cases} \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, & \textit{if } (x,y) \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2, \\ x \wedge y, & \textit{otherwise} \end{cases}$$

is a family of t-norms which are not distributive over each other. Namely, for any $\ell, q \in S_\alpha$, neither T_ℓ is T_q-distributive nor T_q is T_ℓ-distributive.

Proof Firstly, let us show that for every $u \in S_\alpha$, each function T_u is a t-norm.

(i) Since $x \leq 1$, for every element $x \in L$, $1 \notin S_\alpha$. Then it follows $T_u(x,1) = x \wedge 1 = x$ from $(x,1) \notin [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$, that is, the boundary condition is satisfied.

(ii) It can be easily shown that the commutativity holds.

(iii) Considering the monotonicity, suppose that $x \leq y$ for $x, y \in L$. Let $z \in L$ be arbitrary. Then there are the following possible conditions for the couples (x,z), (y,z).

- Let $(x,z), (y,z) \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$. Then we get clearly the equality

$$T_u(x,z) = \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} = T_u(y,z).$$

- Let $(x,z) \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$ and $(y,z) \notin [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$. Then $y \notin [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]$. Clearly, $T_u(x,z) = \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$ and $T_u(y,z) = y \wedge z$. Since $x \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]$ and $x \leq y$, we obtain $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq y$. By $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq z$, we get $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq y \wedge z$, whence $T_u(x,z) \leq T_u(y,z)$.

- Let $(x,z) \notin [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$ and $(y,z) \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$. Then it is clear that $x \notin [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]$. In this case,

$$T_u(x,z) = x \wedge z \quad \text{and} \quad T_u(y,z) = \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}.$$

By $x \leq y$ and $y \leq u$, it is clear that $x \leq u$. Since $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$ is comparable to every element in L, either $x \leq \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$ or $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq x$. If $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq x$, it would be $x \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]$ from $x \leq u$, a contradiction. Thus, it must be $x \leq \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$. Since $z \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]$, $x \wedge z = x$. Thus, the inequality

$$T_u(x,z) = x \wedge z = x \leq \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} = T_u(y,z)$$

holds.

- Let $(x,z), (y,z) \notin [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$. By $x \leq y$, we have that

$$T_u(x,z) = x \wedge z \leq y \wedge z = T_u(y,z).$$

So, the monotonicity holds.

(iv) Now let us show that for every $x, y, z \in L$, the equality $T_u(x, T_u(y,z)) = T_u(T_u(x,y),z)$ holds.

- Let $(x,y), (y,z) \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$. Then

$$T_u\bigl(x, T_u(y,z)\bigr) = \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$$

and

$$T_u\bigl(T_u(x,y),z\bigr) = \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\},$$

whence the equality holds.

- If $(x,y) \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$ and $(y,z) \notin [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$, then it must be $z \notin [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]$. Here, there are two choices for z: either $z \in S_\alpha$ or $z \notin S_\alpha$.

Let $z \in S_\alpha$. Then $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq z$. By the inequality $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq u$, it is clear that $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq u \wedge z$. Since $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq y \leq u$, the following inequalities:

$$\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} = \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \wedge z \leq y \wedge z \leq y \leq u$$

hold, that is, $y \wedge z \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]$. Thus, we have that

$$T_u\bigl(x, T_u(y,z)\bigr) = T_u(x, y \wedge z) = \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$$

and

$$\begin{aligned} T_u\bigl(T_u(x,y),z\bigr) &= T_u\bigl(\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, z\bigr) \\ &= \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \wedge z = \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}. \end{aligned}$$

So, the equality holds again.

Let $z \notin S_\alpha$. Then there exists at least an element v in S_α such that v is comparable to the element z; *i.e.*, either $z \leq v$ or $v \leq z$. Let $v \leq z$. Since $u, v \in S_\alpha$, it is clear that $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq u \wedge v \leq u \wedge z \leq u$. Also, from the inequalities $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq y$ and $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq v \leq z$, it follows $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \leq y \wedge z \leq y \leq u$, *i.e.*, it is obtained that $y \wedge z \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]$. Thus,

$$T_u\big(x, T_u(y,z)\big) = T_u(x, y \wedge z) = \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$$

and

$$\begin{aligned} T_u\big(T_u(x,y), z\big) &= T_u\big(\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, z\big) \\ &= \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \wedge z \\ &= \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}. \end{aligned}$$

Thus, the equality is satisfied.

Now, suppose that $z \leq v$. If $u \leq z$, it would be $u \leq v$, which is a contradiction. Thus, either $z < u$ or z and u are not comparable. If $z < u$, then it must be $z < \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$ since $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$ is comparable to every element in L and $z \notin [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]$. Thus, we have that

$$\begin{aligned} T_u\big(T_u(x,y), z\big) &= T_u\big(\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, z\big) \\ &= \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \wedge z \\ &= z \end{aligned}$$

and

$$\begin{aligned} T_u\big(x, T_u(y,z)\big) &= T_u(x, y \wedge z) \\ &= T_u(x, z) \\ &= x \wedge z = z, \end{aligned}$$

whence the equality holds.

Let z and u be not comparable. Since $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$ is comparable to every element in L, either $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} < z$ or $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} > z$. If $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} > z$, it would be $z < u$, a contradiction. Then it must be $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} < z$. By $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} = \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \wedge y < y \wedge z < y < u$, it is obtained that $y \wedge z \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]$. Then the equalities

$$T_u\big(x, T_u(y,z)\big) = T_u(x, y \wedge z) = \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$$

and

$$\begin{aligned} T_u\big(T_u(x,y), z\big) &= T_u\big(\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, z\big) \\ &= \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\} \wedge z = \inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}. \end{aligned}$$

In this case, the equality is satisfied.

Similarly, one can show that the equality $T_u(x, T_u(y,z)) = T_u(T_u(x,y),z)$ holds when $(x,y) \notin [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$ and $(y,z) \in [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$.

- Now, let us investigate the last condition. If $(x,y),(y,z) \notin [\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}, u]^2$, then it is obvious that

$$T_u\big(x, T_u(y,z)\big) = T_u(x, y \wedge z) = x \wedge (y \wedge z)$$

and

$$T_u\big(T_u(x,y),z\big) = T_u(x \wedge y, z) = (x \wedge y) \wedge z,$$

whence the equality holds.

Consequently, we prove that $(T_u)_{u\in S_\alpha}$ is a family of t-norms on L. Now, we will show that for every $m,n \in S_\alpha$, T_m and T_n are not distributive t-norms over each other.

Suppose that T_m is T_n-distributive. By Proposition 1, it must be $T_m \le T_n$, that is, for every $x,y \in L$, $T_m(x,y) \le T_n(x,y)$. Since m and n are not comparable, it is clear that $n \nleq m$ and $m \nleq n$. Then n must not be in $[\inf\{m \wedge \mu_i | \mu_i \in S_\alpha\}, m]$. Thus,

$$T_m(n,n) = n \wedge n = n.$$

On the other hand, since $n \in [\inf\{n \wedge \mu_i | \mu_i \in S_\alpha\}, n]$,

$$T_n(n,n) = \inf\{n \wedge \mu_i | \mu_i \in S_\alpha\}.$$

Then we have that $T_n(n,n) \neq T_m(n,n)$. Otherwise, we obtain that $n \le m$, which is a contradiction. So, we have that $T_n(n,n) < T_m(n,n)$ contradicts $T_m \le T_n$. Thus, T_m is not T_n-distributive. Similarly, it can be shown that T_n is not T_m-distributive. So, the family given above is a family of t-norms which are not distributive over each other. □

To explain how the family $(S_\alpha)_{\alpha\in I}$ in Theorem 3 can be determined, let us investigate the following example.

Example 5 Let $(L = \{0,a,b,c,d,e,1\}, \le, 0, 1)$ be a bounded lattice as shown in Figure 3.

For the family of $(S_\alpha)_{\alpha\in I}$, there are two choices: one of them must be $S_{\alpha_1} = \{c,d,e\}$ and the other must be $S_{\alpha_2} = \{b,e\}$. Then, by Theorem 3, for every $u \in S_{\alpha_1}$ and $v \in S_{\alpha_2}$, the following

Figure 3 **($L = \{0,a,b,c,d,e,1\}, \le, 0, 1$).**

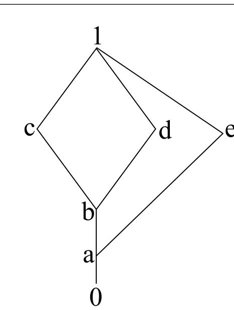
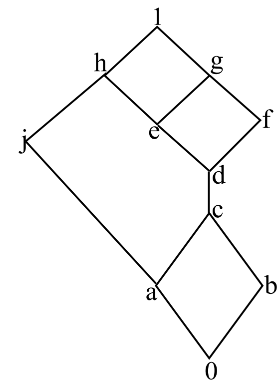


Figure 4 ($L = \{0, a, b, c, d, e, f, g, h, j, 1\}, \leq, 0, 1$).



functions:

$$T_u(x,y) = \begin{cases} a, & \text{if } (x,y) \in [a,u]^2, \\ x \wedge y, & \text{otherwise} \end{cases}$$

and

$$T_v(x,y) = \begin{cases} a, & \text{if } (x,y) \in [a,v]^2, \\ x \wedge y, & \text{otherwise} \end{cases}$$

are two families of t-norms.

Remark 5 In Theorem 3, if the condition that $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$ is comparable to every element in L is canceled, then for any element $u \in S_\alpha$, T_u is not a t-norm. The following is an example showing that T_u is not a t-norm when the condition that for any element $u \in S_\alpha$, $\inf\{u \wedge \mu_i | \mu_i \in S_\alpha\}$ is comparable to every element in L is canceled.

Example 6 Let $(L = \{0, a, b, c, d, e, f, g, h, j, 1\}, \leq, 0, 1)$ be a bounded lattice as displayed in Figure 4.

From Figure 4, it is clear that $\inf\{j, e, f\} = a$ is not comparable to b. However, for the set $S = \{j, e, f\}$, the function defined by

$$T_e(x,y) = \begin{cases} a, & \text{if } (x,y) \in [a,e]^2, \\ x \wedge y, & \text{otherwise} \end{cases}$$

does not satisfy the associativity since $T_e(T_e(c,d), b) = 0$ and $T_e(c, T_e(d,b)) = b$. So, T_e is not a t-norm.

4 Conclusions

In this paper, we introduced the notion of T-distributivity for any t-norm on a bounded lattice and discussed some properties of T-distributivity. We determined a necessary and sufficient condition for T_D to be T-distributive and for T to be $T_\wedge$-distributive. We obtained that T-distributivity is preserved under the isomorphism. We proved that the divisibility of t-norm T_1 requires the divisibility of t-norm T_2 for any two t-norms T_1 and

T_2 where T_1 is T_2-distributive. Also, we constructed a family of t-norms which are not distributive over each other with the help of incomparable elements in a bounded lattice.

Competing interests
The author declares that they have no competing interests.

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