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Research Article

A New Iterative Algorithm for Solving a Class of Matrix Nearness Problem

Xuefeng Duan[1, 2] and Chunmei Li[1]

[1] *College of Mathematics and Computational Science, Guilin University of Electronic Technology, Guilin 541004, China*
[2] *Department of Mathematics, Shanghai University, Shanghai 200444, China*

Correspondence should be addressed to Xuefeng Duan, duanxuefeng@guet.edu.cn

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Based on the alternating projection algorithm, which was proposed by Von Neumann to treat the problem of finding the projection of a given point onto the intersection of two closed subspaces , we propose a new iterative algorithm to solve the matrix nearness problem associated with the matrix equations $AXB = E$, $CXD = F$, which arises frequently in experimental design. If we choose the initial iterative matrix $X_0 = 0$, the least Frobenius norm solution of these matrix equations is obtained. Numerical examples show that the new algorithm is feasible and effective.

1. Introduction

Denoted by $R^{m\times n}$ be the set of $m \times n$ real matrices, A^T and A^+ be the transpose and Moore-Penrose generalized inverse of the matrix A, respectively. For $A, B \in R^{m\times n}$, $\langle A, B\rangle = \operatorname{trace}(B^T A)$ denotes the inner product of the matrix A and B. The induced norm is the so-called Frobenius norm, that is, $\|A\| = \langle A, A\rangle^{1/2}$; then $R^{m\times n}$ is a real Hilbert space. Let M be a closed convex subset in a real Hilbert space H and u be a point in H; then the point in M nearest to u is called the projection of u onto M and denoted by $P_M(u)$; that is to say, $P_M(u)$ is the solution of the following minimization problem (see [1, 2])

$$\min_{x\in M}\|x - u\|, \tag{1}$$

that is,

$$\|P_M(u) - u\| = \min_{x\in M}\|x - u\|. \tag{2}$$

The problem of finding a nearness matrix $\widehat{X}$ in a constraint matrix set to a given matrix $\overline{X}$ is called the matrix nearness problem. Because the preliminary estimation $\overline{X}$ is frequently obtained from experiments, it may not satisfy the given restrictions. Hence it is necessary to find a nearness matrix $\widehat{X}$ in this constraint matrix set to replace the estimation $\overline{X}$ [3]. In the area of structure design, finite element model updating and control theory, and so forth, the matrix set is always the (constraint) solution set or the least square (constraint) solution set of some matrix equations [4–6]. Thus, the problem mentioned above is also called the matrix nearness problem associated with the matrix equation. Recently, there are many discussions on the matrix nearness problem associated with some matrix equations. For instance, see [4, 6–14].

In this paper, we consider the following problem.

Problem 1. Given matrices $A \in R^{p\times n}, B \in R^{n\times q}$, $C \in R^{s\times n}, D \in R^{n\times t}, E \in R^{p\times q}, F \in R^{s\times t}$, and $\overline{X} \in R^{n\times n}$, find $\widehat{X} \in \Omega$ such that

$$\left\|\widehat{X} - \overline{X}\right\| = \min_{X\in\Omega}\left\|X - \overline{X}\right\|, \tag{3}$$

where

$$\Omega = \{X \in R^{n\times n} \mid AXB = E,\ CXD = F\}. \tag{4}$$

Obviously, Ω is the solution set of the matrix equations

$$AXB = E, \qquad CXD = F, \tag{5}$$

and $\widehat{X}$ is the optimal approximate solution of (5) to the given matrix $\overline{X}$. In particular, if $\overline{X} = 0$, then the solution $\widehat{X}$ of

Problem 1 is just the least Frobenius norm solution of the matrix equations (5). It is easy to verify that Ω is convex set; then the solution of Problem 1 is unique.

The matrix equations (5) and its matrix nearness problem have been extensively studied for the past 40 or more years. Wang [15] and Navarra et al. [16] gave some conditions for the existence of a solution and some representations of the general common solution to (5). By the projection theorem and matrix decompositions, Liao et al. [6] gave an analytical expression of the optimal approximate least square symmetric solution of (5). However, these direct methods may be less efficient for the large sparse coefficient matrices due to the limited storages and the speeds of the computers. Therefore, iterative methods for solving the matrix equations (5) have attracted much interests recently. Peng et al. [11] and Chen et al. [7] proposed some iterative methods to compute the symmetric solutions and optimal approximate symmetric solution of (5). An efficient iterative method was presented to solve the matrix nearness Problem 1 associated with the matrix equations (5) in [13]. Ding et al. [17] considered the unique solution of the matrix equations (5) and used gradient-based iterative algorithm to compute the unique solution. The (least square) solution and the optimal approximate (least square) solution of (5), which is constrained as bisymmetric, reflexive, generalized reflexive, generalized centrosymmetric, were studied in [7–10, 12].

The alternating projection algorithm dates back to von Neumann [18], who treated the problem of finding the projection of a given point onto the intersection of two closed subspaces. Later, Bauschke and Borwein [1] extended the analysis of Von Neumann's alternating projection scheme to the case of two closed affine subspaces . There are many variations and extensions of the alternating projection algorithm, and we can use them to find the projection of a given point onto the intersection of k $(k \geq 2)$ closed subspaces [19] and k $(k \geq 2)$ closed convex sets [20, 21]. For a complete discussion on the alternation projection algorithm see Deutsch [2].

In this paper, we propose a new algorithm to solve Problem 1. We state Problem 1 as the minimization of a convex quadratic function over the intersection of two closed affine subspaces in the vector space $R^{n\times n}$. From this point of view, Problem 1 can be solved by the alternating projection algorithm. If we choose the initial iterative matrix $X_0 = 0$, the least Frobenius norm solution of the matrix equations $AXB = E$, $CXD = F$ is obtained. In the end, we use some numerical examples to show that the new algorithm is faster and lower computational cost for each step than the algorithm proposed by Sheng and Chen [13] to solve Problem 1. Especially, the CPU time and iteration steps of our algorithm increase slowly as the dimension of the matrix is increasing; so our algorithm is suitable for large-scale problems.

2. Alternating Projection Algorithm for Solving Problem 1

In this section, we apply the alternating projection algorithm to solve Problem 1. We begin with two lemmas.

Lemma 1 (see [1, Theorem 4.1]). *Let $M_1 = a + \widetilde{M}_1$, $M_2 = b + \widetilde{M}_2$ be closed affine subspaces in a Hilbert space H and u be a point in H. Here, $\widetilde{M}_1$ and $\widetilde{M}_2$ are closed subspaces and $a, b \in H$. If $M_1 \cap M_2 \neq \emptyset$, then the sequences $\{x_k\}$ and $\{y_k\}$ generated by the alternating projection algorithm*

$$x_0 = u, \qquad y_{k+1} = P_{M_1}(x_k),$$
$$x_{k+1} = P_{M_2}(y_{k+1}), \quad k = 0, 1, 2, \ldots \tag{6}$$

both converge to the projection of the point u onto the intersection of M_1 and M_2, that is,

$$x_k \longrightarrow P_{M_1\cap M_2}(u), \qquad y_k \longrightarrow P_{M_1\cap M_2}(u), \qquad k \longrightarrow +\infty. \tag{7}$$

Lemma 2 (see [22, Theorem 9.3.1]). *Let $A \in R^{p\times n}$, $B \in R^{n\times q}$ and $E \in R^{p\times q}$, be known matrices. Then the matrix equation $AXB = E$ has a solution if and only if*

$$AA^+EB^+B = E, \tag{8}$$

and the representation of the solution is

$$X = A^+EB^+ + Y - A^+AYBB^+, \tag{9}$$

where $Y \in R^{n\times n}$ is arbitrary.

Lemma 3 (see [22, Theorem 9.3.2]). *Given $Z \in R^{n\times n}$, set*

$$\mathfrak{R} = \{X \in R^{n\times n} \mid AXB = E,\ A \in R^{p\times n},\ B \in R^{n\times q},\ E \in R^{p\times q}\}, \tag{10}$$

then the solution $\widehat{X}$ of the following problem

$$\min_{X\in\mathfrak{R}} \|X - Z\| \tag{11}$$

is

$$\widehat{X} = Z + A^+(E - AZB)B^+, \tag{12}$$

that is,

$$\left\|\widehat{X} - Z\right\| = \min_{X\in\mathfrak{R}} \|X - Z\|. \tag{13}$$

Now we begin to use the alternating projection algorithm (6) to solve Problem 1. Firstly, we define two sets

$$\Omega_1 = \{X \in R^{n\times n} \mid AXB = E\},$$
$$\Omega_2 = \{X \in R^{n\times n} \mid CXD = F\}. \tag{14}$$

It is easy to know that $\Omega = \Omega_1 \cap \Omega_2$, and if the set Ω is nonempty, then

$$\Omega = \Omega_1 \cap \Omega_2 \neq \emptyset. \tag{15}$$

And by Lemma 2, the sets Ω_1 and Ω_2 can be equivalently written as

$$\begin{aligned}\Omega_1 &= \{X \mid AXB = E\}\\ &= \{X = A^+EB^+ + Y - A^+AYBB^+ \mid Y \in R^{n\times n}\}\\ &= A^+EB^+ + \{Y - A^+AYBB^+ \mid Y \in R^{n\times n}\},\\ \Omega_2 &= \{X \mid CXD = F\}\\ &= \{X = C^+FD^+ + Y - C^+CYDD^+ \mid Y \in R^{n\times n}\}\\ &= C^+FD^+ + \{Y - C^+CYDD^+ \mid Y \in R^{n\times n}\}.\end{aligned} \tag{16}$$

Hence, Ω_1 and Ω_2 are closed affine subspaces.

After defining the sets Ω_1 and Ω_2, Problem 1 can be rewritten as finding $\widehat{X} \in \Omega = \Omega_1 \cap \Omega_2$, such that

$$\left\|\widehat{X} - \overline{X}\right\| = \min_{X\in\Omega_1\cap\Omega_2}\left\|X - \overline{X}\right\|. \tag{17}$$

Noting the equalities (17) and (2), it is easy to find that

$$\widehat{X} = P_{\Omega_1\cap\Omega_2}\left(\overline{X}\right). \tag{18}$$

Therefore, Problem 1 can be converted equivalently into finding the projection $P_{\Omega_1\cap\Omega_2}(\overline{X})$ of a given matrix $\overline{X}$ onto the intersection set $\Omega_1\cap\Omega_2$. Now we will use alternating projection algorithm (6) to compute the projection $P_{\Omega_1\cap\Omega_2}(\overline{X})$. Consequently, we can get the solution $\widehat{X}$ of Problem 1.

By (6) we can see that the key problems to realize the alternating projection algorithm (6) are how to compute the projections $P_{\Omega_1}(Z)$, $P_{\Omega_2}(Z)$ of a matrix Z onto Ω_1 and Ω_2, respectively. Such problems are perfectly solvable in the following theorems.

Theorem 1. *Suppose that the set Ω_1 is nonempty. For a given $n \times n$ matrix Z, we have*

$$P_{\Omega_1}(Z) = Z + A^+(E - AZB)B^+. \tag{19}$$

Proof. By (1) and (2), we know that the projection $P_{\Omega_1}(Z)$ is the solution of the following minimization problem

$$\min_{X\in\Omega_1}\|X - Z\|, \tag{20}$$

and according to Lemma 3 we know that the solution of the minimization problem (20) is $Z + A^+(E - AZB)B^+$. Hence,

$$P_{\Omega_1}(Z) = Z + A^+(E - AZB)B^+. \tag{21}$$

□

Theorem 2. *Suppose that the set Ω_2 is nonempty. For a given $n \times n$ matrix Z, we have*

$$P_{\Omega_2}(Z) = Z + C^+(F - CZD)D^+. \tag{22}$$

Proof. The proof is similar to that of Theorem 1 and is omitted here. □

By the alternation projection algorithm (6) and Theorems 1 and 2, we get a new algorithm to solve Problem 1 which can be stated as follows.

Algorithm 1. One has

(1) set $\widetilde{A} = A^+$, $\widetilde{B} = B^+$, $\widetilde{C} = C^+$, $\widetilde{D} = D^+$;

(2) set $X_0 = \overline{X}$;

(3) for $k = 0, 1, 2, 3, \ldots$

$$\begin{aligned}Y_{k+1} &= P_{\Omega_1}(X_k) = X_k + \widetilde{A}(E - AX_kB)\widetilde{B},\\ X_{k+1} &= P_{\Omega_2}(Y_{k+1}) = Y_{k+1} + \widetilde{C}(F - CY_{k+1}D)\widetilde{D},\end{aligned} \tag{23}$$

end.

By Lemma 1 and (15) and (16), we get the convergence theorem for Algorithm 1.

Theorem 3. *If the set Ω is nonempty, then the matrix sequences $\{X_k\}$ and $\{Y_k\}$ generated by Algorithm 1 both converge to the projection $P_{\Omega_1\cap\Omega_2}(\overline{X})$ of $\overline{X}$ onto the intersection of Ω_1 and Ω_2, that is,*

$$X_k \longrightarrow P_{\Omega_1\cap\Omega_2}\left(\overline{X}\right), \quad Y_k \longrightarrow P_{\Omega_1\cap\Omega_2}\left(\overline{X}\right), \quad k \longrightarrow +\infty. \tag{24}$$

Proof. If the set Ω is nonempty, by (15) we have

$$\Omega_1 \cap \Omega_2 \neq \emptyset. \tag{25}$$

And noting (16), we know that the sets Ω_1 and Ω_2 are closed affine subspaces in Hilbert space $R^{m\times n}$. Hence, by Lemma 1 we derive that the matrix sequences $\{X_k\}$ and $\{Y_k\}$ generated by Algorithm 1 both converge to the projection $P_{\Omega_1\cap\Omega_2}(\overline{X})$ of $\overline{X}$ onto the intersection of Ω_1 and Ω_2, that is,

$$X_k \longrightarrow P_{\Omega_1\cap\Omega_2}\left(\overline{X}\right), \quad Y_k \longrightarrow P_{\Omega_1\cap\Omega_2}\left(\overline{X}\right), \quad k \longrightarrow +\infty. \tag{26}$$

□

Combining Theorem 3 and the equalities (18) and (17), we have the following.

Theorem 4. *If the set Ω is nonempty, then the matrix sequence $\{X_k\}$ and $\{Y_k\}$ generated by Algorithm 1 both converge to the unique solution of Problem 1. Moreover, if the initial matrix $X_0 = \overline{X} = 0$, then the matrix sequence $\{X_k\}$ and $\{Y_k\}$ both converge to the least Frobenius norm solution of the matrix equations $AXB = E$, $CXD = F$.*

3. Numerical Experiments

In this section, we give some numerical examples to illustrate that the new algorithm is feasible and effective to solve Problem 1. All programs are written in MATLAB 7.8. We denote

$$\text{Error} = \|E - AXB\| + \|F - CXD\|, \tag{27}$$

and use the practical stopping criterion Error $\leq 1.0 \times 10^{-10}$.

Example 1. Consider Problem 1 with

$$A = \begin{pmatrix} 1 & 3 & 1 & 3 & 1 \\ 3 & -7 & 3 & -7 & 3 \\ 11 & 6 & 11 & 6 & 11 \\ -5 & 5 & -5 & 5 & -5 \\ 9 & 4 & 9 & 4 & 9 \\ 1 & 3 & 1 & 3 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} -1 & 4 & -1 & 4 & -1 \\ 5 & -1 & 5 & -1 & 5 \\ -1 & -2 & -1 & -2 & -1 \\ 3 & 9 & 3 & 9 & 3 \\ 7 & -8 & 7 & -8 & 7 \end{pmatrix},$$

$$E = \begin{pmatrix} 117 & 18 & 117 & 18 & 117 \\ -65 & -10 & -65 & -10 & -65 \\ 585 & 90 & 585 & 90 & 585 \\ -65 & -10 & -65 & -10 & -65 \\ 455 & 70 & 455 & 70 & 455 \\ 117 & 18 & 117 & 18 & 117 \end{pmatrix}, \quad (28)$$

$$C = \begin{pmatrix} 3 & -5 & 3 & -5 & 7 \\ -4 & 3 & -2 & 9 & 5 \\ 3 & -5 & 3 & -5 & 7 \end{pmatrix},$$

$$D = \begin{pmatrix} 5 & 4 & -1 & 5 & 1 & 4 \\ -2 & 3 & 5 & -2 & 5 & 3 \\ 3 & 5 & -1 & 3 & -1 & 5 \\ 3 & -6 & 3 & 2 & 3 & -6 \\ 1 & 19 & 7 & 1 & 7 & 19 \end{pmatrix},$$

$$F = \begin{pmatrix} 30 & 75 & 39 & 27 & 45 & 75 \\ 110 & 275 & 143 & 99 & 165 & 275 \\ 30 & 75 & 39 & 27 & 45 & 75 \end{pmatrix}.$$

Here we use ones(n) and zeros(n) to stand for $n \times n$ matrix of ones and zeros. It is easy to verify that $X = \text{ones}(5)$ is a solution of the matrix equations (5); that is to say, the set Ω is nonempty. Therefore we can use Algorithm 1 to solve Problem 1.

Let $X_0 = \overline{X} = \text{zeros}(5)$. After 5 iterations of Algorithm 1, we get the optimal approximate solution

$$\hat{X} \approx X_5 = \begin{pmatrix} -0.6817 & 0.7813 & -0.5503 & 0.9637 & 0.7808 \\ 0.0152 & 0.8185 & -0.2866 & 0.9699 & 0.8181 \\ 0.0113 & 0.8178 & -0.2917 & 0.9698 & 0.8173 \\ 0.6091 & 0.9280 & 0.4893 & 0.9980 & 0.9278 \\ 0.9497 & 0.9907 & 0.9342 & 0.9985 & 0.9907 \end{pmatrix}, \quad (29)$$

which is also the least Frobenius norm solution of the matrix equations (5), and its residual error

$$\text{Error} \approx \|E - AX_5B\| + \|F - CX_5D\| = 2.78 \times 10^{-11}. \quad (30)$$

By concrete computations, we know that the distance from $\overline{X}$ to the solution set Ω is

$$\min_{X\in\Omega}\left\|X - \overline{X}\right\| = \left\|\hat{X} - \overline{X}\right\| \approx \left\|X_5 - \overline{X}\right\| = 3.9057. \quad (31)$$

Let $X_0 = \overline{X} = \text{ones}(5)$. After 6 iterations of Algorithm 1, we get the optimal approximate solution

$$\hat{X} \approx X_6 = \begin{pmatrix} 5.7467 & 1.8747 & 7.2013 & 1.1452 & 1.8770 \\ 4.9392 & 1.7259 & 6.1463 & 1.1205 & 1.7278 \\ 4.9548 & 1.7288 & 6.1668 & 1.1209 & 1.7307 \\ 2.5636 & 1.2881 & 3.0427 & 1.478 & 1.2889 \\ 1.2013 & 1.0371 & 1.2630 & 1.0062 & 1.372 \end{pmatrix}, \quad (32)$$

and its residual error

$$\text{Error} \approx \|E - AX_6B\| + \|F - CX_6D\| = 3.89 \times 10^{-11}. \quad (33)$$

By concrete computations, we know that the distance from $\overline{X}$ to the solution set Ω is

$$\min_{X\in\Omega}\left\|X - \overline{X}\right\| = \left\|\hat{X} - \overline{X}\right\| \approx \left\|X_6 - \overline{X}\right\| = 16.0902. \quad (34)$$

Example 1 shows that Algorithm 1 is feasible to solve Problem 1.

Example 2. Consider Problem 1 with

$$\begin{aligned} A &= \text{rand}(100, n), \qquad B = \text{rand}(n, 150), \\ E &= A * \text{ones}(n) * B, \qquad C = \text{rand}(70, n), \\ D &= \text{rand}(n, 120), \qquad F = C * \text{ones}(n) * D, \end{aligned} \quad (35)$$

where rand(s, t) stand for $s \times t$ random matrix. Let the given matrix $\overline{X} = \text{zeros}(n)$. It is easy to verify that $X = \text{ones}(n)$ is the solution of the matrix equations (5); that is, the set Ω is nonempty; therefore, we can use Algorithm 1 and the following algorithm proposed by Sheng and Chen [13] to solve Problem 1.

Algorithm 2. One has

(1) input A, B, C, D, E, F and X_0;

(2) calculate

$$\begin{aligned} R_0 &= E - AX_0B, \\ r_0 &= F - CX_0D, \\ P_0 &= A^T R_0 B^T, \\ Q_0 &= C^T r_0 D^T; \end{aligned} \quad (36)$$

(3) if $R_k = 0$, $r_0 = 0$, then stop; else, $k := k + 1$;

Table 1

		Algorithm 1	Algorithm 2
$n = 50$	IT	5	36
	CPU	0.019262	0.125212
	ERR	3.46×10^{-11}	9.01×10^{-10}
	DIS	50.00	50.00
$n = 80$	IT	8	167
	CPU	0.096335	4.660955
	ERR	2.31×10^{-10}	9.26×10^{-10}
	MIN	80.00	80.0010
$n = 100$	IT	16	1673
	CPU	0.168511	17.604961
	ERR	5.21×10^{-10}	9.8×10^{-10}
	MIN	99.9997	100.0001
$n = 150$	IT	79	—
	CPU	2.589103	—
	ERR	5.21×10^{-10}	—
	MIN	150.0261	—

(4) calculate

$$X_k = X_{k-1} + \frac{\|R_{k-1}\|^2 + \|r_{k-1}\|^2}{\|P_{k-1} + Q_{k-1}\|^2}(P_{k-1} + Q_{k-1}),$$
$$R_k = E - AX_kB,$$
$$r_k = F - CX_kD, \tag{37}$$
$$P_k = A^T R_k B^T + \frac{\|R_k\|^2 + \|r_k\|^2}{\|R_{k-1}\|^2 + \|r_{k-1}\|^2},$$
$$Q_k = C^T r_k D^T + \frac{\|R_k\|^2 + \|r_k\|^2}{\|R_{k-1}\|^2 + \|r_{k-1}\|^2} Q_{k-1};$$

(5) go to step 3.

It is easy to see that Algorithm 1 has lower computational cost for each step in the comparison with Algorithm 2. Experiments show that Algorithm 1 and Algorithm 2 are feasible to solve Problem 1. We list the iteration steps (denoted by IT), CPU time (denoted by CPU), residual error (denoted by ERR), and the distance $\|X_{\mathrm{IT}} - \overline{X}\|$ (denoted by DIS) in Table 1.

From Table 1 , we can see that Algorithm 1 outperforms Algorithm 2 in iteration step and CPU time. Therefore our algorithm is faster than the algorithm proposed by Sheng and Chen [13]. Especially, the CPU time and iteration steps of our algorithm increase slowly as the dimension n is increasing; so our algorithm is suitable for large-scale problems.

4. Conclusion

The alternating projection algorithm dates back to von Neumann [18], who treated the problem of finding the projection of a given point onto the intersection of two closed subspace. In this paper, we first apply the alternating projection algorithm to solve Problem 1, which occurs frequently in experimental design [23]. If we choose the initial matrix $X_0 = 0$, the least Frobenius norm solution of the matrix equations $AXB = E$, $CXD = F$ can be obtained. Numerical examples show that the new algorithm is faster and lower computational cost for each step than the algorithm proposed by Sheng and Chen [13] to solve Problem 1. Especially, the CPU time and iteration steps of the new algorithm increase slowly as the matrix's dimension is increasing; so the alternating projection algorithm is suitable for large-scale matrix nearness problems.

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