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Spectra and fine spectra of certain lower triangular double-band matrices as operators on c_0

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Abstract

In this paper we determine the fine spectrum of the generalized difference operator $\Delta_{a,b}$ defined by a lower triangular double-band matrix over the sequence space c_0. The class of the operator $\Delta_{a,b}$ contains as special cases many operators that have been studied recently in the literature. Illustrative examples showing the advantage of the present results are also given.

MSC: Primary 47A10; secondary 47B37

Keywords: spectrum of an operator; infinite lower triangular matrices; sequence spaces

1 Introduction

Several authors have studied the spectrum and fine spectrum of linear operators defined by lower and upper triangular matrices over some sequence spaces [1–21].

Let X be a Banach space. By $R(T)$, T^*, X^*, $B(X)$, $\sigma(T,X)$, $\sigma_p(T,X)$, $\sigma_r(T,X)$ and $\sigma_c(T,X)$, we denote the range of T, the adjoint operator of T, the space of all continuous linear functionals on X, the space of all bounded linear operators on X into itself, the spectrum of T on X, the point spectrum of T on X, the residual spectrum of T on X and the continuous spectrum of T on X, respectively. We shall write c and c_0 for the spaces of all convergent and null sequences, respectively. Also by l_1 we denote the space of all absolutely summable sequences.

We assume here some familiarity with basic concepts of spectral theory and we refer to Kreyszig [22, pp.370-372] for basic definitions such as spectrum, point spectrum, residual spectrum, and continuous spectrum of linear operators in normed spaces. Also, we refer to Goldberg [23, pp.58-71] for Goldberg's classification of spectra.

Now, let (a_k) and (b_k) be two convergent sequences of nonzero real numbers with

$$\lim_{k\to\infty} a_k = a \quad \text{and} \quad \lim_{k\to\infty} b_k = b \neq 0. \tag{1}$$

We consider the operator $\Delta_{a,b} : c_0 \to c_0$, which is defined as follows:

$$\Delta_{a,b}x = \Delta_{a,b}(x_k) = (a_k x_k + b_{k-1}x_{k-1})_{k=0}^{\infty} \quad \text{with } x_{-1} = b_{-1} = 0. \tag{2}$$

It is easy to verify that the operator $\Delta_{a,b}$ can be represented by a lower triangular double-band matrix of the form

$$\Delta_{a,b} = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \cdots \\ 0 & b_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3}$$

We begin by determining when a matrix A induces a bounded linear operator from c_0 to itself.

Lemma 1.1 (*cf.* [24, p.129]) *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if*

(1) *the rows of A are in l_1 and their l_1 norms are bounded,*

(2) *the columns of A are in c_0.*

The operator norm of T is the supremum of the l_1 norms of the rows.

As a consequence of the above lemma, we have the following corollary for the bounded linearity of the operator $\Delta_{a,b}$ on the space c_0.

Corollary 1.1 *The operator $\Delta_{a,b} : c_0 \longrightarrow c_0$ is a bounded linear operator with the norm $\|\Delta_{a,b}\|_{c_0} = \sup_k(|a_k| + |b_{k-1}|)$.*

The rest of the paper is organized as follows. In Section 2, we analyze the spectrum of the operator $\Delta_{a,b}$ on the sequence space c_0. In Section 3 we give some illustrative examples. Finally, Section 4 concludes with remarks and some special cases.

2 Fine spectrum of the operator $\Delta_{a,b}$ on c_0

In this section we examine the spectrum, the point spectrum, the residual spectrum and the continuous spectrum of the operator $\Delta_{a,b}$ on the sequence space c_0.

Theorem 2.1 *Let $D = \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\}$ and $E = \{a_k : k \in \mathbb{N}, |a_k - a| > |b|\}$. Then $\sigma(\Delta_{a,b}, c_0) = D \cup E$.*

Proof First, we prove that $(\Delta_{a,b} - \lambda I)^{-1}$ exists and is in $B(c_0)$ for $\lambda \notin D \cup E$ and then the operator $\Delta_{a,b} - \lambda I$ is not invertible for $\lambda \in D \cup E$.

Let $\lambda \notin D \cup E$. Then $|\lambda - a| > |b|$ and $\lambda \neq a_k$, for all $k \in \mathbb{N}$. So, $(\Delta_{a,b} - \lambda I)$ is triangle and hence $(\Delta_{a,b} - \lambda I)^{-1}$ exists. We can calculate that

$$(\Delta_{a,b} - \lambda I)^{-1} = (s_{kj}) = \begin{pmatrix} \frac{1}{(a_0-\lambda)} & 0 & 0 & \cdots \\ \frac{-b_0}{(a_0-\lambda)(a_1-\lambda)} & \frac{1}{(a_1-\lambda)} & 0 & \cdots \\ \frac{b_0 b_1}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)} & \frac{-b_1}{(a_1-\lambda)(a_2-\lambda)} & \frac{1}{(a_2-\lambda)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now, for each $k \in \mathbb{N}$, the series $S_k = \sum_j |s_{kj}|$ is convergent since it is finite. Next, we prove that $\sup_k S_k$ is finite.

Since $\lim_{k\to\infty}|\frac{b_k}{a_k-\lambda}| = |\frac{b}{a-\lambda}| < 1$, then there exist $k_0 \in \mathbb{N}$ and $q_0 < 1$ such that $|\frac{b_k}{a_k-\lambda}| < q_0$, for all $k \geq k_0$. Then, for each $k > k_0$,

$$\begin{aligned} S_k &= \frac{1}{|a_k-\lambda|}\biggl[1 + \frac{|b_{k-1}|}{|a_{k-1}-\lambda|} + \frac{|b_{k-1}||b_{k-2}|}{|a_{k-1}-\lambda||a_{k-2}-\lambda|} \\ &\quad + \cdots + \frac{|b_{k-1}||b_{k-2}|\cdots|b_{k_0}|}{|a_{k-1}-\lambda||a_{k-2}-\lambda|\cdots|a_{k_0}-\lambda|} \\ &\quad + \cdots + \frac{|b_{k-1}||b_{k-2}|\cdots|b_{k_0+1}||b_{k_0}|\cdots|b_0|}{|a_{k-1}-\lambda||a_{k-2}-\lambda|\cdots|a_{k_0+1}-\lambda||a_{k_0}-\lambda|\cdots|a_0-\lambda|}\biggr] \\ &\leq \frac{1}{|a_k-\lambda|}\biggl[1 + q_0 + q_0^2 + \cdots + q_0^{k-k_0} + q_0^{k-k_0}\frac{|b_{k_0-1}|}{|a_{k_0-1}-\lambda|} + q_0^{k-k_0}\frac{|b_{k_0-1}||b_{k_0-2}|}{|a_{k_0-1}-\lambda||a_{k_0-2}-\lambda|} \\ &\quad + \cdots + q_0^{k-k_0}\frac{|b_{k_0-1}||b_{k_0-2}|\cdots|b_0|}{|a_{k_0-1}-\lambda||a_{k_0-2}-\lambda|\cdots|a_0-\lambda|}\biggr]. \end{aligned}$$

Therefore

$$S_k \leq \frac{1}{|a_k-\lambda|}\bigl[1 + q_0 + q_0^2 + \cdots + q_0^{k-k_0} m_{k_0}\bigr],$$

where

$$m_{k_0} = 1 + \frac{|b_{k_0-1}|}{|a_{k_0-1}-\lambda|} + \frac{|b_{k_0-1}||b_{k_0-2}|}{|a_{k_0-1}-\lambda||a_{k_0-2}-\lambda|} + \cdots + \frac{|b_{k_0-1}||b_{k_0-2}|\cdots|b_0|}{|a_{k_0-1}-\lambda||a_{k_0-2}-\lambda|\cdots|a_0-\lambda|}.$$

Then $m_{k_0} > 1$ and so

$$S_k \leq \frac{m_{k_0}}{|a_k-\lambda|}\bigl[1 + q_0 + q_0^2 + \cdots + q_0^{k-k_0}\bigr].$$

But there exist $k_1 \in \mathbb{N}$ and a real number $q_1 < \frac{1}{|b|}$ such that $\frac{1}{|a_k-\lambda|} < q_1$, for all $k \geq k_1$. Then

$$S_k \leq \frac{q_1 m_{k_0}}{1-q_0},$$

for all $k > \max\{k_0, k_1\}$. Thus $\sup_k S_k < \infty$.

Also, it is easy to see that $\lim_{k\to\infty}|s_{kj}| = 0$, for all $j \in \mathbb{N}$, since

$$\lim_{k\to\infty}\left|\frac{s_{k+1,j}}{s_{k,j}}\right| = \lim_{k\to\infty}\left|\frac{b_k}{a_{k+1}-\lambda}\right| = \left|\frac{b}{a-\lambda}\right| < 1.$$

So, the sequence $(s_{0j}, s_{1j}, s_{2j}, \ldots)$ converges to zero, for each $j \in \mathbb{N}$. This shows that the columns of $(\Delta_{a,b} - \lambda I)^{-1}$ are in c_0. Then, from Lemma 1.1, $(\Delta_{a,b} - \lambda I)^{-1} \in B(c_0)$ and so, $\lambda \notin \sigma(\Delta_{a,b}, c_0)$. Thus $\sigma(\Delta_{a,b}, c_0) \subseteq D \cup E$.

Conversely, suppose that $\lambda \notin \sigma(\Delta_{a,b}, c_0)$. Then $(\Delta_{a,b} - \lambda I)^{-1} \in B(c_0)$. Since the $(\Delta_{a,b} - \lambda I)^{-1}$ transform of the unite sequence $e_0 = (1, 0, 0, \ldots)$ is in c_0, we have $\lim_{k\to\infty}|\frac{b_k}{a_{k+1}-\lambda}| = |\frac{b}{a-\lambda}| \leq 1$ and $\lambda \neq a_k$, for all $k \in \mathbb{N}$. Then $\{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \subseteq \sigma(\Delta_{a,b}, c_0)$ and $\{a_k : k \in \mathbb{N}\} \subseteq \sigma(\Delta_{a,b}, c_0)$. But $\sigma(\Delta_{a,b}, c_0)$ is a compact set, and so it is closed. Then $D = \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\} \subseteq \sigma(\Delta_{a,b}, c_0)$ and $E = \{a_k : k \in \mathbb{N}, |a_k - a| > |b|\} \subseteq \sigma(\Delta_{a,b}, c_0)$. This completes the proof. □

Theorem 2.2 $\sigma_p(\Delta_{a,b}, c_0) = E \cup K$, *where*

$$K = \left\{ a_j : j \in \mathbb{N}, |a_j - a| = |b|, \prod_{i=m}^{\infty} \frac{b_{i-1}}{a_j - a_i} \text{ diverges to zero for some } m \in \mathbb{N} \right\}.$$

Proof Suppose $\Delta_{a,b}x = \lambda x$ for any $x \in c_0$. Then we obtain

$$(a_0 - \lambda)x_0 = 0 \quad \text{and} \quad b_k x_k + (a_{k+1} - \lambda)x_{k+1} = 0, \quad \text{for all } k \in \mathbb{N}.$$

If the sequence (a_k) is constant, then we can easily see that $x = \theta$ and so, $\sigma_p(\Delta_{a,b}, c_0) = \emptyset$ and the result follows immediately. Now, if the sequence (a_k) is not constant, then for all $\lambda \notin \{a_k : k \in \mathbb{N}\}$, we have $x_k = 0$ for all $k \in \mathbb{N}$. So, $\lambda \notin \sigma_p(\Delta_{a,b}, c_0)$. Also, we can easily prove that $a \notin \sigma_p(\Delta_{a,b}, c_0)$. Thus $\sigma_p(\Delta_{a,b}, c_0) \subseteq \{a_k : k \in \mathbb{N}\} \backslash \{a\}$. Now, we will prove that

$$\lambda \in \sigma_p(\Delta_{a,b}, c_0) \quad \text{if and only if} \quad \lambda \in E \cup K.$$

If $\lambda \in \sigma_p(\Delta_{a,b}, c_0)$, then $\lambda = a_j \neq a$ for some $j \in \mathbb{N}$ and there exists $x \in c_0$, $x \neq \theta$ such that $\Delta_{a,b}x = a_j x$. Then

$$\lim_{k \to \infty} \left| \frac{x_{k+1}}{x_k} \right| = \left| \frac{b}{a - a_j} \right| \leq 1.$$

Then $\lambda = a_j \in E$ or $|a_j - a| = |b|$. In the case when $|a_j - a| = |b|$, we have

$$x_k = \frac{b_{k-1} b_{k-2} \cdots b_{m-1}}{(a_j - a_k)(a_j - a_{k-1}) \cdots (a_j - a_m)} x_{m-1} = x_{m-1} \prod_{i=m}^{k} \frac{b_{i-1}}{a_j - a_i}, \quad k \geq m.$$

Then $\prod_{i=m}^{\infty} \frac{b_{i-1}}{a_j - a_i}$ diverges to 0, since $x \in c_0$. Therefore $\lambda \in E \cup K$. Thus $\sigma_p(\Delta_{a,b}, c_0) \subseteq E \cup K$.

Conversely, let $\lambda \in E \cup K$. If $\lambda \in E$, then there exists $i \in \mathbb{N}$ such that $\lambda = a_i \neq a$ and so we can take $x \neq \theta$ such that $\Delta_{a,b}x = a_i x$ and

$$\lim_{k \to \infty} \left| \frac{x_{k+1}}{x_k} \right| = \left| \frac{b}{a - a_i} \right| < 1,$$

that is, $x \in c_0$. Also, if $\lambda \in K$, then there exists $j \in \mathbb{N}$ such that $\lambda = a_j \neq a$ and $|a_j - a| = |b|$, $\prod_{i=m}^{\infty} \frac{b_{i-1}}{a_j - a_i}$ diverges to 0, for some $m \in \mathbb{N}$. Then we can take $x \in c_0$, $x \neq \theta$ such that $\Delta_{a,b}x = a_j x$. Thus $E \cup K \subseteq \sigma_p(\Delta_{a,b}, c_0)$. This completes the proof. □

Theorem 2.3 $\sigma_p(\Delta^*_{a,b}, c^*_0) = \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup E \cup H$, *where*

$$H = \left\{ \lambda \in \mathbb{C} : |\lambda - a| = |b|, \sum_{k=0}^{\infty} \left| \prod_{i=0}^{k} \frac{\lambda - a_i}{b_i} \right| < \infty \right\}.$$

Proof Suppose that $\Delta^*_{a,b}f = \lambda f$ for $f = (f_0, f_1, f_2, \ldots) \neq \theta$ in $c_0^* \cong l_1$. Then, by solving the system of linear equations

$$\begin{aligned} &a_0f_0 + b_0f_1 = \lambda f_0, \\ &a_1f_1 + b_1f_2 = \lambda f_1, \\ &a_2f_2 + b_2f_3 = \lambda f_2, \\ &\vdots \end{aligned}$$

we obtain

$$f_{k+1} = \frac{\lambda - a_k}{b_k} f_k, \quad k \in \mathbb{N}.$$

Then we must take $f_0 \neq 0$ since otherwise we would have $f = \theta$. It is clear that for all $k \in \mathbb{N}$, the vector $f = (f_0, f_1, \ldots, f_k, 0, 0, \ldots)$ is an eigenvector of the operator $\Delta^*_{a,b}$ corresponding to the eigenvalue $\lambda = a_k$, where $f_0 \neq 0$ and $f_n = \frac{\lambda - a_{n-1}}{b_{n-1}} f_{n-1}$, for all $n = 1, 2, 3, \ldots, k$. Then $\{a_k : k \in \mathbb{N}\} \subseteq \sigma_p(\Delta^*_{a,b}, c_0^*)$. Also, if $\lambda \neq a_k$ for all $k \in \mathbb{N}$, then $f_k \neq 0$, for all $k \geq 1$ and $\sum_{k=0}^{\infty} |f_k| < \infty$ if $\lim_{k\to\infty} |\frac{f_{k+1}}{f_k}| = |\frac{\lambda - a}{b}| < 1$. Also, if $|\lambda - a| = |b|$, we can easily see that

$$|f_k| = \left|\frac{(\lambda - a_0)(\lambda - a_1)\cdots(\lambda - a_{k-1})}{b_0 b_1 \cdots b_{k-1}}\right| |f_0| = |f_0| \prod_{i=0}^{k-1} \left|\frac{\lambda - a_i}{b_i}\right|, \quad \text{for all } k \geq 1,$$

and so $\sum_{k=0}^{\infty} |f_k| < \infty$ if $\sum_{k=0}^{\infty} |\prod_{i=0}^{k} \frac{\lambda - a_i}{b_i}| < \infty$. This implies that $H \subseteq \sigma_p(\Delta^*_{a,b}, c_0^*)$. Thus

$$\{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup E \cup H \subseteq \sigma_p(\Delta^*_{a,b}, c_0^*).$$

The second inclusion can be proved analogously. □

The following lemma is required in the proof of the next theorem.

Lemma 2.1 [23, p.59] *T has a dense range if and only if T^* is one to one.*

Theorem 2.4 $\sigma_r(\Delta_{a,b}, c_0) = \sigma_p(\Delta^*_{a,b}, c_0^*) \backslash \sigma_p(\Delta_{a,b}, c_0)$.

Proof For $\lambda \in \sigma_p(\Delta^*_{a,b}, c_0^*) \backslash \sigma_p(\Delta_{a,b}, c_0)$, the operator $\Delta_{a,b} - \lambda I$ is one to one and hence has an inverse. But $\Delta^*_{a,b} - \lambda I$ is not one to one. Now, Lemma 2.1 yields the fact that the range of the operator $\Delta_{a,b} - \lambda I$ is not dense in c_0. This implies that $\lambda \in \sigma_r(\Delta_{a,b}, c_0)$. □

Theorem 2.5 $\sigma_r(\Delta_{a,b}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup (H \backslash K)$.

Proof The proof follows immediately from Theorems 2.2, 2.3, and 2.4. □

Theorem 2.6 $\sigma_c(\Delta_{a,b}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - a| = |b|\} \backslash H$.

Proof Since $\sigma(\Delta_{a,b}, c_0)$ is the disjoint union of the parts $\sigma_p(\Delta_{a,b}, c_0)$, $\sigma_r(\Delta_{a,b}, c_0)$ and $\sigma_c(\Delta_{a,b}, c_0)$ we must have $\sigma_c(\Delta_{a,b}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - a| = |b|\} \backslash H$. □

Also, we have the following result.

Theorem 2.7 $\sigma_c(\Delta_{a,b}, c_0) = \sigma(\Delta_{a,b}, c_0) \backslash \sigma_p(\Delta^*_{a,b}, c^*_0)$.

Proof The proof is obvious and so is omitted. □

With respect to Goldberg's classification of the spectrum of an operator (see [23, pp.58-71]), the spectrum is partitioned into nine states, which are I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, and III_3. For the operator $\Delta_{a,b} : c_0 \to c_0$, we have

$$I_3\sigma(\Delta_{a,b}, c_0) = II_3\sigma(\Delta_{a,b}, c_0) = \emptyset,$$

since $\sigma_p(\Delta_{a,b}, c_0) \subseteq \sigma_p(\Delta^*_{a,b}, c^*_0)$. Also, $I_2\sigma(\Delta_{a,b}, c_0) = \emptyset$, by the closed graph theorem. Thus we have to discuss the states II_2, III_1, III_2, and III_3.

Theorem 2.8 $\lambda \in \sigma_p(\Delta_{a,b}, c_0)$ *if and only if* $\lambda \in III_3\sigma(\Delta_{a,b}, c_0)$.

Proof The proof is obvious and so is omitted. □

Theorem 2.9 $\lambda \in \sigma_c(\Delta_{a,b}, c_0)$ *if and only if* $\lambda \in II_2\sigma(\Delta_{a,b}, c_0)$.

Proof Let $\lambda \in \sigma_c(\Delta_{a,b}, c_0)$. By Theorem 2.7, $\Delta^*_{a,b} - \lambda I$ is one to one. By Lemma 2.1, $\Delta_{a,b} - \lambda I$ has dense range. Additionally, $\lambda \notin \sigma_p(\Delta_{a,b}, c_0)$ implies that the operator $\Delta_{a,b} - \lambda I$ has inverse. Therefore, $\lambda \in II_2\sigma(\Delta_{a,b}, c_0)$ or $\lambda \in I_2\sigma(\Delta_{a,b}, c_0)$. But $I_2\sigma(\Delta_{a,b}, c_0) = \emptyset$. Thus $\lambda \in II_2\sigma(\Delta_{a,b}, c_0)$. □

Theorem 2.10 $\lambda \in \sigma_r(\Delta_{a,b}, c_0)$ *if and only if* $\lambda \in III_1\sigma(\Delta_{a,b}, c_0) \cup III_2\sigma(\Delta_{a,b}, c_0)$.

Proof Let $\lambda \in \sigma_r(\Delta_{a,b}, c_0)$. By Theorem 2.4, $\Delta^*_{a,b} - \lambda I$ is not one to one. By Lemma 2.1, $\Delta_{a,b} - \lambda I$ has not a dense range. Additionally, $\lambda \notin \sigma_p(\Delta_{a,b}, c_0)$ implies that the operator $\Delta_{a,b} - \lambda I$ has inverse. Therefore, $\lambda \in III_1\sigma(\Delta_{a,b}, c_0) \cup III_2\sigma(\Delta_{a,b}, c_0)$. □

3 Illustrative examples

In this section we provide some illustrative examples in support of our new results.

Example 3.1 Consider the sequences (a_k) and (b_k) defined by the following recurrence relations:

$$a_0 = \sqrt{2}, \qquad a_{k+1} = \sqrt{2 + a_k},$$
$$b_0 = \sqrt{2}, \qquad b_{k+1} = \sqrt{2b_k},$$

for all $k \in \mathbb{N}$. Then (a_k) and (b_k) are monotonically increasing sequences and $\lim_{k\to\infty} a_k = a = 2$ and $\lim_{k\to\infty} b_k = b = 2$. Also, $a_k \geq b_k$ for all $k \in \mathbb{N}$. Thus, for all $\lambda \in \mathbb{C}$ with $|\lambda - 2| = 2$, one can prove that $|\frac{\lambda - a_k}{b_k}| \geq 1$ for all $k \in \mathbb{N}$. This implies that $H = \emptyset$. Also, we can prove that $E = K = \emptyset$. Using Theorems 2.1, 2.2, 2.5, and 2.6, we have

$$\sigma(\Delta_{a,b}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 2| \leq 2\},$$
$$\sigma_p(\Delta_{a,b}, c_0) = \emptyset,$$

$$\sigma_r(\Delta_{a,b}, c_0) = \bigl\{\lambda \in \mathbb{C} : |\lambda - 2| < 2\bigr\},$$

$$\sigma_c(\Delta_{a,b}, c_0) = \bigl\{\lambda \in \mathbb{C} : |\lambda - 2| = 2\bigr\}.$$

Example 3.2 Let $a_k = \frac{k+1}{k+2}$ and $b_k = \frac{k+1}{k+3}$ for all $k \in \mathbb{N}$. Then $\lim_{k\to\infty} a_k = a = 1$ and $\lim_{k\to\infty} b_k = b = 1$. Similarly, as in Example 3.1, we can prove that $E = K = H = \emptyset$ and so

$$\sigma(\Delta_{a,b}, c_0) = \bigl\{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\bigr\},$$

$$\sigma_p(\Delta_{a,b}, c_0) = \emptyset,$$

$$\sigma_r(\Delta_{a,b}, c_0) = \bigl\{\lambda \in \mathbb{C} : |\lambda - 1| < 1\bigr\},$$

$$\sigma_c(\Delta_{a,b}, c_0) = \bigl\{\lambda \in \mathbb{C} : |\lambda - 1| = 1\bigr\}.$$

Example 3.3 Consider the sequences (a_k) and (b_k) defined by the following recurrence relations:

$$a_0 = 4, \qquad a_1 = 2, \qquad a_k = 1 \quad \text{for } k \geq 2,$$

$$b_0 = 1, \qquad b_1 = 1, \qquad b_k = \left(\frac{k+1}{k}\right)^2 \quad \text{for } k \geq 2.$$

Therefore, $\lim_{k\to\infty} a_k = a = 1$ and $\lim_{k\to\infty} b_k = b = 1$. Then $E = \{4\}$, $K = \emptyset$ and $H = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\}$, and so

$$\sigma(\Delta_{a,b}, c_0) = \bigl\{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\bigr\} \cup \{4\},$$

$$\sigma_p(\Delta_{a,b}, c_0) = \{4\},$$

$$\sigma_r(\Delta_{a,b}, c_0) = \bigl\{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\bigr\},$$

$$\sigma_c(\Delta_{a,b}, c_0) = \emptyset.$$

Remark 3.1 From the above examples, we note that the spectrum of the operator $\Delta_{a,b}$ on the space c_0 may include also a finite number of points outside a region enclosed by a circle. Also, we may have $\sigma_p(\Delta_{a,b}, c_0) \neq \emptyset$.

Example 3.4 Let the sequences (a_k) and (b_k) be taken such that $a_k = -b_k = \frac{(k+2)^2}{(k+2)^2+(k+3)^2}$, $k \in \mathbb{N}$. Then we can prove that $E = K = H = \emptyset$ and so we have

$$\sigma(\Delta_{a,b}, c_0) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\},$$

$$\sigma_p(\Delta_{a,b}, c_0) = \emptyset,$$

$$\sigma_r(\Delta_{a,b}, c_0) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\},$$

$$\sigma_c(\Delta_{a,b}, c_0) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| = \frac{1}{2}\right\}.$$

4 Remarks and some special cases

In this section we are going to give some special cases of the operator $\Delta_{a,b}$ which has been studied recently. More precisely, we show that special conditions on the sequences (a_k) and (b_k) characterize certain special cases of the operator $\Delta_{a,b}$.

The difference operator Δ: If $a_k = 1$ and $b_k = -1$ for all $k \in \mathbb{N}$, then the operator $\Delta_{a,b}$ reduces to the backward difference operator Δ (*cf.* [7]).

The generalized difference operator $B(r,s)$: If $a_k = r$ and $b_k = s \neq 0$ for all $k \in \mathbb{N}$, then the operator $\Delta_{a,b}$ reduces to the operator $B(r,s)$ (*cf.* [8]).

The generalized difference operator Δ_v: If $a_k = -b_k = v_k$ for all $k \in \mathbb{N}$, then the operator $\Delta_{a,b}$ reduces to the operator Δ_v (*cf.* [19]).

The generalized difference operator Δ_{uv}: If (a_k) is a sequence of positive real numbers such that $a_k \neq 0$ for all $k \in \mathbb{N}$ with $\lim_{k\to\infty} a_k = U \neq 0$ and (b_k) is either constant or strictly decreasing sequence of positive real numbers with $\lim_{k\to\infty} b_k = V \neq 0$ and $\sup_k a_k < U + V$, then the operator $\Delta_{a,b}$ reduces to the operator Δ_{uv} (*cf.* [12]).

Remark 4.1 If (a_k) and (b_k) are convergent sequences of nonzero real numbers such that

$$\lim_{k\to\infty} a_k = a > 0, \tag{4}$$

$$\lim_{k\to\infty} b_k = b; \quad |b| = a, \tag{5}$$

and

$$\sup_k a_k \leq a, \qquad b_k^2 \leq a_k^2 \quad \text{for all } k \in \mathbb{N}, \tag{6}$$

then we can prove that $H = \emptyset$ and so we have:

$$\begin{aligned}
&\sigma(\Delta_{a,b}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - a| \leq a\} \cup \{a_k : k \in \mathbb{N}, |a_k - a| > a\},\\
&\sigma_p(\Delta_{a,b}, c_0) = \{a_k : k \in \mathbb{N}, |a_k - a| > a\},\\
&\sigma_p(\Delta_{a,b}^*, c_0^*) = \{\lambda \in \mathbb{C} : |\lambda - a| < a\} \cup \{a_k : k \in \mathbb{N}, |a_k - a| > a\},\\
&\sigma_r(\Delta_{a,b}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - a| < a\},\\
&\sigma_c(\Delta_{a,b}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - a| = a\}.
\end{aligned}$$

It is immediate that our new results cover a wider class of linear operators which are represented by infinite lower triangular double-band matrices on the sequence space c_0. For this reason, our study is more general and more comprehensive than the previous work. We note that our new results in this paper improve and generalize the results which have been stated in [3, 12].

Competing interests

The author declares that he has no competing interests.

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