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Endpoint estimates for vector-valued multilinear commutator of fractional area integral operator

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Abstract
In this paper, we prove the endpoint estimates for vector-valued multilinear commutator of fractional area integral operator.

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1 Introduction

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator, the commutator $[b, T]$ generated by b and T is defined by

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberb and Weiss (see [1]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$ $(1 < p < \infty)$. In [2–4], the boundedness properties of the commutators for the extreme values of p are obtained. In this paper, we introduce vector-valued multilinear commutator of fractional area integral operator and prove the endpoint estimates for the commutator $|S^{\vec{b}}_{\psi,\delta}|_r$ generated by the fractional area integral operator $S_{\psi,\delta}$ and BMO functions.

2 Notations and results

We give the following definitions (see [2, 3, 5–7]).

Definition 1 Let $0 < \delta < n$, a function ψ satisfies:

(1) $\int_{R^n} \psi(x)\,dx = 0$;
(2) $|\psi(x)| \le C(1 + |x|)^{-(n+1-\delta)}$;
(3) $|\psi(x + y) - \psi(x)| \le C|y|^{\varepsilon}(1 + |x|)^{-(n+2-\delta)}$, $2|y| < |x|$.

Suppose that $1 < r < \infty$, b_j $(j = 1, \dots, m)$ are the fixed locally integrable functions on R^n. Set $\Gamma(x) = \{(y, t) \in R^{n+1}_+ : |x - y| \le t\}$ and the eigenfunction by $\chi_{\Gamma(x)}$. We define the vector-valued multilinear commutator of fractional area integral operator by

$$\left|S^{\vec{b}}_{\psi,\delta}(f)(x)\right|_r = \left(\sum_{i=1}^{\infty}\left(S^{\vec{b}}_{\psi,\delta}(f_i)(x)\right)^r\right)^{1/r},$$

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where

$$S_{\psi,\delta}^{\vec{b}}(f)(x) = \left(\int_{\Gamma(x)} \left| F_t^{\vec{b}}(f)(x,y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2}$$

and

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \left[\prod_{j=1}^{m} \left(b_j(x) - b_j(z) \right) \right] \psi_t(y-z) f(z)\,dz.$$

Definition 2 We call a locally integrable function b in the central BMO space, namely $CMO(R^n)$, if the function b satisfies

$$\|b\|_{CMO} = \sup_{r>1} \left| Q(0,r) \right|^{-1} \int_Q \left| b(y) - b_Q \right| dy < \infty.$$

We have

$$\|b\|_{CMO} \approx \sup_{r>1} \inf_{c \in C} \left| Q(0,r) \right|^{-1} \int_Q \left| b(y) - c \right| dy.$$

Definition 3 Let $0 < \delta < n$, $1 < p < n/\delta$. We call a locally integrable function b in $B_p^\delta(R^n)$, if the function b satisfies

$$\|b\|_{B_p^\delta} = \sup_{r>1} r^{-n(1/p-\delta/n)} \|b\chi_{Q(0,r)}\|_{L^p} < \infty.$$

Now we state our theorems as follows.

Theorem 1 *Suppose* $1 < r < \infty$, $0 < \delta < n$, *and* $\vec{b} = (b_1, \dots, b_m)$ *for* $b_j \in BMO$, $1 \le j \le m$. *Then* $|S_{\psi,\delta}^{\vec{b}}|_r$ *is bounded from* $L^{n/\delta}$ *to* $BMO(R^n)$.

Theorem 2 *Let* $1 < r < \infty$, $0 < \delta < n$, $1 < p < n/\delta$, *and* $\vec{b} = (b_1, \dots, b_m)$, *with* $b_j \in BMO(R^n)$, *for* $1 \le j \le m$. *Then* $|S_{\psi,\delta}^{\vec{b}}|_r$ *is bounded from* $B_p^\delta(R^n)$ *to* $CMO(R^n)$.

3 Proofs of theorems

We begin with a preliminaries lemma.

Lemma 1 (see [3, 4]) *Let* $1 < r < \infty$, $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$. *Then* $|S_{\psi,\delta}|_r$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$.

Proof of Theorem 1 It is only to prove that there exists a constant C_Q, the following inequality holds:

$$\frac{1}{|Q|} \int_Q \left| \left| S_{\psi,\delta}^{\vec{b}}(f)(x) \right|_r - C_Q \right| dx \le C \left\| |f|_r \right\|_{L^{n/\delta}}.$$

Fix a cube $Q = Q(x_0, r)$, let $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i \chi_Q$, $h_i = f_i \chi_{(Q)^c}$.
When $m = 1$, set $(b_1)_Q = |Q|^{-1} \int_Q b_1(y)\,dy$, then

$$F_t^{b_1}(f_i)(x,y) = \left(b_1(x) - (b_1)_Q \right) F_t(f_i)(y) - F_t\left(\left(b_1 - (b_1)_Q \right) g_i \right)(y) - F_t\left(\left(b_1 - (b_1)_Q \right) h_i \right)(y),$$

so

$$
\begin{aligned}
&\left|S_{\psi,\delta}^{b_1}(f)(x)\right|_r - \left|S_{\psi,\delta}\big(\big((b_1)_{2Q} - b_1\big)h\big)(x_0)_r\right| \\
&\quad\le \left(\sum_{i=1}^{\infty}\left\|\chi_{\Gamma(x)}\big(b_1(x) - (b_1)_Q\big)F_t(f_i)(y)\right\|^r\right)^{1/r} \\
&\qquad + \left(\sum_{i=1}^{\infty}\left\|\chi_{\Gamma(x)}F_t\big(\big((b_1)_Q - b_1\big)g_i\big)(y)\right\|^r\right)^{1/r} \\
&\qquad + \left\|\chi_{\Gamma(x)}F_t\big(\big(b_1 - (b_1)_Q\big)f_2\big)(y) - \chi_{\Gamma(x_0)}F_t\big(\big(b_1 - (b_1)_Q\big)h\big)(y)\right\|_r \\
&\quad= A(x) + B(x) + C(x).
\end{aligned}
$$

For $A(x)$, suppose $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $1/q + 1/q' = 1$, by the Hölder inequality, then

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q\left|A(x)\right|dx &= \frac{1}{|Q|}\int_Q\left|b_1(x) - (b_1)_Q\right|\left|S_{\psi,\delta}(f)(x)\right|_r dx \\
&\le \left(\frac{1}{|Q|}\int_Q\left|b_1(x) - (b_1)_Q\right|^{q'} dx\right)^{1/q'} \\
&\quad \times \left(\frac{1}{|Q|}\int_{R^n}\left|S_{\psi,\delta}(f)(x)\right|_r^q \chi_Q(x)\,dx\right)^{1/q} \\
&\le C\|b_1\|_{BMO}|Q|^{-1/q}\left(\int_{R^n}|f(x)|_r^p \chi_Q(x)\,dx\right)^{1/p} \\
&\le C\|b_1\|_{BMO}|Q|^{-1/q} \\
&\quad \times \left[\left(\int_{R^n}|f(x)|_r^{n/\delta}\,dx\right)^{\delta p/n}\left(\int_Q \chi_Q(x)\,dx\right)^{1-\delta p/n}\right]^{1/p} \\
&\le C\|b_1\|_{BMO}|Q|^{-1/q}\left\||f|_r\right\|_{L^{n/\delta}}|Q|^{(1-\delta p/n)/p} \\
&\le C\|b_1\|_{BMO}\left\||f|_r\right\|_{L^{n/\delta}}.
\end{aligned}
$$

For $B(x)$, fix $1 < u < n/\delta$, $1/v = 1/u - \delta/n$, by the Hölder inequality, then

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q\left|B(x)\right|dx \\
&\quad= \frac{1}{|Q|}\int_Q\left|S_{\psi,\delta}\big(\big(b_1 - (b_1)_Q\big)g\big)(x)\right|_r dx \\
&\quad\le \left(\frac{1}{|Q|}\int_{R^n}\left|S_{\psi,\delta}\big(\big(b_1 - (b_1)_Q\big)g\big)(x)\big)^v\,dx\right|_r\right)^{1/v} \\
&\quad\le C|Q|^{-1/v}\left(\int_{R^n}\left|b_1(x) - (b_1)_Q\right|^u|f(x)|_r^u \chi_Q(x)\,dx\right)^{1/u} \\
&\quad\le C\left(\frac{1}{|Q|}\int_Q\left|b_1(x) - (b_1)_Q\right|^u dx\right)^{1/u}\left\||f|_r\right\|_{L^{n/\delta}} \\
&\quad\le C\|b_1\|_{BMO}\left\||f|_r\right\|_{L^{n/\delta}}.
\end{aligned}
$$

For $C(x)$, we have

$$\begin{aligned}
C(x) &= \big\| \chi_{\Gamma(x)} F_t\big((b_1-(b_1)_Q)f_2\big)(y) - \chi_{\Gamma(x_0)} F_t\big((b_1-(b_1)_Q)h\big)(y)\big\|_r \\
&\le \left[\int\int_{R_+^{n+1}} \left(\int_{Q^c} |\chi_{\Gamma(x)} - \chi_{\Gamma(x_0)}\| b_1(z)-(b_1)_Q\|\psi_t(y-z)\|f(z)|_r\,dz\right)^2 \frac{dy\,dt}{t^{n+1}}\right]^{1/2} \\
&\le C\int_{Q^c} |b_1(z)-(b_1)_Q||f(z)|_r \\
&\quad\times \left|\int\int_{|x-y|\le t} \frac{t^{1-n}\,dy\,dt}{(t+|y-z|)^{2n+2-2\delta}} - \int\int_{|x_0-y|\le t} \frac{t^{1-n}\,dy\,dt}{(t+|y-z|)^{2n+2-2\delta}}\right|^{1/2} dz \\
&\le C\int_{Q^c} |b_1(z)-(b_1)_Q||f(z)|_r \\
&\quad\times \left(\int\int_{|y|\le t, |x+y-z|\le t} \left|\frac{1}{(t+|x+y-z|)^{2n+2-2\delta}} \right.\right. \\
&\quad\left.\left. - \frac{1}{(t+|x_0+y-z|)^{2n+2-2\delta}}\right| \frac{dy\,dt}{t^{n-1}}\right)^{1/2} dz \\
&\le \int_{Q^c} |b_1(z)-(b_1)_Q||f(z)|_r \left(\int\int_{|y|\le t, |x+y-z|\le t} \frac{|x-x_0|t^{1-n}}{(t+|x+y-z|)^{2n+3-2\delta}}\,dy\,dt\right)^{1/2} dz.
\end{aligned}$$

Notice that when $|y| \le t$, $2t + |x+y-z| \ge 2t + |x-z| - |y| \ge t + |x-z|$, and

$$\int_0^\infty \frac{t\,dt}{(t+|x-z|)^{2n+3-2\delta}} = C|x-z|^{-2n-1+2\delta},$$

then, for $x \in Q$,

$$\begin{aligned}
C(x) &\le \int_{Q^c} |b_1(z)-(b_1)_Q||f(z)|_r \left(\int\int_{|y|\le t} \frac{2^{2n+3-2\delta}|x_0-x|t^{1-n}\,dy\,dt}{(2t+2|x+y-z|)^{2n+3-2\delta}}\right)^{1/2} dz \\
&\le C\int_{Q^c} |b_1(z)-(b_1)_Q||f(z)|_r |x-x_0|^{1/2} \left(\int\int_{|y|\le t} \frac{t^{1-n}\,dy\,dt}{(t+|x-z|)^{2n+3-2\delta}}\right)^{1/2} dz \\
&\le C\int_{Q^c} |b_1(z)-(b_1)_Q||f(z)|_r |x-x_0|^{1/2} \left(\int_0^\infty \frac{t\,dt}{(t+|x-z|)^{2n+3-2\delta}}\right)^{1/2} dz \\
&\le C\int_{Q^c} |b_1(z)-(b_1)_Q||f(z)|_r \frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2-\delta}}\,dz \\
&\le C\sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |b_1(z)-(b_1)_Q||f(z)|_r \frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2-\delta}}\,dz \\
&\le C\sum_{k=1}^\infty 2^{-k/2}|2^{k+1}Q|^{-1+\delta/n} \int_{2^{k+1}Q} |b_1(z)-(b_1)_Q||f(z)|_r\,dz \\
&\le C\|b_1\|_{BMO} \sum_{k=1}^\infty k2^{-k/2} \big\||f|_r\big\|_{L^{n/\delta}} \\
&\le C\|b_1\|_{BMO} \big\||f|_r\big\|_{L^{n/\delta}},
\end{aligned}$$

so that

$$\frac{1}{|Q|}\int_Q \left|C(x)\right| dx \le C\|b_1\|_{BMO}\left\||f|_r\right\|_{L^{n/\delta}}.$$

When $m > 1$, let $\vec{b}_Q = ((b_1)_Q, \dots, (b_m)_Q) \in R^n$, where

$$(b_j)_Q = |Q|^{-1}\int_Q b_j(y)\,dy, \quad 1 \le j \le m,$$

let $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i\chi_Q$, $h_i = f_i\chi_{(Q)^c}$. We have

$$\begin{aligned}
&F_t^{\vec{b}}(f_i)(x,y)\\
&\quad= \int_{R^n}\left[\prod_{j=1}^m \big(b_1(x) - b_1(z)\big)\right]\psi_t(y-z)f_i(z)\,dz\\
&\quad= \big(b_1(x) - (b_1)_Q\big)\cdots\big(b_m(x) - (b_m)_Q\big)F_t(f_i)(y)\\
&\qquad+ (-1)^m F_t\big(\big(b_1 - (b_1)_Q\big)\cdots\big(b_m - (b_m)_Q\big)f_i\big)(y)\\
&\qquad+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}\big(\vec{b}(x) - \vec{b}_Q\big)_\sigma\\
&\qquad\times \int_{R^n}\big(\vec{b}(z) - \vec{b}_Q\big)_{\sigma^c}\psi_t(y-z)f_i(z)\,dz\\
&\quad= \big(b_1(x) - (b_1)_Q\big)\cdots\big(b_m(x) - (b_m)_Q\big)F_t(f_i)(y)\\
&\qquad+ (-1)^m F_t\big(\big(b_1 - (b_1)_Q\big)\cdots\big(b_m - (b_m)_Q\big)g_i\big)(y)\\
&\qquad+ (-1)^m F_t\big(\big(b_1 - (b_1)_Q\big)\cdots\big(b_m - (b_m)_Q\big)h_i\big)(y)\\
&\qquad+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}\big(\vec{b}(x) - \vec{b}_Q\big)_\sigma F_t\big((\vec{b} - \vec{b}_Q)_{\sigma^c}f_i\big)(x,y),
\end{aligned}$$

by the Minkowski inequality, we have

$$\begin{aligned}
&\left|\left|S_{\psi,\delta}^{\vec{b}}(f)(x)\right|_r - \left|S_{\psi,\delta}\big(\big((b_1)_Q - b_1\big)\cdots\big((b_m)_Q - b_m\big)h\big)(x_0)\right|_r\right|\\
&\quad\le \left\|\chi_{\Gamma(x)}\big(b_1(x) - (b_1)_Q\big)\cdots\big(b_m(x) - (b_m)_Q\big)F_t(f)(y)\right\|_r\\
&\qquad+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left\|\chi_{\Gamma(x)}\big(\vec{b}(x) - \vec{b}_Q\big)_\sigma F_t\big((\vec{b} - \vec{b}_Q)_{\sigma^c}f\big)(x,y)\right\|_r\\
&\qquad+ \left\|\chi_{\Gamma(x)}F_t\big(\big(b_1 - (b_1)_Q\big)\cdots\big(b_m - (b_m)_Q\big)g\big)(y)\right\|_r\\
&\qquad+ \left\|\chi_{\Gamma(x)}F_t\left(\prod_{j=1}^m (b_j - (b_j)_Q)h\right)(y) - \chi_{\Gamma(x_0)}F_t\left(\prod_{j=1}^m (b_j - (b_j)_Q)h\right)(y)\right\|_r\\
&\quad= M_1(x) + M_2(x) + M_3(x) + M_4(x).
\end{aligned}$$

For $M_1(x)$, similar to the proof of $m = 1$, we take $1 < p < n/\delta$, $1/q = 1/p - \delta/n$, by the Hölder inequality and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q M_1(x)\,dx \\
&\quad\le \left(\frac{1}{|Q|}\int_Q \left|\prod_{j=1}^m \big(b_j(x)-(b_j)_Q\big)\right|^{q'} dx\right)^{1/q'} \left(\frac{1}{|Q|}\int_Q \big|S_{\psi,\delta}(f)(x)\big|_r^q\,dx\right)^{1/q} \\
&\quad\le C\|\vec{b}\|_{BMO}|Q|^{-1/q}\left(\int_Q \big|f(x)\big|_r^p\,dx\right)^{1/p} \\
&\quad\le C\|\vec{b}\|_{BMO}|Q|^{-1/q}\left(\int_Q \big|f(x)\big|_r^{n/\delta}\,dx\right)^{\delta/n}|Q|^{(1-(\delta p/n))/p} \\
&\quad\le C\|\vec{b}\|_{BMO}\big\||f|_r\big\|_{L^{n/\delta}}.
\end{aligned}$$

For $M_2(x)$, taking $1 < p < n/\delta$, $1/q = 1/p - \delta/n$, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q M_2(x)\,dx \\
&\quad\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}|Q|^{-1/q}\left(\int_{R^n}\big|\big(b(x)-b_Q\big)_{\sigma^c}f(x)\big|_r^p\chi_Q(x)\,dx\right)^{1/p} \\
&\quad\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}\left(\frac{1}{|Q|}\int_Q\big|\big(b(x)-b_Q\big)_{\sigma^c}\big|^q\,dx\right)^{1/q}\big\||f|_r\big\|_{L^{n/\delta}} \\
&\quad\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}\|\vec{b}_{\sigma^c}\|_{BMO}\|f\|_{L^{n/\delta}} \\
&\quad\le C\|\vec{b}\|_{BMO}\big\||f|_r\big\|_{L^{n/\delta}}.
\end{aligned}$$

For $M_3(x)$, taking $1 < p < n/\delta$, $1/q = 1/p - \delta/n$, we obtain

$$\begin{aligned}
\frac{1}{|Q|}\int_Q M_3(x)\,dx &\le \left(\frac{1}{|Q|}\int_Q\big|S_{\psi,\delta}\big(\big(b_1-(b_1)_Q\big)\cdots\big(b_m-(b_m)_Q\big)g\big)(x)\big|_r^q\,dx\right)^{1/q} \\
&\le C|Q|^{-1/q}\big\|\big(b_1(x)-(b_1)_Q\big)\cdots\big(b_m(x)-(b_m)_Q\big)\big|g(x)\big|_r\big\|_{L^p} \\
&\le C\left(\frac{1}{|Q|}\int_Q\big|\big(b_1(x)-(b_1)_Q\big)\cdots\big(b_m(x)-(b_m)_Q\big)\big|^q\,dx\right)^{1/q}\big\||f|_r\big\|_{L^{n/\delta}} \\
&\le C\|\vec{b}\|_{BMO}\big\||f|_r\big\|_{L^{n/\delta}}.
\end{aligned}$$

For $M_4(x)$, we have

$$\begin{aligned}
M_4(x) &\le C\int_{Q^c}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}\left|\prod_{j=1}^m\big(b_j(z)-(b_j)_Q\big)\right|\big|f(z)\big|_r\,dz \\
&\le C\sum_{k=1}^\infty\int_{2^kQ\setminus 2^{k-1}Q}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}\left|\prod_{j=1}^m\big(b_j(z)-(b_j)_Q\big)\right|\big|f(z)\big|_r\,dz
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{k=1}^{\infty}\int_{2^kQ\setminus 2^{k-1}Q}\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2-\delta}}\left|\prod_{j=1}^{m}\big(b_j(z)-(b_j)_Q\big)\right|\big|f(z)\big|_r\,dz\\
&\le C\sum_{k=1}^{\infty}2^{-k/2}\frac{1}{|2^kQ|^{1-\delta/n}}\int_{2^kQ}\left|\prod_{j=1}^{m}\big(b_j(z)-(b_j)_Q\big)\right|\big|f(z)\big|_r\,dz\\
&\le C\sum_{k=1}^{\infty}2^{-k/2}\left(\int_{2^kQ}\big|f(z)\big|_r^{n/\delta}\,dz\right)^{\delta/n}\\
&\quad\times\left(\frac{1}{|2^kQ|}\int_{2^kQ}\left|\prod_{j=1}^{m}\big(b_j(z)-(b_j)_Q\big)\right|^{n/(n-\delta)}dz\right)^{(n-\delta)/n}\\
&\le C\sum_{k=1}^{\infty}k^m 2^{-k/2}\prod_{j=1}^{m}\|b_j\|_{BMO}\big\||f|_r\big\|_{L^{n/\delta}}\\
&\le C\|\vec{b}\|_{BMO}\big\||f|_r\big\|_{L^{n/\delta}},
\end{aligned}$$

so

$$\frac{1}{|Q|}\int_Q\big|M_4(x)\big|\,dx\le C\|\vec{b}\|_{BMO}\big\||f|_r\big\|_{L^{n/\delta}}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2 It is only to prove that there exists a constant C_Q, for any of the cubes $Q=Q(0,d)$ $(d>1)$, the following inequality holds:

$$\frac{1}{|Q|}\int_Q\big|\big|S_{\psi,\delta}^{\vec{b}}(f)(x)\big|_r-C_Q\big|\,dx\le C\|f\|_{B_p^\delta}.$$

Fix a cube $Q=Q(0,d)$ $(d>1)$. Let $f=g+h=\{g_i\}+\{h_i\}$, where $g_i=f_i\chi_Q$, $h_i=f_i\chi_{(Q)^c}$ and $\vec{b}_Q=((b_1)_Q,\dots,(b_m)_Q)$. For $(b_j)_Q=|Q|^{-1}\int_Q|b_j(y)|\,dy$, $1\le j\le m$, we have

$$\begin{aligned}
F_t^{\vec{b}}(f_i)(x,y)&=\int_{R^n}\left[\prod_{j=1}^{m}\big(b_1(x)-b_1(z)\big)\right]\psi_t(y-z)f_i(z)\,dz\\
&=\big(b_1(x)-(b_1)_Q\big)\cdots\big(b_m(x)-(b_m)_Q\big)F_t(f_i)(y)\\
&\quad+(-1)^mF_t\big(\big(b_1-(b_1)_Q\big)\cdots\big(b_m-(b_m)_Q\big)g_i\big)(y)\\
&\quad+(-1)^mF_t\big(\big(b_1-(b_1)_Q\big)\cdots\big(b_m-(b_m)_Q\big)h_i\big)(y)\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}\big(\vec{b}(x)-\vec{b}_Q\big)_\sigma F_t\big((\vec{b}-\vec{b}_Q)_{\sigma^c}f_i\big)(x,y).
\end{aligned}$$

By the Minkowski inequality, we have

$$\begin{aligned}
&\big|\big|S_{\psi,\delta}^{\vec{b}}(f)(x)\big|_r-\big|S_{\psi,\delta}\big(\big((b_1)_Q-b_1\big)\cdots\big((b_m)_Q-b_m\big)h\big)(x_0)\big|_r\big|\\
&\quad\le\big\|\chi_{\Gamma(x)}\big(b_1(x)-(b_1)_Q\big)\cdots\big(b_m(x)-(b_m)_Q\big)F_t(f)(y)\big\|_r\\
&\qquad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\big\|\chi_{\Gamma(x)}\big(\vec{b}(x)-\vec{b}_Q\big)_\sigma F_t\big((\vec{b}-\vec{b}_Q)_{\sigma^c}f\big)(x,y)\big\|_r
\end{aligned}$$

$$+ \left\| \chi_{\Gamma(x)} F_t\big((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)g\big)(y)\right\|_r$$
$$+ \left\| \chi_{\Gamma(x)} F_t\left(\prod_{j=1}^m (b_j-(b_j)_Q)h\right)(y) - \chi_{\Gamma(x_0)} F_t\left(\prod_{j=1}^m (b_j-(b_j)_Q)h\right)(y)\right\|_r$$
$$= H_1(x)+H_2(x)+H_3(x)+H_4(x).$$

For $H_1(x)$, take $1/q = 1/p - \delta/n$, by the Hölder inequality and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q H_1(x)\,dx \\
&\le \left(\frac{1}{|Q|}\int_Q \left|\prod_{j=1}^m (b_j(x)-(b_j)_Q)\right|^{q'} dx\right)^{1/q'} \left(\frac{1}{|Q|}\int_Q \left|S_{\psi,\delta}(f)(x)\right|_r^q dx\right)^{1/q} \\
&\le C\|\vec{b}\|_{BMO}|Q|^{-1/q}\left(\int_{R^n} |f(x)|_r^p \chi_Q(x)\,dx\right)^{1/p} \\
&\le C\|\vec{b}\|_{BMO} d^{-n(1/p-\delta/n)}\big\||f|_r\chi_Q\big\|_{L^p} \\
&\le C\|\vec{b}\|_{BMO}\big\||f|_r\big\|_{B_p^\delta}.
\end{aligned}$$

For $H_2(x)$, taking $1<u<p<n/\delta$, $1/v = 1/u - \delta/n$, we get

$$\begin{aligned}
\frac{1}{|Q|}\int_Q H_2(x)\,dx &\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q \left|(b(x)-b_Q)_\sigma\right|^{v'} dx\right)^{1/v'} \\
&\quad\times\left(\frac{1}{|Q|}\int_Q \left|S_{\psi,\delta}\big((b-b_Q)_{\sigma^c}f\big)(x)\right|_r^u dx\right)^{1/u} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}|Q|^{-1/v}\left(\int_{R^n}\left|(b(x)-b_Q)_{\sigma^c}f(x)\right|_r^u\chi_Q(x)\,dx\right)^{1/u} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}|Q|^{(\delta/n-1/p)}\big\||f|_r\chi_Q\big\|_{L^p} \\
&\quad\times\left(\frac{1}{|Q|}\int_Q\left|(b(x)-b_Q)_{\sigma^c}\right|^{pr/(p-r)}dx\right)^{(p-u)/pu} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}\|\vec{b}_{\sigma^c}\|_{BMO} d^{-n(1/p-\delta/n)}\big\||f|_r\chi_Q\big\|_{L^p} \\
&\le C\|\vec{b}\|_{BMO}\big\||f|_r\big\|_{B_p^\delta}.
\end{aligned}$$

For $H_3(x)$, taking $1<u<p<n/\delta$, $1/v = 1/u - \delta/n$, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q H_3(x)\,dx \\
&\le \left(\frac{1}{|Q|}\int_Q \left|S_{\psi,\delta}\big((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)g\big)(x)\right|_r^v dx\right)^{1/v}
\end{aligned}$$

$$\le C|Q|^{-1/v}\left(\int_Q\left|\big(b_1(x)-(b_1)_Q\big)\cdots\big(b_m(x)-(b_m)_Q\big)g(x)\right|_r^u dx\right)^{1/u}$$
$$\le C|Q|^{-1/v}\left\|\big(b_1(x)-(b_1)_Q\big)\cdots\big(b_m(x)-(b_m)_Q\big)|f|_r\chi_Q\right\|_{L^v}$$
$$\le C\|\vec{b}\|_{BMO}\left\||f|_r\right\|_{B_p^\delta}.$$

For $H_4(x)$, we have

$$I_4(x)\le\left[\int\int_{R_+^{n+1}}\left(\int_{Q^c}|\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)}|\prod_{j=1}^m\left|b_j(z)-(b_j)_Q\right|\left|\psi_t(y-z)\right|\left|f(z)\right|_r dz\right)^2\frac{dy\,dt}{t^{n+1}}\right]^{1/2}$$
$$\le C\sum_{k=0}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-2\delta)}\left|\prod_{j=1}^m\big(b_j(z)-(b_j)_Q\big)\right|\left|f(z)\right|_r dz$$
$$\le C\sum_{k=1}^\infty 2^{-k/2}\left|2^{k+1}Q\right|^{-1+\delta/n}\int_{2^{k+1}Q}\left|\prod_{j=1}^m\big(b_j(z)-(b_j)_Q\big)\right|\left|f(z)\right|_r dz$$
$$\le C\sum_{k=1}^\infty k^m 2^{-k/2}\left|2^kQ\right|^{-(1/p-\delta/n)}\|\vec{b}\|_{BMO}\left\||f|_r\chi_{2^kQ}\right\|_{L^p}$$
$$\le C\|\vec{b}\|_{BMO}\left\||f|_r\right\|_{B_p^\delta},$$

so

$$\frac{1}{|Q|}\int_Q\left|H_4(x)\right|dx\le C\|\vec{b}\|_{BMO}\left\||f|_r\right\|_{B_p^\delta}.$$

This completes the proof of Theorem 2. □

Competing interests
The author declares that they have no competing interests.

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