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Strong convergence theorems for a countable family of totally quasi-ϕ-asymptotically nonexpansive nonself mappings in Banach spaces with applications

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Abstract

In this paper, we introduce a class of totally quasi-ϕ-asymptotically nonexpansive nonself mappings and study the strong convergence under a limit condition only in the framework of Banach spaces. Meanwhile, our results are applied to study the approximation problem of a solution to a system of equilibrium problems. The results presented in the paper improve and extend the corresponding results of Chang *et al.* (Appl. Math. Comput. 218:7864-7870, 2012).

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1 Introduction

Assume that X is a real Banach space with the dual X^*, D is a nonempty closed convex subset of X. We also denote by J the normalized duality mapping from X to 2^{X^*} which is defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Let D be a nonempty closed subset of a real Banach space X. A mapping $T : D \to D$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \to D$ if $p = T(p)$. The set of fixed points of T is represented by $F(T)$.

A Banach space X is said to be strictly convex if $\|\frac{x+y}{2}\| \leq 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\} \subset X$ with $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 0$.

The norm of a Banach space X is said to be Gâteaux differentiable if for each $x, y \in S(x)$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

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exists, where $S(x) = \{x : \|x\| = 1, x \in X\}$. In this case, X is said to be smooth. The norm of a Banach space X is said to be Fréchet differentiable if for each $x \in S(x)$, the limit (1.1) is attained uniformly for $y \in S(x)$ and the norm is uniformly Fréchet differentiable if the limit (1.1) is attained uniformly for $x, y \in S(x)$. In this case, X is said to be uniformly smooth.

A subset D of X is said to be a retract of X if there exists a continuous mapping $P : X \to D$ such that $Px = x$ for all $x \in X$. It is well known that every nonempty closed convex subset of a uniformly convex Banach space X is a retract of X. A mapping $P : X \to D$ is said to be a retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. A mapping $P : X \to D$ is said to be a nonexpansive retraction if it is nonexpansive and it is a retraction from X to D.

Next, we assume that X is a smooth, strictly convex and reflexive Banach space and D is a nonempty closed convex subset of X. In the sequel, we always use $\phi : X \times X \to R^+$ to denote the Lyapunov functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad x, y \in X. \tag{1.2}$$

It is obvious from the definition of the function ϕ that

$$\bigl(\|x\| - \|y\|\bigr)^2 \le \phi(x,y) \le \bigl(\|x\| + \|y\|\bigr)^2, \tag{1.3}$$

$$\phi(y,x) = \phi(y,z) + \phi(z,x) + 2\langle z - y, Jx - Jz\rangle, \quad x, y, z \in X, \tag{1.4}$$

and

$$\phi\bigl(x, J^{-1}\bigl(\lambda Jy + (1-\lambda)Jz\bigr)\bigr) \le \lambda\phi(x,y) + (1-\lambda)\phi(x,z) \tag{1.5}$$

for all $\lambda \in [0,1]$ and $x, y, z \in X$.

Following Alber [1], the generalized projection $\Pi_D : X \to D$ is defined by

$$\Pi_D(x) = \arg\inf_{y \in D} \phi(y,x), \quad \forall x \in X. \tag{1.6}$$

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a nonexpansive mapping.

In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Lemma 1.1 (see [1]) *Let X be a smooth, strictly convex and reflexive Banach space and D be a nonempty closed convex subset of X. Then the following conclusions hold*:

(a) $\phi(x,y) = 0$ *if and only if* $x = y$;
(b) $\phi(x, \Pi_D y) + \phi(\Pi_D y, y) \le \phi(x,y)$, $\forall x, y \in D$;
(c) *if $x \in X$ and $z \in D$, then $z = \Pi_D x$ if and only if* $\langle z - y, Jx - Jz\rangle \ge 0$, $\forall y \in D$.

Remark 1.1 (see [2]) Let Π_D be the generalized projection from a smooth, reflexive and strictly convex Banach space X onto a nonempty closed convex subset D of X. Then Π_D is a closed and quasi-ϕ-nonexpansive from X onto D.

Remark 1.2 (see [2]) If H is a real Hilbert space, then $\phi(x,y) = \|x - y\|^2$, and Π_D is the metric projection of H onto D.

Definition 1.1 Let $P : X \to D$ be a nonexpansive retraction.

(1) A nonself mapping $T : D \to X$ is said to be quasi-ϕ-nonexpansive if $F(T) \neq \Phi$, and

$$\phi\big(p, T(PT)^{n-1}x\big) \leq \phi(p,x), \quad \forall x \in D, p \in F(T), \forall n \geq 1; \tag{1.7}$$

(2) A nonself mapping $T : D \to X$ is said to be quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \Phi$, and there exists a real sequence $k_n \subset [1,+\infty)$, $k_n \to 1$ (as $n \to \infty$), such that

$$\phi\big(p, T(PT)^{n-1}x\big) \leq k_n\phi(p,x), \quad \forall x \in D, p \in F(T), \forall n \geq 1; \tag{1.8}$$

(3) A nonself mapping $T : D \to X$ is said to be totally quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \Phi$, and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ such that

$$\phi\big(p, T(PT)^{n-1}x\big) \leq \phi(p,x) + \nu_n\zeta\big[\phi(p,x)\big] + \mu_n, \quad \forall x \in D, \forall n \geq 1, p \in F(T). \tag{1.9}$$

Remark 1.3 From the definitions, it is obvious that a quasi-ϕ-nonexpansive nonself mapping is a quasi-ϕ-asymptotically nonexpansive nonself mapping, and a quasi-ϕ-asymptotically nonexpansive nonself mapping is a totally quasi-ϕ-asymptotically nonexpansive nonself mapping, but the converse is not true.

Next, we present an example of a quasi-ϕ-nonexpansive nonself mapping.

Example 1.1 (see [2]) Let H be a real Hilbert space, D be a nonempty closed and convex subset of H and $f : D \times D \to R$ be a bifunction satisfying the conditions: (A1) $f(x,x) = 0$, $\forall x \in D$; (A2) $f(x,y) + f(y,x) \leq 0$, $\forall x, y \in D$; (A3) for each $x, y, z \in D$, $\lim_{t \to 0} f(tz + (1-t)x, y) \leq f(x,y)$; (A4) for each given $x \in D$, the function $y \longmapsto f(x,y)$ is convex and lower semicontinuous. The so-called equilibrium problem for f is to find an $x^* \in D$ such that $f(x^*, y) \geq 0$, $\forall y \in D$. The set of its solutions is denoted by $EP(f)$.

Let $r > 0$, $x \in H$ and define a mapping $T_r : D \to D \subset H$ as follows:

$$T_r(x) = \left\{ z \in D, f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in D \right\}, \quad \forall x \in D \subset H, \tag{1.10}$$

then (1) T_r is single-valued, and so $z = T_r(x)$; (2) T_r is a relatively nonexpansive nonself mapping, therefore it is a closed quasi-ϕ-nonexpansive nonself mapping; (3) $F(T_r) = EP(f)$ and $F(T_r)$ is a nonempty and closed convex subset of D; (4) $T_r : D \to D$ is nonexpansive. Since $F(T_r)$ is nonempty, and so it is a quasi-ϕ-nonexpansive nonself mapping from D to H, where $\phi(x,y) = \|x - y\|^2$, $x, y \in H$.

Now, we give an example of a totally quasi-ϕ-asymptotically nonexpansive nonself mapping.

Example 1.2 (see [2]) Let D be a unit ball in a real Hilbert space l^2, and let $T : D \to l^2$ be a nonself mapping defined by

$$T : (x_1, x_2, \ldots) \to \big(0, x_1^2, a_2x_2, a_3x_3, \ldots\big) \in l^2, \quad \forall (x_1, x_2, \ldots) \in D,$$

where $\{a_i\}$ is a sequence in $(0,1)$ such that $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$.

It is proved in Goebal and Kirk [3] that

(i) $\|Tx - Ty\| \le 2\|x - y\|$, $\forall x, y \in D$;

(ii) $\|T^n x - T^n y\| \le 2\prod_{j=2}^n a_j$, $\forall x, y \in D$, $n \ge 2$.

Let $\sqrt{k_1} = 2$, $\sqrt{k_n} = 2\prod_{j=2}^n a_j$, $n \ge 2$, then $\lim_{n\to\infty} k_n = 1$. Letting $\nu_n = k_n - 1$ $(n \ge 2)$, $\zeta(t) = t$ $(t \ge 0)$ and $\{\mu_n\}$ be a nonnegative real sequence with $\mu_n \to 0$, then from (i) and (ii) we have

$$\|T^n x - T^n y\|^2 \le \|x - y\|^2 + \nu_n \zeta(\|x - y\|^2) + \mu_n, \quad \forall x, y \in D.$$

Since D is a unit ball in a real Hilbert space l^2, it follows from Remark 1.2 that $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in D$. The above inequality can be written as

$$\phi(T^n x, T^n y) \le \phi(x, y) + \nu_n \zeta(\phi(x, y)) + \mu_n, \quad \forall x, y \in D.$$

Again, since $0 \in D$ and $0 \in F(T)$, this implies that $F(T) \neq \Phi$. From above inequality, we get that

$$\phi(p, T(PT)^{n-1}x) \le \phi(p, x) + \nu_n \zeta(\phi(p, x)) + \mu_n, \quad \forall p \in F(T), x \in D,$$

where P is the nonexpansive retraction. This shows that the mapping T defined as above is a totally quasi-ϕ-asymptotically nonexpansive nonself mapping.

Lemma 1.2 (see [4]) *Let X be a uniformly convex and smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\{x_n\}$ and $\{y_n\}$ are bounded; if $\phi(x_n, y_n) \to 0$, then $\|x_n - y_n\| \to 0$.*

Lemma 1.3 *Let X be a smooth, strictly convex and reflexive Banach space and D be a nonempty closed convex subset of X. Let $T : D \to X$ be a totally quasi-ϕ-asymptotically nonexpansive nonself mapping with $\mu_1 = 0$, then $F(T)$ is a closed and convex subset of D.*

Proof Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to p$. Since T is a totally quasi-ϕ-asymptotically nonexpansive nonself mapping, we have

$$\phi(x_n, Tp) \le \phi(x_n, p) + \nu_1 \zeta(\phi(x_n, Tp))$$

for all $n \in N$. Therefore,

$$\phi(p, Tp) = \lim_{n\to\infty} \phi(x_n, Tp) \le \lim_{n\to\infty} \phi(x_n, p) + \nu_1 \zeta(\phi(x_n, p)) = \phi(p, p) = 0.$$

By Lemma 1.2, we obtain $Tp = p$. So, we have $p \in F(T)$. This implies $F(T)$ is closed.

Let $p, q \in F(T)$ and $t \in (0, 1)$, and put $w = tp + (1 - t)q$. We prove that $w \in F(T)$. Indeed, in view of the definition of ϕ, let $\{u_n\}$ be a sequence generated by $u_1 = Tw$, $u_2 = T(PT)w$, $u_3 = T(PT)^2 w, \ldots, u_n = T(PT)^{n-1}w = TPu_{n-1}$, we have

$$\begin{aligned}\phi(w, u_n) &= \|w\|^2 - 2\langle w, Ju_n\rangle + \|u_n\|^2 \\ &= \|w\|^2 - 2\langle tp + (1 - t)q, Ju_n\rangle + \|u_n\|^2 \\ &= \|w\|^2 + t\phi(p, u_n) + (1 - t)\phi(q, u_n) - t\|p\|^2 - (1 - t)\|q\|^2. \end{aligned} \tag{1.11}$$

Since

$$\begin{aligned} &t\phi(p,u_n) + (1-t)\phi(q,u_n) \\ &\quad \le t\big[\phi(p,w) + \nu_n\zeta\big[\phi(p,w)\big] + \mu_n\big] + (1-t)\big[\phi(q,w) + \nu_n\zeta\big[\phi(q,w)\big] + \mu_n\big] \\ &\quad = t\big\{\|p\|^2 - 2\langle p,Jw\rangle + \|w\|^2 + \nu_n\zeta\big[\phi(p,w)\big] + \mu_n\big\} \\ &\qquad + (1-t)\big\{\|q\|^2 - 2\langle q,Jw\rangle + \|w\|^2 + \nu_n\zeta\big[\phi(q,w)\big] + \mu_n\big\} \\ &\quad = t\|p\|^2 + (1-t)\|q\|^2 - \|w\|^2 + t\nu_n\zeta\big[\phi(p,w)\big] + (1-t)\nu_n\zeta\big[\phi(q,w)\big] + \mu_n. \end{aligned} \tag{1.12}$$

Substituting (1.10) into (1.11) and simplifying it, we have

$$\phi(w,u_n) \le t\nu_n\zeta\big[\phi(p,w)\big] + (1-t)\nu_n\zeta\big[\phi(q,w)\big] + \mu_n \to 0 \quad (\text{as } n \to \infty).$$

Hence, we have $u_n \to w$. This implies that $u_{n+1} \to w$. Since TP is closed and $u_{n+1} = T(PT)^n w = TPu_n$, we have $TPw = w$. Since $w \in C$, and so $Tw = w$, *i.e.*, $w \in F(T)$. This implies $F(T)$ is convex. This completes the proof of Lemma 1.3. □

Definition 1.2 (1) (see [5]) A countable family of nonself mappings $\{T_i\} : D \to X$ is said to be uniformly quasi-ϕ-asymptotically nonexpansive if $\bigcap_{i=1}^{\infty} F(T_i) \neq \Phi$, and there exist nonnegative real sequences $k_n \subset [1,+\infty)$, $k_n \to 1$, such that for each $i \ge 1$,

$$\phi\big(p, T_i(PT_i)^{n-1}x\big) \le k_n\phi(p,x), \quad \forall x \in D, \forall n \ge 1, p \in F(T). \tag{1.13}$$

(2) A countable family of nonself mappings $\{T_i\} : D \to X$ is said to be uniformly totally quasi-ϕ-asymptotically nonexpansive if $\bigcap_{i=1}^{\infty} F(T_i) \neq \Phi$, and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ such that for each $i \ge 1$,

$$\phi\big(p, T_i(PT_i)^{n-1}x\big) \le \phi(p,x) + \nu_n\zeta\big[\phi(p,x)\big] + \mu_n, \quad \forall x \in D, \forall n \ge 1, p \in F(T). \tag{1.14}$$

(3) (see [5]) A nonself mapping $T : D \to X$ is said to be uniformly L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\big\|T(PT)^{n-1}x - T(PT)^{n-1}y\big\| \le L\|x-y\|, \quad \forall x,y \in D, \forall n \ge 1. \tag{1.15}$$

Considering the strong and weak convergence of asymptotically nonexpansive self or nonself mappings, relatively nonexpansive, quasi-ϕ-nonexpansive and quasi-ϕ-asymptotically nonexpansive self or nonself mappings have been considered extensively by several authors in the setting of Hilbert or Banach spaces (see [4–19]).

The purpose of this paper is to modify the Halpern and Mann-type iteration algorithm for a family of totally quasi-ϕ-asymptotically nonexpansive nonself mappings to have the strong convergence under a limit condition only in the framework of Banach spaces. As an application, we utilize our results to study the approximation problem of solution to a system of equilibrium problems. The results presented in the paper improve and extend the corresponding results of Chang *et al.* [5, 6, 20, 21], Su *et al.* [16], Kiziltunc *et al.* [10], Yildirim *et al.* [11], Yang *et al.* [22], Wang [18, 19], Pathak *et al.* [14], Thianwan [17], Qin *et al.* [15], Hao *et al.* [9], Guo *et al.* [7], Nilsrakoo *et al.* [13] and others.

2 Main results

Theorem 2.1 *Let X be a real uniformly smooth and uniformly convex Banach space, D be a nonempty closed convex subset of X. Let $\{T_i\}: D \to X$ be a family of uniformly totally quasi-ϕ-asymptotically nonexpansive nonself mappings with sequences $\{\nu_n\}$, $\{\mu_n\}$, with $\nu_n, \mu_n \to 0$ (as $n \to \infty$), and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ such that for each $i \ge 1$, $\{T_i\}: D \to X$ is uniformly L_i-Lipschitz continuous. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ and $\{\beta_n\}$ be a sequence in $(0,1)$ satisfying the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \lim_{n\to\infty} \inf \beta_n \le \lim_{n\to\infty} \sup \beta_n < 1$.

Let x_n be a sequence generated by

$$\begin{cases} x_1 \in X \text{ is arbitrary}; \qquad D_1 = D, \\ y_{n,i} = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)JT_i(PT_i)^{n-1}x_n)] \quad (i \ge 1), \\ D_{n+1} = \{z \in D_n : \sup_{i\ge1} \phi(z, y_{n,i}) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots), \end{cases} \tag{2.1}$$

where $\xi_n = \nu_n \sup_{p\in\mathcal{F}} \zeta(\phi(p, x_n)) + \mu_n$, $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$, $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1}. If $\mathcal{F}$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Proof (I) First, we prove that $\mathcal{F}$ and D_n are closed and convex subsets in D.

In fact, by Lemma 1.3 for each $i \ge 1$, $F(T_i)$ is closed and convex in D. Therefore, $\mathcal{F}$ is a closed and convex subset in D. By the assumption that $D_1 = D$ is closed and convex, suppose that D_n is closed and convex for some $n \ge 1$. In view of the definition of ϕ, we have

$$\begin{aligned} D_{n+1} &= \Big\{z \in D_n : \sup_{i\ge1} \phi(z, y_{n,i}) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\Big\} \\ &= \bigcap_{i\ge1} \Big\{z \in D : \sup_{i\ge1} \phi(z, y_{n,i}) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\Big\} \cap D_n \\ &= \bigcap_{i\ge1} \big\{z \in D : 2\alpha_n \langle z, Jx_1\rangle + 2(1-\alpha_n)\langle z, Jx_n\rangle - 2\langle z, Jy_{n,i}\rangle \\ &\qquad \le \alpha_n \|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_{n,i}\|^2\big\} \cap D_n. \end{aligned}$$

This shows that D_{n+1} is closed and convex. The conclusions are proved.

(II) Next, we prove that $\mathcal{F} \subset D_n$ for all $n \ge 1$.

In fact, it is obvious that $\mathcal{F} \subset D_1$. Suppose that $\mathcal{F} \subset D_n$.

Let $w_{n,i} = J^{-1}(\beta_n Jx_n + (1-\beta_n)JT_i(PT_i)^{n-1}x_n)$. Hence for any $u \in \mathcal{F} \subset D_n$, by (1.5), we have

$$\begin{aligned} \phi(u, y_{n,i}) &= \phi\big(u, J^{-1}\big(\alpha_n Jx_1 + (1-\alpha_n)Jw_{n,i}\big)\big) \\ &\le \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, w_{n,i}) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \phi(u, w_{n,i}) &= \phi\big(u, J^{-1}\big(\beta_n Jx_n + (1-\beta_n)JT_i(PT_i)^{n-1}x_n\big)\big) \\ &\le \beta_n \phi(u, x_n) + (1-\beta_n)\phi\big(u, T_i(PT_i)^{n-1}x_n\big) \end{aligned}$$

$$\begin{aligned}&\le \beta_n\phi(u,x_n) + (1-\beta_n)\big\{\phi(u,x_n) + \nu_n\zeta\big[\phi(u,x_n)\big] + \mu_n\big\}\\&= \phi(u,x_n) + (1-\beta_n)\nu_n\zeta\big[\phi(u,x_n)\big] + (1-\beta_n)\mu_n.\end{aligned}\tag{2.3}$$

Therefore, we have

$$\begin{aligned}\sup_{i\ge1}\phi(u,y_{n,i}) &\le \alpha_n\phi(u,x_1) + (1-\alpha_n)\big[\phi(u,x_n) + (1-\beta_n)\nu_n\zeta\big[\phi(u,x_n)\big] + (1-\beta_n)\mu_n\big]\\&\le \alpha_n\phi(u,x_1) + (1-\alpha_n)\phi(u,x_n) + \nu_n\sup_{p\in\mathcal{F}}\zeta\big[\phi(p,x_n)\big]\\&= \alpha_n\phi(z,x_1) + (1-\alpha_n)\phi(z,x_n) + \xi_n,\end{aligned}\tag{2.4}$$

where $\xi_n = \nu_n\sup_{p\in\mathcal{F}}\zeta(\phi(p,x_n)) + \mu_n$. This shows that $u \in \mathcal{F} \subset D_{n+1}$ and so $\mathcal{F} \subset D_n$. The conclusion is proved.

(III) Now, we prove that $\{x_n\}$ converges strongly to some point p^*.

Since $x_n = \Pi_{D_n}x_1$, from Lemma 1.1(c), we have

$$\langle x_n - y, Jx_1 - Jx_n\rangle \ge 0, \quad \forall y \in D_n.$$

Again since $\mathcal{F} \subset D_n$, we have

$$\langle x_n - u, Jx_1 - Jx_n\rangle \ge 0, \quad \forall u \in \mathcal{F}.$$

It follows from Lemma 1.1(b) that for each $u \in \mathcal{F}$ and for each $n \ge 1$,

$$\phi(x_n,x_1) = \phi(\Pi_{D_n}x_1,x_1) \le \phi(u,x_1) - \phi(u,x_n) \le \phi(u,x_1).\tag{2.5}$$

Therefore, $\{\phi(x_n,x_1)\}$ is bounded, and so is $\{x_n\}$. Since $x_n = \Pi_{D_n}x_1$ and $x_{n+1} = \Pi_{D_{n+1}}x_1 \in D_{n+1} \subset D_n$, we have $\phi(x_n,x_1) \le \phi(x_{n+1},x_1)$. This implies that $\{\phi(x_n,x_1)\}$ is nondecreasing. Hence, $\lim_{n\to\infty}\phi(x_n,x_1)$ exists.

By the construction of $\{D_n\}$, for any $m \ge n$, we have $D_m \subset D_n$ and $x_m = \Pi_{D_m}x_1 \in D_n$. This shows that

$$\phi(x_m,x_n) = \phi(x_m,\Pi_{D_n}x_1) \le \phi(x_m,x_1) - \phi(x_n,x_1) \to 0 \quad (\text{as } n\to\infty).$$

It follows from Lemma 1.2 that $\lim_{n\to\infty}\|x_m - x_n\| = 0$. Hence, $\{x_n\}$ is a Cauchy sequence in D. Since D is complete, without loss of generality, we can assume that $\lim_{n\to\infty}x_n = p^*$ (some point in D).

By the assumption, it is easy to see that

$$\lim_{n\to\infty}\xi_n = \lim_{n\to\infty}\Big[\nu_n\sup_{p\in\mathcal{F}}\zeta\big(\phi(p,x_n)\big) + \mu_n\Big] = 0.\tag{2.6}$$

(IV) Now, we prove that $p^* \in \mathcal{F}$.

Since $x_{n+1} \in D_{n+1}$, from (2.1) and (2.6), we have

$$\sup_{i\ge1}\phi(x_{n+1},y_{n,i}) \le \alpha_n\phi(x_{n+1},x_1) + (1-\alpha_n)\phi(x_{n+1},x_n) + \xi_n \to 0.\tag{2.7}$$

Since $x_n \to p^*$, it follows from (2.7) and Lemma 1.2 that

$$y_{n,i} \to p^*. \tag{2.8}$$

Since $\{x_n\}$ is bounded and $\{T_i\}$ is a family of uniformly total quasi-ϕ-asymptotically nonexpansive nonself mappings, we have

$$\phi\big(p, T_i(PT_i)^{n-1}x_n\big) \le \phi(p, x_n) + \nu_n\zeta\big[\phi(p, x_n)\big] + \mu_n, \quad \forall x \in D, \forall n, i \ge 1, p \in F(T_i).$$

This implies that $\{T_i(PT_i)^{n-1}x_n\}$ is uniformly bounded.

Since

$$\begin{aligned} \|w_{n,i}\| &= \big\|J^{-1}\big(\beta_n Jx_n + (1-\beta_n)JT_i(PT_i)^{n-1}x_n\big)\big\| \\ &\le \beta_n\|x_n\| + (1-\beta_n)\big\|T_i(PT_i)^{n-1}x_n\big\| \\ &\le \|x_n\| + \big\|T_i(PT_i)^{n-1}x_n\big\|, \end{aligned}$$

this implies that $\{w_{n,i}\}$ is also uniformly bounded.

In view of $\alpha_n \to 0$, from (2.1), we have that

$$\lim_{n\to\infty} \|Jy_{n,i} - Jw_{n,i}\| = \lim_{n\to\infty} \alpha_n\|Jx_1 - Jw_{n,i}\| = 0 \tag{2.9}$$

for each $i \ge 1$.

Since J^{-1} is uniformly continuous on each bounded subset of X^*, it follows from (2.8) and (2.9) that

$$w_{n,i} \to p^* \tag{2.10}$$

for each $i \ge 1$. Since J is uniformly continuous on each bounded subset of X, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \big\|Jw_{n,i} - JP^*\big\| \\ &= \lim_{n\to\infty} \big\|\big(\beta_n Jx_n + (1-\beta_n)JT_i(PT_i)^{n-1}x_n\big) - Jp^*\big\| \\ &= \lim_{n\to\infty} \big\|\beta_n(Jx_n - Jp^*) + (1-\beta_n)\big(JT_i(PT_i)^{n-1}x_n - Jp^*\big)\big\| \\ &= \lim_{n\to\infty} (1-\beta_n)\big\|JT_i(PT_i)^{n-1}x_n - Jp^*\big\|. \end{aligned} \tag{2.11}$$

By condition (ii), we have that

$$\lim_{n\to\infty} \big\|JT_i(PT_i)^{n-1}x_n - JP^*\big\| = 0.$$

Since J is uniformly continuous, this shows that

$$\lim_{n\to\infty} T_i(PT_i)^{n-1}x_n = P^* \tag{2.12}$$

for each $i \geq 1$. Again, by the assumption that $\{T_i\} : D \to X$ is uniformly L_i-Lipschitz continuous for each $i \geq 1$, thus we have

$$\begin{aligned}
&\left\| T_i(PT_i)^n x_n - T_i(PT_i)^{n-1} x_n \right\| \\
&\quad \leq \left\| T_i(PT_i)^n x_n - T_i(PT_i)^n x_{n+1} \right\| + \left\| T_i(PT_i)^n x_{n+1} - x_{n+1} \right\| \\
&\qquad + \|x_{n+1} - x_n\| + \left\| x_n - T_i(PT_i)^{n-1} x_n \right\| \\
&\quad \leq (L_i + 1)\|x_{n+1} - x_n\| + \left\| T_i(PT_i)^n x_{n+1} - x_{n+1} \right\| + \left\| x_n - T_i(PT_i)^{n-1} x_n \right\|
\end{aligned} \tag{2.13}$$

for each $i \geq 1$.

We get $\lim_{n\to\infty} \|T_i(PT_i)^n x_n - T_i(PT_i)^{n-1} x_n\| = 0$. Since $\lim_{n\to\infty} T_i(PT_i)^{n-1} x_n = P^*$ and $\lim_{n\to\infty} x_n = p^*$, we have $\lim_{n\to\infty} T_i P T_i (PT_i)^{n-1} x_n = p^*$.

In view of the continuity of $T_i P$, it yields that $T_i P p^* = p^*$. Since $p^* \in C$, it implies that $T_i p^* = p^*$. By the arbitrariness of $i \geq 1$, we have $p^* \in \mathcal{F}$.

(V) Finally, we prove that $p^* = \Pi_{\mathcal{F}} x_1$ and so $x_n \to \Pi_{\mathcal{F}} x_1 = p^*$.

Let $w = \Pi_{\mathcal{F}} x_1$. Since $w \in \mathcal{F} \subset D_n$ and $x_n = \Pi_{D_n} x_1$, we have $\phi(x_n, x_1) \leq \phi(w, x_1)$. This implies that

$$\phi(p^*, x_1) = \lim_{n\to\infty} \phi(x_n, x_1) \leq \phi(w, x_1), \tag{2.14}$$

which yields that $p^* = w = \Pi_{\mathcal{F}} x_1$. Therefore, $x_n \to \Pi_{\mathcal{F}} x_1$. The proof of Theorem 3.1 is completed. □

By Remark 1.3, the following corollary is obtained.

Corollary 2.1 *Let X, D, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem 2.1. Let $\{T_i\} : D \to X$ be a family of uniformly quasi-ϕ-asymptotically nonexpansive nonself mappings with the sequence $k_n \subset [1, +\infty)$, $k_n \to 1$, such that for each $i \geq 1$, $\{T_i\} : D \to X$ is uniformly L_i-Lipschitz continuous.*

Let x_n be a sequence generated by

$$\begin{cases}
x_1 \in X \text{ is arbitrary}; \qquad D_1 = D, \\
y_{n,i} = J^{-1}[\alpha_n J x_1 + (1-\alpha_n)(\beta_n J x_n + (1-\beta_n) J T_i (PT_i)^{n-1} x_n)] \quad (i \geq 1), \\
D_{n+1} = \{z \in D_n : \sup_{i\geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\}, \\
x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots),
\end{cases} \tag{2.15}$$

where $\xi_n = (k_n - 1)\sup_{p\in\mathcal{F}} \phi(p, x_n)$, $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$, $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1}. If $\mathcal{F}$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

3 Application

In this section we utilize Corollary 2.1 to study a modified Halpern iterative algorithm for a system of equilibrium problems. We have the following result.

Theorem 3.1 *Let H be a real Hilbert space, D be a nonempty closed and convex subset of H. $\{\alpha_n\}$, (β_n) be the same as in Theorem 2.1. Let $\{f_i\} : D \times D \to R$ be a countable family of* *bifunctions satisfying conditions* (A1)-(A4) *as given in Example* 1.1. *Let* $\{T_{r,i} : D \to D \subset H\}$ *be the family of mappings defined by* (1.9), *i.e.*,

$$T_{r,i}(x) = \left\{ z \in D, f_i(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in D \right\}, \quad \forall x \in D \subset H.$$

Let $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} x_1 \in D \text{ is arbitrary}; \qquad D_1 = D, \\ f_i(u_{n,i}, y) + \frac{1}{r}\langle y - u_{n,i}, u_{n,i} - x_n\rangle \geq 0, \quad \forall y \in D, r > 0, i \geq 1, \\ y_{n,i} = \alpha_n x_1 + (1-\alpha_n)[\beta_n x_n + (1-\beta_n)u_{n,i}], \\ D_{n+1} = \{z \in D_n : \sup_{i\geq 1} \|z - y_{n,i}\|^2 \leq \alpha_n \|z - x_1\|^2 + (1-\alpha_n)\|z - x_n\|^2\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots). \end{cases} \tag{3.1}$$

If $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_{r,i}) \neq \Phi$, *then* $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}} x_1$, *which is a common solution of the system of equilibrium problems for* f.

Proof In Example 1.1, we have pointed out that $u_{n,i} = T_{r,i}(x_n)$, $F(T_{r,i}) = EP(f_i)$ is nonempty and convex for all $i \geq 1$, $T_{r,i}$ is a countable family of quasi-ϕ-nonexpansive nonself mappings. Since $F(T_{r,i})$ is nonempty, so $T_{r,i}$ is a countable family of quasi-ϕ-nonexpansive mappings and for all $i \geq 1$, $T_{r,i}$ is a uniformly 1-Lipschitzian mapping. Hence, (3.1) can be rewritten as follows:

$$\begin{cases} x_1 \in H \text{ is arbitrary}; \qquad D_1 = D, \\ y_{n,i} = \alpha_n x_1 + (1-\alpha_n)[\beta_n x_n + (1-\beta_n)T_{r,i} x_n], \\ D_{n+1} = \{z \in D_n : \sup_{i\geq 1} \|z - y_{n,i}\|^2 \leq \alpha_n \|z - x_1\|^2 + (1-\alpha_n)\|z - x_n\|^2\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots). \end{cases} \tag{3.2}$$

Therefore, the conclusion of Theorem 3.1 can be obtained from Corollary 2.1. □

Competing interests
The author declares that they have no competing interests.

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