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# Anisotropic curl-free wavelets with boundary conditions

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**Abstract**

This paper deals with the construction of anisotropic curl-free wavelets that satisfy the tangent boundary conditions on bounded domains. Based on some assumptions, we first obtain the desired curl-free Riesz wavelet bases through the orthogonal decomposition of vector-valued $L^2$. Next, the characterization of Sobolev spaces is studied. Finally, we give the concrete construction of wavelets satisfying the initial assumptions.

**MSC:** 42C20

**Keywords:** anisotropic; curl-free; wavelets; bounded domains; boundary conditions

## 1 Instruction

Due to their potential use in many physical problems, like the simulation of incompressible fluids or electromagnetism, curl-free wavelet bases have been advocated in several papers and all results focus on the cases of $R^2$ and $R^3$ [1–4]. Moreover, it is questionable whether they are appropriately called bases and whether they can be used to characterize Sobolev spaces. However, it is reasonable to study the corresponding wavelet bases on bounded domains because of some practical use. At the same time, the boundary conditions, the stability and the characterization of Sobolev spaces are also necessary in some applications such as adaptive wavelet methods. In references [5, 6], anisotropic divergence-free wavelets which satisfy the specific boundary conditions on the hypercube are studied. Inspired by the fact that a div-free space and a curl-free space form the orthogonal Helmholtz decomposition, we mainly study the anisotropic curl-free wavelet bases satisfying the tangent boundary conditions on bounded domains in this paper, which is organized as follows. In Section 2, based on some assumption, the desired curl-free wavelets are constructed through the orthogonal decomposition of vector-valued $L^2$. Section 3 is devoted to studying the characterization of Sobolev spaces. We give the concrete construction of wavelets satisfying the initial assumption in the final section.

For two 2D vectors $\vec{u} = (u_1, u_2)^T$ and $\vec{v} = (v_1, v_2)^T$, $\vec{u} \times \vec{v}$ is defined as

$$\vec{u} \times \vec{v} =: u_1 v_2 - u_2 v_1.$$

Then for $\vec{u}(x, y) = (u_1(x, y), u_2(x, y))^T$, we define the 2D curl-operator by

$$curl\vec{u} =: (\partial_1, \partial_2) \times \vec{u} = \partial_1 u_2 - \partial_2 u_1$$

and for $\vec{u}(x,y,z) = (u_1,u_2,u_3)^T$, the 3D curl-operator is defined by

$$curl\vec{u} = (\partial_1,\partial_2,\partial_3) \times \vec{u} = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^T.$$

## 2 Decomposition of $L^2(I^n)^n$

In this part, we will construct curl-free wavelets that satisfy tangent boundary conditions by the orthogonal decomposition of vector-valued $L^2$.

Let $I = (0,1)$. For $n = 2,3$, we firstly define the following spaces:

$$H(curl;I^n) =: \{\vec{u} \in L^2(I^n)^n : curl\vec{u} \in L^2(I^2) \text{ or } L^2(I^3)^3\},$$
$$H_0(curl;I^n) =: \{\vec{u} \in H(curl;I^n) : \vec{u} \times \vec{n} = 0 \text{ or } \vec{0} \text{ on boundary } \Gamma\},$$
$$\mathcal{H}(I^n) =: H_0(curl0;I^n) =: \{\vec{u} \in H_0(curl;I^n) : curl\vec{u} = 0 \text{ or } \vec{0}\}.$$

For a scalar function $\phi(x,y)$, define $\overrightarrow{curl}\phi =: (\partial_2\phi, -\partial_1\phi)^T$. Then integration by parts shows

$$\mathcal{H}(I^2) \perp \overrightarrow{curl}H^1(I^2) \quad \text{and} \quad \mathcal{H}(I^3) \perp curlH^1(I^3)^3.$$

Let $L^2 =: L^2(I)$, $L^{2,0} =: \{u \in L^2 : \int_0^1 u(x)\,dx = 0\}$. Furthermore, set

$$\widehat{L^2(I^2)^2} =: L^{2,0} \otimes L^2 \times L^2 \otimes L^{2,0};$$
$$\widehat{L^2(I^3)^3} =: L^{2,0} \otimes L^2 \otimes L^2 \times L^2 \otimes L^{2,0} \otimes L^2 \times L^2 \otimes L^2 \otimes L^{2,0};$$
$$\widehat{H}^s(I^n) =: H^s(I^n) \cap (L^{2,0} \otimes \cdots \otimes L^{2,0}), \quad n = 1,2.$$

For $n = 3$, we define $\widehat{H}_1^s(I^3) =: H^s(I^3) \cap (L^2 \otimes L^{2,0} \otimes L^{2,0})$ and

$$\widehat{H}_2^s(I^3) =: H^s(I^3) \cap (L^{2,0} \otimes L^2 \otimes L^{2,0}), \qquad \widehat{H}_3^s(I^3) =: H^s(I^3) \cap (L^{2,0} \otimes L^{2,0} \otimes L^2).$$

Finally, let $\widehat{\mathcal{H}}(I^n) =: \mathcal{H}(I^n) \cap \widehat{L^2(I^n)^n}$, $n = 2,3$.

The following result will be proved in Section 4:

**Assumption 2.1** There exist bi-orthogonal Riesz bases $\Psi^{(n)} = \Psi_{curl}^{(n)} \cup \Psi_{comp}^{(n)}$ and $\widetilde{\Psi}^{(n)} = \widetilde{\Psi}_{curl}^{(n)} \cup \widetilde{\Psi}_{comp}^{(n)}$ for $\widehat{L^2(I^n)^n}$ (of wavelet type) such that

$$\Psi_{curl}^{(n)} \subset \widehat{\mathcal{H}}(I^n) \quad (n = 2,3), \qquad \widetilde{\Psi}_{comp}^{(2)} \subset \overrightarrow{curl}\widehat{H}^1(I^2) \quad \text{or}$$
$$\widetilde{\Psi}_{comp}^{(3)} \subset curl(\widehat{H}_1^1(I^3) \times \widehat{H}_2^1(I^3) \times \widehat{H}_3^1(I^3)).$$

**Proposition 2.1** *It holds that $\Psi_{curl}^{(n)}$, $\widetilde{\Psi}_{comp}^{(2)}$ and $\widetilde{\Psi}_{comp}^{(3)}$ are Riesz bases for $\widehat{\mathcal{H}}(I^n)$ $(n = 2,3)$, $\overrightarrow{curl}\widehat{H}^1(I^2)$ and $curl(\widehat{H}_1^1(I^3) \times \widehat{H}_2^1(I^3) \times \widehat{H}_3^1(I^3))$, respectively.*

*Proof* For any $\vec{u} \in \widehat{\mathcal{H}}(I^n)$ $(n = 2,3)$, we know

$$\widehat{\mathcal{H}}(I^2) \perp \overrightarrow{curl}\widehat{H}^1(I^2) \quad \text{and} \quad \widehat{\mathcal{H}}(I^3) \perp curl(\widehat{H}_1^1(I^3) \times \widehat{H}_2^1(I^3) \times \widehat{H}_3^1(I^3)),$$

then $\vec{u} = \langle \vec{u}, \widetilde{\Psi}^{(n)} \rangle_{L^2(I^n)^n} \Psi^{(n)} = \langle \vec{u}, \widetilde{\Psi}^{(n)}_{curl} \rangle_{L^2(I^n)^n} \Psi^{(n)}_{curl}$ with $\|\vec{u}\|_{L^2(I^n)^n} \simeq \| \langle \vec{u}, \widetilde{\Psi}^{(n)}_{curl} \rangle_{L^2(I^n)^n} \|_{\ell^2}$. Finally, it is easy to verify by the definition of $\overrightarrow{curl}$ and $curl$ that

$$\overrightarrow{curl}\widehat{H}^1(I^2) \subset \widehat{L^2(I^2)^2} \quad \text{and} \quad curl(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3)) \subset \widehat{L^2(I^3)^3},$$

the remaining results can be proved similarly. □

**Proposition 2.2** *The following decompositions hold*:

$$\widehat{L^2(I^2)^2} = \widehat{\mathcal{H}}(I^2) \oplus^\perp \overrightarrow{curl}\widehat{H}^1(I^2),$$
$$\widehat{L^2(I^3)^3} = \widehat{\mathcal{H}}(I^3) \oplus^\perp curl(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3)).$$

*Proof* We only prove the case of $n = 3$, the others can be proved similarly. Since

$$\vec{u} = \langle \vec{u}, \Psi^{(3)}_{curl} \rangle_{L^2(I^3)^3} \widetilde{\Psi}^{(3)}_{curl} + \langle \vec{u}, \Psi^{(3)}_{comp} \rangle_{L^2(I^3)^3} \widetilde{\Psi}^{(3)}_{comp} = \langle \vec{u}, \Psi^{(3)}_{comp} \rangle_{L^2(I^3)^3} \widetilde{\Psi}^{(3)}_{comp}$$

for any $\vec{u} \in \widehat{\mathcal{H}}(I^3)^\perp$, then $\widehat{\mathcal{H}}(I^3)^\perp \subseteq curl(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3))$. On the other hand, since $\widehat{\mathcal{H}}(I^3) \perp curl(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3))$, then $curl(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3)) \subseteq \widehat{\mathcal{H}}(I^3)^\perp$. Therefore, $\widehat{L^2(I^3)^3} = \widehat{\mathcal{H}}(I^3) \oplus^\perp curl(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3))$. □

Now, we consider the orthogonal decomposition of $L^2(I^n)^n$. Let $L^2 = L^{2,0} \oplus^\perp \Lambda$. Then there are the following orthogonal decompositions:

$$L^2 \otimes L^2 = L^{2,0} \otimes L^2 \oplus^\perp \Lambda \otimes L^2, \qquad L^2 \otimes L^2 = L^2 \otimes L^{2,0} \oplus^\perp L^2 \otimes \Lambda;$$
$$L^2 \otimes L^2 \otimes L^2 = L^{2,0} \otimes L^2 \otimes L^2 \oplus^\perp \Lambda \otimes L^2 \otimes L^2;$$
$$L^2 \otimes L^2 \otimes L^2 = L^2 \otimes L^{2,0} \otimes L^2 \oplus^\perp L^2 \otimes \Lambda \otimes L^2;$$
$$L^2 \otimes L^2 \otimes L^2 = L^2 \otimes L^2 \otimes L^{2,0} \oplus^\perp L^2 \otimes L^2 \otimes \Lambda.$$

Therefore, we obtain the following decomposition:

$$L^2(I^2)^2 = \widehat{L^2(I^2)^2} \oplus^\perp \begin{pmatrix} \Lambda \otimes L^2 \\ 0 \end{pmatrix} \oplus^\perp \begin{pmatrix} 0 \\ L^2 \otimes \Lambda \end{pmatrix}, \tag{2.1}$$

$$L^2(I^3)^3 = \widehat{L^2(I^3)^3} \oplus^\perp \begin{pmatrix} \Lambda \otimes L^2 \otimes L^2 \\ 0 \\ 0 \end{pmatrix} \oplus^\perp \begin{pmatrix} 0 \\ L^2 \otimes \Lambda \otimes L^2 \\ 0 \end{pmatrix} \oplus^\perp \begin{pmatrix} 0 \\ 0 \\ L^2 \otimes L^2 \otimes \Lambda \end{pmatrix}. \tag{2.2}$$

By Proposition 2.2, $\widehat{L^2(I^2)^2} = \widehat{\mathcal{H}}(I^2) \oplus^\perp \overrightarrow{curl}\widehat{H}^1(I^2)$, $\widehat{L^2(I^3)^3} = \widehat{\mathcal{H}}(I^3) \oplus^\perp curl(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3))$. Moreover,

$$\begin{pmatrix} \Lambda \otimes L^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ L^2 \otimes \Lambda \end{pmatrix} \subset \overrightarrow{curl}H^1(I^2),$$

$$\begin{pmatrix} \Lambda \otimes L^2 \otimes L^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ L^2 \otimes \Lambda \otimes L^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ L^2 \otimes L^2 \otimes \Lambda \end{pmatrix} \subset curlH^1(I^3)^3.$$

Finally, we obtain $L^2(I^2)^2 = \mathcal{H}(I^2) \oplus^\perp \overrightarrow{curl}H^1(I^2)$ and $L^2(I^3)^3 = \mathcal{H}(I^3) \oplus^\perp curlH^1(I^3)^3$.

Now, we will construct Riesz bases for $\mathcal{H}(I^n)$ ($n = 2, 3$), $\overrightarrow{curl}H^1(I^2)$ and $curlH^1(I^3)^3$. For $n = 2$, we define the embedding $E^{(2)}_{\{1\}}, E^{(2)}_{\{2\}} : L^2(I) \longrightarrow L^2(I^2)^2$ by

$$\left(E^{(2)}_{\{1\}}v\right)(x_1, x_2) = v(x_2)\vec{e}_1, \qquad \left(E^{(2)}_{\{2\}}v\right)(x_1, x_2) = v(x_1)\vec{e}_2,$$

$$E^{(2)}_{\{1,2\}} : \widehat{L^2(I^2)^2} \longrightarrow L^2(I^2)^2 \quad \text{by} \quad \left(E^{(2)}_{\{1,2\}}\vec{v}\right)(x_1, x_2) = \sum_{i=1}^{2} v_i(x_1, x_2)\vec{e}_i.$$

For $n = 3$, define $E^{(3)}_{\{1\}}, E^{(3)}_{\{2\}}, E^{(3)}_{\{3\}} : L^2(I) \longrightarrow L^2(I^3)^3$ by

$$\left(E^{(3)}_{\{1\}}v\right) = v(x_2)v(x_3)\vec{e}_1, \qquad \left(E^{(3)}_{\{2\}}v\right) = v(x_1)v(x_3)\vec{e}_2, \qquad \left(E^{(3)}_{\{3\}}v\right) = v(x_1)v(x_2)\vec{e}_3,$$

$$E^{(3)}_{\{1,2,3\}} : \widehat{L^2(I^3)^3} \longrightarrow L^2(I^3)^3 \quad \text{by} \quad \left(E^{(3)}_{\{1,2,3\}}\vec{v}\right)(x_1, x_2, x_3) = \sum_{i=1}^{3} v_i(x_1, x_2, x_3)\vec{e}_i.$$

It is obvious that $E^{(2)}_{\{1,2\}} = I$, $E^{(3)}_{\{1,2,3\}} = I$. Moreover, the image satisfies

$$ImE^{(2)}_{\{1\}} = \left(\Lambda \otimes L^2, 0\right)^T, \qquad ImE^{(2)}_{\{2\}} = \left(0, L^2 \otimes \Lambda\right)^T;$$
$$ImE^{(3)}_{\{1\}} = \left(\Lambda \otimes L^2 \otimes L^2, 0, 0\right)^T, \qquad ImE^{(3)}_{\{2\}} = \left(0, L^2 \otimes \Lambda \otimes L^2, 0\right)^T,$$
$$ImE^{(3)}_{\{3\}} = \left(0, 0, L^2 \otimes L^2 \otimes \Lambda\right)^T.$$

Furthermore, we know from (2.1) and (2.2) that $L^2(I^2)^2 = ImE^{(2)}_{\{1\}} \oplus^\perp ImE^{(2)}_{\{2\}} \oplus^\perp ImE^{(2)}_{\{1,2\}}$,

$$L^2(I^3)^3 = ImE^{(3)}_{\{1\}} \oplus^\perp ImE^{(3)}_{\{2\}} \oplus^\perp ImE^{(3)}_{\{3\}} \oplus^\perp ImE^{(3)}_{\{1,2,3\}}.$$

Since $Im(E^{(2)}_{\{1,2\}}|_{\widehat{\mathcal{H}}(I^2)}) \subset \mathcal{H}(I^2)$, $Im(E^{(2)}_{\{1,2\}}|_{\overrightarrow{curl}\widehat{H}^1(I^2)}) \subset \overrightarrow{curl}H^1(I^2)$ and

$$Im\left(E^{(3)}_{\{1,2,3\}}|_{\widehat{\mathcal{H}}(I^3)}\right) \subset \mathcal{H}(I^3), \qquad Im\left(E^{(3)}_{\{1,2,3\}}|_{curl(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3))}\right) \subset curlH^1(I^3)^3,$$

we obtain $L^2(I^2)^2 = Im(E^{(2)}_{\{1,2\}}|_{\widehat{\mathcal{H}}(I^2)}) \oplus^\perp Im(E^{(2)}_{\{1,2\}}|_{\overrightarrow{curl}\widehat{H}^1(I^2)}) \oplus^\perp ImE^{(2)}_{\{1\}} \oplus^\perp ImE^{(2)}_{\{2\}}$ and $L^2(I^3)^3 = Im(E^{(3)}_{\{1,2,3\}}|_{\widehat{\mathcal{H}}(I^3)}) \oplus^\perp Im(E^{(3)}_{\{1,2,3\}}|_{curl(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3))}) \oplus^\perp ImE^{(3)}_{\{1\}} \oplus^\perp ImE^{(3)}_{\{2\}} \oplus^\perp ImE^{(3)}_{\{3\}}$.

In view of Proposition 2.1, we obtain

**Theorem 2.1** *In the situation of Assumption* 2.1, *the collections* $\Psi_{curl} =: \Psi^{(n)}_{curl}$ ($n = 2, 3$) *are Riesz bases for* $\mathcal{H}(I^n)$ ($n = 2, 3$). $\widetilde{\Psi}^{(2)}_{comp} \cup E^{(2)}_{\{1\}}\widetilde{\Psi}^{(1)} \cup E^{(2)}_{\{2\}}\widetilde{\Psi}^{(1)}$ *and* $\widetilde{\Psi}^{(3)}_{comp} \cup E^{(3)}_{\{1\}}\widetilde{\Psi}^{(1)} \cup E^{(3)}_{\{2\}}\widetilde{\Psi}^{(1)} \cup E^{(3)}_{\{3\}}\widetilde{\Psi}^{(1)}$ *are Riesz bases for* $\overrightarrow{curl}H^1(I^2)$ *and* $curlH^1(I^3)^3$, *respectively.*

**Note 2.1** In fact, $\widehat{\mathcal{H}}(I^n) = \mathcal{H}(I^n)$ for $n = 2, 3$. $\widetilde{\Psi}^{(1)} =: \widetilde{\Psi}^- = \{\widetilde{\psi}^-_\lambda : \lambda \in \nabla\}$, which is defined in Section 4.

## 3 Characterization of $H^m(I^n)^n$

This part will show that the curl-free wavelets constructed above can be used to characterize Sobolev spaces. For $n = 2, 3$ and $m \in N$, define the following Sobolev spaces:

$$\vec{H}_0^m(I^n) =: \{\vec{u} \in H^m(I^n)^n : \vec{u} \times \vec{n} = 0 \text{ or } \vec{0} \text{ on } \Gamma\},$$
$$\vec{V}(I^n) =: \vec{H}_0^m(I^n) \cap \mathcal{H}(I^n).$$

The following result will be verified in Section 4:

**Assumption 3.1** The collection $\Psi^{(n)}$ from Assumption 2.1 can be constructed so that, normalized in $H^m(I^n)^n$, it is a Riesz basis for

$$\widehat{\vec{H}_0^m(I^n)} =: \vec{H}_0^m(I^n) \cap \widehat{L^2(I^n)}^n.$$

Based on this assumption, we obtain:

**Theorem 3.1** *In the situation of Assumptions* 2.1 *and* 3.1, *the collection* $\Psi_{curl} =: \Psi_{curl}^{(n)}$, *normalized in* $H^m(I^n)^n$, *is a Riesz basis for* $\vec{V}(I^n)$.

*Proof* Since $\mathcal{H}(I^n) = \widehat{\mathcal{H}}(I^n) = \mathcal{H}(I^n) \cap \widehat{L^2(I^n)^n}$, then for any $\vec{u} \in \vec{V}(I^n)$, we know $\vec{u} \in \widehat{\vec{H}_0^m(I^n)}$ and by Assumption 3.1,

$$\vec{u} = \langle \vec{u}, \widetilde{\Psi}^{(n)} \rangle_{L^2(I^n)^n} \Psi^{(n)} \quad \text{in } H^m(I^n)^n$$

with $\|\vec{u}\|_{H^m(I^n)^n}^2 \simeq \sum_{\widetilde{\psi} \in \widetilde{\Psi}^{(n)}} |\langle \vec{u}, \widetilde{\psi} \rangle_{L^2(I^n)^n}|^2 \cdot \|\psi_{\widetilde{\psi}}\|_{H^m(I^n)^n}^2$, where $\psi_{\widetilde{\psi}} \in \Psi^{(n)}$ denotes the primal wavelets corresponding to $\widetilde{\psi}$. Furthermore, since $\vec{u} \in \mathcal{H}(I^n)$, then

$$\vec{u} = \langle \vec{u}, \widetilde{\Psi}_{curl}^{(n)} \rangle_{L^2(I^n)^n} \Psi_{curl}^{(n)} \quad \text{in } H^m(I^n)^n$$

with $\|\vec{u}\|_{H^m(I^n)^n}^2 \simeq \sum_{\widetilde{\psi} \in \widetilde{\Psi}_{curl}^{(n)}} |\langle \vec{u}, \widetilde{\psi} \rangle_{L^2(I^n)^n}|^2 \cdot \|\psi_{\widetilde{\psi}}\|_{H^m(I^n)^n}^2$. □

## 4 Construction of wavelets

In this section, we will give the construction of wavelets satisfying Assumptions 2.1 and 3.1.

**Lemma 4.1** ([5, Corollary 3.3]) *Suppose that the collections* $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ *and* $\widetilde{\Psi} = \{\widetilde{\psi}_\lambda : \lambda \in \nabla\}$ *are bi-orthogonal in* $L^{2,0}(I)$. *In addition, for some* $m < \gamma < d \in N$, $2 < \widetilde{\gamma} < \widetilde{d} \in N$,

$$\inf_{v \in span\{\psi_\lambda : |\lambda| \le \ell\}} \|u - v\|_{L^2(I)} \preceq 2^{-\ell d} \|u\|_{H^d(I)} \quad \big(u \in \widehat{H}^d(I)\big),$$
$$\inf_{v \in span\{\widetilde{\psi}_\lambda : |\lambda| \le \ell\}} \|u - v\|_{L^2(I)} \preceq 2^{-\ell \widetilde{d}} \|u\|_{H^{\widetilde{d}}(I)} \quad \big(u \in \widehat{H}^{\widetilde{d}}(I)\big),$$
$$\text{for } s < \gamma, \quad \|\cdot\|_{H^s(I)} \preceq 2^{\ell s} \|\cdot\|_{L^2(I)} \quad \text{on } span\{\psi_\lambda : |\lambda| \le \ell\},$$
$$\text{for } s < \widetilde{\gamma}, \quad \|\cdot\|_{H^s(I)} \preceq 2^{\ell s} \|\cdot\|_{L^2(I)} \quad \text{on } span\{\widetilde{\psi}_\lambda : |\lambda| \le \ell\}.$$

*Define the collections* $\Psi^+ = \{\psi_\lambda^+ : \lambda \in \nabla\}$ *and* $\widetilde{\Psi}^- = \{\widetilde{\psi}_\lambda^- : \lambda \in \nabla\}$ *by*

$$\psi_\lambda^+(x) =: 2^{|\lambda|} \int_0^x \psi_\lambda(y)\, dy \quad \text{and} \quad \widetilde{\psi}_\lambda^-(x) =: -2^{-|\lambda|} \dot{\widetilde{\psi}}_\lambda.$$

*Then it holds that*

$$\begin{aligned}
&\left\{2^{-|\lambda|s}\psi_\lambda : \lambda \in \nabla\right\} \textit{ is a Riesz basis for } \widehat{H}^s(I), \quad s \in [0,\gamma),\\
&\left\{2^{-|\lambda|s}\widetilde{\psi}_\lambda : \lambda \in \nabla\right\} \textit{ is a Riesz basis for } \widehat{H}^s(I), \quad s \in [0,\widetilde{\gamma}),\\
&\left\{2^{-|\lambda|s}\psi^+_\lambda : \lambda \in \nabla\right\} \textit{ is a Riesz basis for } \mathcal{H}^s_0(I), \quad s \in [0,\gamma+1),\\
&\left\{2^{-|\lambda|s}\widetilde{\psi}^-_\lambda : \lambda \in \nabla\right\} \textit{ is a Riesz basis for } H^s(I), \quad s \in [0,\widetilde{\gamma}-1),
\end{aligned}$$

*where*

$$\mathcal{H}^s_0(I) =: \begin{cases} [L^2(I), H^1_0(I)]_{s,2}, & s \in [0,1];\\ H^s(I) \cap H^1_0(I), & s \geq 1.\end{cases}$$

*Moreover,* $\Psi^+$ *and* $\widetilde{\Psi}^-$ *are bi-orthogonal.*

**Note 4.1** It has been pointed out in [5] that such wavelet bases can be obtained by taking standard bi-orthogonal wavelet bases for $L^2(I)$ that satisfy the corresponding Jackson and Bernstein assumptions of $d$, $\widetilde{d}$, $\gamma$ and $\widetilde{\gamma}$ with $\widehat{H}^d(I)$ and $\widehat{H}^{\widetilde{d}}(I)$ reading as $H^d(I)$ and $H^{\widetilde{d}}(I)$ (see [7]), and then removing those scaling functions without a vanishing moment.

The following result can be proved by the same method as Corollary 3.7 of [5].

**Corollary 4.1** *For* $0 \leq s < \gamma$ *and* $0 \leq \widetilde{s} < \widetilde{\gamma} - 1$, *the sets*

$$\left\{\left(\sum_{i=1}^n 4^{|\lambda_i|}\right)^{-\frac{s}{2}} \psi^+_{\lambda_1} \otimes \cdots \otimes \psi_{\lambda_k} \otimes \cdots \otimes \psi^+_{\lambda_n} : \lambda = (\lambda_1,\ldots,\lambda_n) \in \vec{\nabla} = (\nabla)^n\right\},$$

$$\left\{\left(\sum_{i=1}^n 4^{|\lambda_i|}\right)^{-\frac{\widetilde{s}}{2}} \widetilde{\psi}^-_{\lambda_1} \otimes \cdots \otimes \widetilde{\psi}_{\lambda_k} \otimes \cdots \otimes \widetilde{\psi}^-_{\lambda_n} : \lambda = (\lambda_1,\ldots,\lambda_n) \in \vec{\nabla} = (\nabla)^n\right\}$$

*are Riesz bases for*

$$\begin{gathered}
\qquad\qquad\qquad \downarrow k\text{th position}\\
\mathcal{H}^s_0 \otimes L^2 \otimes \cdots \otimes L^2 \otimes L^{2,0} \otimes L^2 \otimes \cdots \otimes L^2 \cap\\
\vdots\\
k\text{th position } L^2 \otimes \cdots \otimes L^2 \otimes \widehat{H}^s \otimes L^2 \otimes \cdots \otimes L^2 \cap\\
\vdots\\
L^2 \otimes L^2 \otimes \cdots \otimes L^2 \otimes L^{2,0} \otimes L^2 \otimes \cdots \otimes \mathcal{H}^s_0
\end{gathered}$$

*and*

$$\begin{gathered}
\qquad\qquad\qquad \downarrow k\text{th position}\\
H^{\widetilde{s}} \otimes L^2 \otimes \cdots \otimes L^2 \otimes L^{2,0} \otimes L^2 \otimes \cdots \otimes L^2 \cap\\
\vdots\\
k\text{th position } L^2 \otimes \cdots \otimes L^2 \otimes \widehat{H}^{\widetilde{s}} \otimes L^2 \otimes \cdots \otimes L^2 \cap\\
\vdots\\
L^2 \otimes L^2 \otimes \cdots \otimes L^2 \otimes L^{2,0} \otimes L^2 \otimes \cdots \otimes H^{\widetilde{s}},
\end{gathered}$$

*respectively. For* $s = \widetilde{s} = 0$, *the corresponding collections are bi-orthogonal in* $L^2 \otimes \cdots \otimes L^2 \otimes L^{2,0} \otimes L^2 \otimes \cdots \otimes L^2$.

For $\lambda \in \vec{\nabla}$, we define the vector-valued wavelets

$$\underline{\psi}^{(n)}_{\lambda,k} =: \psi^+_{\lambda_1} \otimes \cdots \otimes \psi_{\lambda_k} \otimes \cdots \otimes \psi^+_{\lambda_n} \vec{e}_k, \qquad \underline{\widetilde{\psi}}^{(n)}_{\lambda,k} =: \widetilde{\psi}^-_{\lambda_1} \otimes \cdots \otimes \widetilde{\psi}_{\lambda_k} \otimes \cdots \otimes \widetilde{\psi}^-_{\lambda_n} \vec{e}_k.$$

From Corollary 4.1 and the definition of $\widehat{\vec{H}^1_0(I^n)}$, we obtain

**Proposition 4.1** *For* $0 \le s < \gamma$ *and* $0 \le \widetilde{s} < \widetilde{\gamma} - 1$, *the sets*

$$\left\{ \left( \sum_{i=1}^{n} 4^{|\lambda_i|} \right)^{-\frac{s}{2}} \underline{\psi}^{(n)}_{\lambda,k} : 1 \le k \le n, \lambda \in \vec{\nabla} \right\} \quad \textit{and}$$

$$\left\{ \left( \sum_{i=1}^{n} 4^{|\lambda_i|} \right)^{-\frac{\widetilde{s}}{2}} \underline{\widetilde{\psi}}^{(n)}_{\lambda,k} : 1 \le k \le n, \lambda \in \vec{\nabla} \right\}$$

*are Riesz bases for the vector spaces*

$$\begin{cases} [\widehat{L^2(I^n)^n}, \widehat{\vec{H}^1_0(I^n)}]_{s,2}, & s \in [0,1]; \\ \widehat{\vec{H}^1_0(I^n)} \cap H^s(I^n)^n, & s \ge 1 \end{cases}$$

*and* $\widehat{L^2(I^n)^n} \cap H^{\widetilde{s}}(I^n)^n$, *respectively. For* $s = \widetilde{s} = 0$, *the collections are bi-orthogonal Riesz bases for* $\widehat{L^2(I^n)^n}$.

Now, we are in the position to apply the basis transform. Let $A^\lambda$ be an orthogonal matrix with its 1st row given by

$$A^\lambda_1 =: \frac{1}{(\sum_{i=1}^n 4^{|\lambda_i|})^{\frac{1}{2}}} \left( 2^{|\lambda_1|}, \ldots, 2^{|\lambda_n|} \right) = (\alpha_1, \ldots, \alpha_n) =: \alpha^T.$$

Such an example is known as the Householder transform

$$A^\lambda = I - \frac{2(\alpha - \vec{e}_1)(\alpha - \vec{e}_1)^T}{(\alpha - \vec{e}_1)^T(\alpha - \vec{e}_1)},$$

which is

$$A^\lambda = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & -\alpha_1 \end{pmatrix} \quad \text{and} \quad A^\lambda = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & 1 - \frac{\alpha_2^2}{1-\alpha_1} & -\frac{\alpha_2\alpha_3}{1-\alpha_1} \\ \alpha_3 & -\frac{\alpha_2\alpha_3}{1-\alpha_1} & 1 - \frac{\alpha_3^2}{1-\alpha_1} \end{pmatrix}$$

in the case $n = 2$ and $n = 3$. Defining

$$\begin{pmatrix} \psi^{(n)}_{\lambda,1} \\ \vdots \\ \psi^{(n)}_{\lambda,n} \end{pmatrix} =: A^\lambda \begin{pmatrix} \underline{\psi}^{(n)}_{\lambda,1} \\ \vdots \\ \underline{\psi}^{(n)}_{\lambda,n} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \widetilde{\psi}^{(n)}_{\lambda,1} \\ \vdots \\ \widetilde{\psi}^{(n)}_{\lambda,n} \end{pmatrix} =: A^\lambda \begin{pmatrix} \underline{\widetilde{\psi}}^{(n)}_{\lambda,1} \\ \vdots \\ \underline{\widetilde{\psi}}^{(n)}_{\lambda,n} \end{pmatrix}.$$

Set

$$\Psi^{(n)} =: \{\psi^{(n)}_{\lambda,k} : 1 \le k \le n, \lambda \in \vec{\nabla}\} \quad \text{and} \quad \widetilde{\Psi}^{(n)} =: \{\widetilde{\psi}^{(n)}_{\lambda,k} : 1 \le k \le n, \lambda \in \vec{\nabla}\}.$$

Applying the property of an orthogonal transform, we infer the following result.

**Proposition 4.2** *For $0 \le s < \gamma$ and $0 \le \widetilde{s} < \widetilde{\gamma} - 1$, the sets*

$$\left\{\left(\sum_{i=1}^{n} 4^{|\lambda_i|}\right)^{-\frac{s}{2}} \psi^{(n)}_{\lambda,k} : 1 \le k \le n, \lambda \in \vec{\nabla}\right\} \quad \textit{and}$$
$$\left\{\left(\sum_{i=1}^{n} 4^{|\lambda_i|}\right)^{-\frac{\widetilde{s}}{2}} \widetilde{\psi}^{(n)}_{\lambda,k} : 1 \le k \le n, \lambda \in \vec{\nabla}\right\}$$

*are Riesz bases for the vector spaces*

$$\begin{cases} [\widehat{L^2(I^n)^n}, \widehat{\vec{H}^1_0(I^n)}]_{s,2}, & s \in [0,1]; \\ \widehat{\vec{H}^1_0(I^n)} \cap H^s(I^n)^n, & s \ge 1 \end{cases}$$

*and $\widehat{L^2(I^n)^n} \cap H^{\widetilde{s}}(I^n)^n$, respectively. In particular, for $s = \widetilde{s} = 0$, the collections $\Psi^{(n)}$ and $\widetilde{\Psi}^{(n)}$ are bi-orthogonal Riesz bases for $\widehat{L^2(I^n)^n}$.*

In the following, we are mainly concerned with the cases $n = 2$ and $n = 3$ because of the complicated form of curl operators in $n > 3$.

**Theorem 4.1** *Let $\Psi^{(n)}_{curl} =: \{\psi^{(n)}_{\lambda,1} : \lambda \in \vec{\nabla}\}$ and $\widetilde{\Psi}^{(n)}_{comp} =: \{\widetilde{\psi}^{(n)}_{\lambda,k} : 2 \le k \le n, \lambda \in \vec{\nabla}\}$. Then*

(i) $\Psi^{(n)}_{curl} \subset \widehat{\mathcal{H}}(I^n)$ $(n = 2,3)$, $\widetilde{\Psi}^{(2)}_{comp} \subset \overrightarrow{curl}\widehat{H}^1(I^2)$ *or* $\widetilde{\Psi}^{(3)}_{comp} \subset curl(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3))$.

(ii) $\{(\sum_{i=1}^{n} 4^{|\lambda_i|})^{-\frac{m}{2}} \psi^{(n)}_{\lambda,k} : 1 \le k \le n, \lambda \in \vec{\nabla}\}$ *is a Riesz basis for the vector valued space* $\widehat{\vec{H}^m_0(I^n)} =: \vec{H}^m_0(I^n) \cap \widehat{L^2(I^n)^n}$.

*Proof* (i) It is easy to see that $\underline{\psi}^{(n)}_{\lambda,k} \in H_0(curl; I^n)$ for $1 \le k \le n$, then

$$\psi^{(n)}_{\lambda,1} = \alpha_1 \underline{\psi}^{(n)}_{\lambda,1} + \alpha_2 \underline{\psi}^{(n)}_{\lambda,2} + \cdots + \alpha_n \underline{\psi}^{(n)}_{\lambda,n} \in H_0(curl; I^n).$$

Furthermore, $\overrightarrow{curl}\psi^{(2)}_{\lambda,1} = 0$ and $curl\psi^{(3)}_{\lambda,1} = \vec{0}$. Therefore, $\Psi^{(n)}_{curl} \subset \widehat{\mathcal{H}}(I^n)$ $(n = 2,3)$. In addition,

$$\widetilde{\psi}^{(2)}_{\lambda,2} = \alpha_2 \underline{\widetilde{\psi}}^{(2)}_{\lambda,1} - \alpha_1 \underline{\widetilde{\psi}}^{(2)}_{\lambda,2} = \begin{pmatrix} \alpha_2 \widetilde{\psi}_{\lambda_1} \otimes \widetilde{\psi}^-_{\lambda_2} \\ -\alpha_1 \widetilde{\psi}^-_{\lambda_1} \otimes \widetilde{\psi}_{\lambda_2} \end{pmatrix}$$
$$= -\frac{1}{(4^{|\lambda_1|} + 4^{|\lambda_2|})^{\frac{1}{2}}} \overrightarrow{curl}\widetilde{\psi}_{\lambda_1} \otimes \widetilde{\psi}_{\lambda_2} \in \overrightarrow{curl}\widehat{H}^1(I^2).$$

Therefore, we obtain $\widetilde{\Psi}^{(2)}_{comp} \subset \overrightarrow{curl}\widehat{H}^1(I^2)$. Finally, suppose that $a$, $b$ and $c$ are the solutions of

$$\begin{cases} b2^{|\lambda_3|} - c2^{|\lambda_2|} = A_{21}, \\ -a2^{|\lambda_3|} + c2^{|\lambda_1|} = A_{22}, \\ -b2^{|\lambda_1|} + a2^{|\lambda_2|} = A_{23}, \end{cases}$$

whose existence can be guaranteed by the orthogonality of $A^\lambda$. Then

$$\begin{aligned} \widetilde{\psi}^{(3)}_{\lambda,2} &= A_{21}\underline{\widetilde{\psi}}^{(3)}_{\lambda,1} + A_{22}\underline{\widetilde{\psi}}^{(3)}_{\lambda,2} + A_{23}\underline{\widetilde{\psi}}^{(3)}_{\lambda,3} = \begin{pmatrix} A_{21}\widetilde{\psi}_{\lambda_1} \otimes \widetilde{\psi}^-_{\lambda_2} \otimes \widetilde{\psi}^-_{\lambda_3} \\ A_{22}\widetilde{\psi}^-_{\lambda_1} \otimes \widetilde{\psi}_{\lambda_2} \otimes \widetilde{\psi}^-_{\lambda_3} \\ A_{23}\widetilde{\psi}^-_{\lambda_1} \otimes \widetilde{\psi}^-_{\lambda_2} \otimes \widetilde{\psi}_{\lambda_3} \end{pmatrix} \\ &= curl \begin{pmatrix} a\widetilde{\psi}^-_{\lambda_1} \otimes \widetilde{\psi}_{\lambda_2} \otimes \widetilde{\psi}_{\lambda_3} \\ b\widetilde{\psi}_{\lambda_1} \otimes \widetilde{\psi}^-_{\lambda_2} \otimes \widetilde{\psi}_{\lambda_3} \\ c\widetilde{\psi}_{\lambda_1} \otimes \widetilde{\psi}_{\lambda_2} \otimes \widetilde{\psi}^-_{\lambda_3} \end{pmatrix} \in curl\big(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3)\big). \end{aligned}$$

Similarly, if $a$, $b$ and $c$ are the solutions of the equation

$$\begin{cases} b2^{|\lambda_3|} - c2^{|\lambda_2|} = A_{31}, \\ -a2^{|\lambda_3|} + c2^{|\lambda_1|} = A_{32}, \\ -b2^{|\lambda_1|} + a2^{|\lambda_2|} = A_{33}, \end{cases}$$

then we can also obtain

$$\widetilde{\psi}^{(3)}_{\lambda,3} = curl \begin{pmatrix} a\widetilde{\psi}^-_{\lambda_1} \otimes \widetilde{\psi}_{\lambda_2} \otimes \widetilde{\psi}_{\lambda_3} \\ b\widetilde{\psi}_{\lambda_1} \otimes \widetilde{\psi}^-_{\lambda_2} \otimes \widetilde{\psi}_{\lambda_3} \\ c\widetilde{\psi}_{\lambda_1} \otimes \widetilde{\psi}_{\lambda_2} \otimes \widetilde{\psi}^-_{\lambda_3} \end{pmatrix} \in curl\big(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3)\big).$$

Therefore, $\widetilde{\Psi}^{(3)}_{comp} \subset curl(\widehat{H}^1_1(I^3) \times \widehat{H}^1_2(I^3) \times \widehat{H}^1_3(I^3))$.

(ii) Since $\gamma > m$, taking $s = m$ in Proposition 4.2, we know the set $\{(\sum_{i=1}^n 4^{|\lambda_i|})^{-\frac{m}{2}} \psi^{(n)}_{\lambda,k} : 1 \le k \le n, \lambda \in \vec{\nabla}\}$ is a Riesz basis for $\widehat{\vec{H}^1_0(I^n)} \cap H^m(I^n)^n$. Furthermore, it is easy to verify

$$\widehat{\vec{H}^1_0(I^n)} \cap H^m(I^n)^n = \vec{H}^m_0(I^n) \cap \widehat{L^2(I^n)^n} = \widehat{\vec{H}^m_0(I^n)}. \qquad \square$$

#### Competing interests

The author declares that they have no competing interests.

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