View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Crossref

He and Huang *Journal of Inequalities and Applications* 2014, 2014:114
http://www.journalofinequalitiesandapplications.com/content/2014/1/114

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Inequalities for M-tensors

Jun He* and Ting-Zhu Huang

*Correspondence: hejunfan1@163.com
School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, P.R. China

Abstract
In this paper, we establish some important properties of *M*-tensors. We derive upper and lower bounds for the minimum eigenvalue of *M*-tensors, bounds for eigenvalues of *M*-tensors except the minimum eigenvalue are also presented; finally, we give the Ky Fan theorem for *M*-tensors.

MSC: 15A18; 15A69; 65F15; 65F10

Keywords: *M*-tensors; nonnegative tensor; spectral radius; eigenvalues

1 Introduction

Eigenvalue problems of higher-order tensors have become an important topic of study in a new applied mathematics branch, numerical multilinear algebra, and they have a wide range of practical applications [1–7].

If there are a complex number λ and a nonzero complex vector x that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called the eigenvalue of $\mathcal{A}$ and x the eigenvector of $\mathcal{A}$ associated with λ, where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose ith component is

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_n} \right)_{1\le i\le n},$$
$$x^{[m-1]} := \left(x_i^{m-1}\right)_{1\le i\le n}.$$

This definition was introduced by Qi and Lim [8, 9] where they supposed that $\mathcal{A}$ is an order m dimension n symmetric tensor and m is even. First, we introduce some results of nonnegative tensors [10–12], which are generalized from nonnegative matrices.

Definition 1.1 The tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $\mathbb{J} \subset \{1, 2, \ldots, n\}$ such that $a_{i_1,i_2,\ldots,i_m} = 0$, $\forall i_1 \in \mathbb{J}$, $\forall i_2, \ldots, i_m \notin \mathbb{J}$. If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ to be irreducible.

Let $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda$ is an eigenvalue of $\mathcal{A}\}$, where $|\lambda|$ denotes the modulus of λ. We call $\rho(\mathcal{A})$ the spectral radius of tensor $\mathcal{A}$.

©2014 He and Huang; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Theorem 1.2 *If $\mathcal{A}$ is irreducible and nonnegative, then there exists a number $\rho(\mathcal{A}) > 0$ and a vector $x_0 > 0$ such that $\mathcal{A}x_0^{m-1} = \rho(\mathcal{A})x_0^{[m-1]}$. Moreover, if λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \rho(\mathcal{A})$. If λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \le \rho(\mathcal{A})$.*

The authors in [13, 14] extended the notion of M-matrices to higher-order tensors and introduced the definition of an M-tensor.

Definition 1.3 Let $\mathcal{A}$ be an m-order and n-dimensional tensor. $\mathcal{A}$ is called an M-tensor if there exist a nonnegative tensor $\mathcal{B}$ and a real number $c > \rho(\mathcal{B})$, where $\mathcal{B}$ is the spectral radius of $\mathcal{B}$, such that

$$\mathcal{A} = c\mathcal{I} - \mathcal{B}.$$

Theorem 1.4 *Let $\mathcal{A}$ be an M-tensor and denote by $\tau(\mathcal{A})$ the minimal value of the real part of all eigenvalues of $\mathcal{A}$. Then $\tau(\mathcal{A}) > 0$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector. Moreover, there exist a nonnegative tensor $\mathcal{B}$ and a real number $c > \rho(\mathcal{B})$ such that $\mathcal{A} = c\mathcal{I} - \mathcal{B}$. If $\mathcal{A}$ is irreducible, then $\tau(\mathcal{A})$ is the unique eigenvalue with a positive eigenvector.*

In this paper, let $N = \{1, 2, \ldots, n\}$, we define the ith row sum of $\mathcal{A}$ as $R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m}$, and denote the largest and the smallest row sums of $\mathcal{A}$ by

$$R_{\max}(\mathcal{A}) = \max_{i=1,\ldots,n} R_i(\mathcal{A}), \qquad R_{\min}(\mathcal{A}) = \min_{i=1,\ldots,n} R_i(\mathcal{A}).$$

Furthermore, a real tensor of order m dimension n is called the unit tensor, if its entries are $\delta_{i_1\cdots i_m}$ for $i_1, \ldots, i_m \in N$, where

$$\delta_{i_1\cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

And we define $\sigma(\mathcal{A})$ as the set of all the eigenvalues of $\mathcal{A}$ and

$$r_i(\mathcal{A}) = \sum_{\delta_{ii_2\cdots i_m}=0} |a_{ii_2\cdots i_m}|, \qquad r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| = r_i(\mathcal{A}) - |a_{ij\cdots j}|.$$

In this paper, we continue this research on the eigenvalue problems for tensors. In Section 2, some bounds for the minimum eigenvalue of M-tensors are obtained, and proved to be tighter than those in Theorem 1.1 in [15]. In Section 3, some bounds for eigenvalues of M-tensors except the minimum eigenvalue are given. Moreover, the Ky Fan theorem for M-tensors is presented in Section 4.

2 Bounds for the minimum eigenvalue of *M*-tensors

Theorem 2.1 *Let $\mathcal{A}$ be an irreducible M-tensor. Then*

$$\tau(\mathcal{A}) \le \min\{a_{i\cdots i}\}, \tag{1}$$

$$R_{\min}(\mathcal{A}) \le \tau(\mathcal{A}) \le R_{\max}(\mathcal{A}). \tag{2}$$

Proof Let $x > 0$ be an eigenvector of $\mathcal{A}$ corresponding to $\tau(\mathcal{A})$, *i.e.*, $\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}$. For each $i \in N$, we can get

$$\big(a_{i\cdots i} - \tau(\mathcal{A})\big)x_i^{m-1} = -\sum_{\delta_{ii_2\cdots i_m}=0} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m} \geq 0,$$

then

$$\tau(\mathcal{A}) \leq \min\{a_{i\cdots i}\}.$$

Assume that x_s is the smallest component of x,

$$\big(a_{s\cdots s} - \tau(\mathcal{A})\big)x_s^{m-1} = -\sum_{\delta_{si_2\cdots i_m}=0} a_{si_2\cdots i_m} x_{i_2}\cdots x_{i_m} \geq 0.$$

That is,

$$\tau(\mathcal{A}) \leq \sum_{\delta_{si_2\cdots i_m}=0} a_{si_2\cdots i_m} + a_{s\cdots s},$$

so

$$\tau(\mathcal{A}) \leq R_{\max}(\mathcal{A}).$$

Similarly, if we assume that $x_t = \{\max x_i, i \in N\}$, then we can get

$$\tau(\mathcal{A}) \geq \sum_{\delta_{ti_2\cdots i_m}=0} a_{ti_2\cdots i_m} + a_{t\cdots t} \geq R_{\min}(\mathcal{A}).$$

Thus, we complete the proof. □

Theorem 2.2 *Let $\mathcal{A}$ be an irreducible M-tensor. Then*

$$\min_{i,j\in N, j\neq i} \frac{1}{2}\big\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\big\}$$
$$\leq \tau(\mathcal{A}) \leq \max_{i,j\in N, j\neq i} \frac{1}{2}\big\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\big\}, \quad (3)$$

where

$$\Delta_{i,j}(\mathcal{A}) = \big(a_{i\cdots i} - a_{j\cdots j} + r_i^j(\mathcal{A})\big)^2 - 4a_{ij\cdots j}r_j(\mathcal{A}).$$

Proof Because $\tau(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$, from Theorem 2.1 in [15], there are $i, j \in N$, $j \neq i$, such that

$$\big(\big|\tau(\mathcal{A}) - a_{i\cdots i}\big| - r_i^j(\mathcal{A})\big)\big|\tau(\mathcal{A}) - a_{j\cdots j}\big| \leq |a_{ij\cdots j}|r_j(\mathcal{A}).$$

From Theorem 2.1, we can get

$$\big(a_{i\cdots i} - \tau(\mathcal{A}) - r_i^j(\mathcal{A})\big)\big(a_{j\cdots j} - \tau(\mathcal{A})\big) \leq -a_{ij\cdots j}r_j(\mathcal{A}),$$

equivalently,

$$\tau(\mathcal{A})^2 - \bigl(a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A})\bigr)\tau(\mathcal{A}) + a_{j\cdots j}\bigl(a_{i\cdots i} - r_i^j(\mathcal{A})\bigr) + a_{ij\cdots j}r_j(\mathcal{A}) \le 0.$$

Then, solving for $\tau(\mathcal{A})$,

$$\tau(\mathcal{A}) \ge \frac{1}{2}\bigl\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\bigr\} \ge \min_{i,j\in N, j\neq i} \frac{1}{2}\bigl\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\bigr\}.$$

Let $x > 0$ be an eigenvector of $\mathcal{A}$ corresponding to $\tau(\mathcal{A})$, *i.e.*, $\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}$, x_s is the smallest component of x. For each $s, t \in N$, $s \neq t$, we can get

$$\bigl(a_{t\cdots t} - \tau(\mathcal{A})\bigr)x_t^{m-1} = -\sum_{\delta_{ti_2\cdots i_m}=0} a_{ti_2\cdots i_m}x_{i_2}\cdots x_{i_m} \ge r_t(\mathcal{A})x_s^{m-1}, \tag{4}$$

$$\bigl(a_{s\cdots s} - \tau(\mathcal{A})\bigr)x_s^{m-1} = -\sum_{\substack{\delta_{ti_2\cdots i_m}=0,\\ \delta_{si_2\cdots i_m}=0}} a_{ti_2\cdots i_m}x_{i_2}\cdots x_{i_m} - a_{st\cdots t}x_t^{m-1} \ge r_t^s(\mathcal{A})x_s^{m-1} - a_{st\cdots t}x_t^{m-1},$$

$$\bigl(a_{s\cdots s} - \tau(\mathcal{A}) - r_t^s(\mathcal{A})\bigr)x_s^{m-1} \ge -a_{st\cdots t}x_t^{m-1}. \tag{5}$$

Multiplying equations (4) and (5), we get

$$\bigl(a_{t\cdots t} - \tau(\mathcal{A})\bigr)\bigl(a_{s\cdots s} - \tau(\mathcal{A}) - r_t^s(\mathcal{A})\bigr) \ge -a_{st\cdots t}r_t(\mathcal{A}).$$

Then, solving for $\tau(\mathcal{A})$,

$$\tau(\mathcal{A}) \le \frac{1}{2}\bigl\{a_{t\cdots t} + a_{s\cdots s} - r_t^s(\mathcal{A}) - \Delta_{t,s}^{\frac{1}{2}}(\mathcal{A})\bigr\} \le \max_{i,j\in N, j\neq i} \frac{1}{2}\bigl\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\bigr\}.$$

Thus, we complete the proof. □

We now show that the bounds in Theorem 2.2 are tight and sharper than those in Theorem 1.1 in [15] by the following example. Consider the M-tensor $\mathcal{A} = (a_{ijkl})$ of order 4 dimension 2 with entries defined as follows:

$$a_{1111} = 3, \qquad a_{1222} = -1,$$
$$a_{2111} = -2, \qquad a_{2222} = 2,$$

other $a_{ijkl} = 0$. By Theorem 1.1 in [15], we have

$$-2 \le \tau(\mathcal{A}) \le 4.$$

By Theorem 2.1, we have

$$0 \le \tau(\mathcal{A}) \le 2.$$

By Theorem 2.2, we have

$$\frac{1}{2}(5 - \sqrt{17}) \le \tau(\mathcal{A}) \le \frac{1}{2}(5 - \sqrt{5}).$$

In fact, $\tau(\mathcal{A}) = 1$. Hence, the bounds in Theorem 2.2 are tight and sharper than those in Theorem 1.1 in [15].

3 Bounds for eigenvalues of *M*-tensors except the minimum eigenvalue

In this section, we introduce the stochastic M-tensor, which is a generalization of the nonnegative stochastic tensor.

Definition 3.1 An M-tensor $\mathcal{A}$ of order m dimension n is called stochastic provided

$$R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} \equiv 1, \quad i = 1,\ldots,n.$$

Obviously, when $\mathcal{A}$ is a stochastic M-tensor, 1 is the minimum eigenvalue of $\mathcal{A}$ and e is an eigenvector corresponding to 1, where e is an all-ones vector.

Theorem 3.2 *Let $\mathcal{A}$ be an order m dimension n irreducible M-tensor. Then there exists a diagonal matrix D with positive main diagonal entries such that*

$$\tau(\mathcal{A}) \cdot \mathcal{B} = \mathcal{A} \cdot D^{(1-m)} \cdot \overbrace{D \cdot \ldots \cdot D}^{m-1},$$

where B is a stochastic irreducible M-tensor. Furthermore, B is unique, and the diagonal entries of D are exactly the components of the unique positive eigenvector corresponding to $\tau(\mathcal{A})$.

Proof Let x be the unique positive eigenvector corresponding to $\tau(\mathcal{A})$, *i.e.*,

$$\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}.$$

Let D be the diagonal matrix such that its diagonal entries are components of x, let us check the tensor $\mathcal{C} = \mathcal{A} \cdot D^{(1-m)} \cdot D \cdot \ldots \cdot D$. It is clear that for $i = 1, 2, \ldots, n$,

$$\sum_{i_2,\ldots,i_m=1}^{n} \mathcal{C}_{ii_2\cdots i_m} = \left(\mathcal{C}e^{m-1}\right)_i = \left(\mathcal{A} \cdot D^{(1-m)} \cdot \overbrace{D \cdot \ldots \cdot D}^{m-1} e^{m-1}\right)_i = \tau(\mathcal{A}).$$

Hence $\mathcal{B} = \mathcal{C}/\tau(\mathcal{A})$ is the desired stochastic M-tensor. Since the positive eigenvector is unique, then B is unique, and the diagonal entries of D are exactly the components of the unique positive eigenvector corresponding to $\tau(\mathcal{A})$. □

Theorem 3.3 *Let $\mathcal{A}$ be an order m dimension n stochastic irreducible nonnegative tensor, $\omega = \min a_{i\cdots i}$, $\lambda \in \sigma(\mathcal{A})$. Then*

$$|\lambda - \omega| \leq 1 - \omega.$$

Proof Let λ be an eigenvalue of the stochastic irreducible nonnegative tensor $\mathcal{A}$, x is the eigenvector corresponding to λ, *i.e.*,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

Assume that $0 < |x_s| = \max_i |x_i|$, then we can get

$$(\lambda - a_{s\cdots s})x_s^{m-1} = \sum_{\delta_{si_2\cdots i_m}=0} a_{si_2\cdots i_m} x_{i_2}\cdots x_{i_m}.$$

Then

$$|\lambda - a_{s\cdots s}| \leq \sum_{\delta_{si_2\cdots i_m}=0} a_{si_2\cdots i_m} = r_s(\mathcal{A}) = 1 - a_{s\cdots s},$$

and therefore,

$$\begin{aligned}
|\lambda - \omega| &\leq |\lambda - a_{s\cdots s} + a_{s\cdots s} - \omega| \\
&\leq |\lambda - a_{s\cdots s}| + |a_{s\cdots s} - \omega| \\
&\leq (1 - a_{s\cdots s}) + (a_{s\cdots s} - \omega) \\
&= 1 - \omega. \qquad (6)
\end{aligned}$$

Thus, we complete the proof. □

Theorem 3.4 *Let $\mathcal{A}$ be an order m dimension n irreducible M-tensor,* $\Omega = \max a_{i\cdots i}$, $\lambda \in \sigma(\mathcal{A})$. *Then*

$$|\Omega - \lambda| \leq \Omega - \tau(\mathcal{A}).$$

Proof From Theorem 3.2, we may evidently take $\tau(\mathcal{A}) = 1$, and after performing a similarity transformation with a positive diagonal matrix, we may assume that $\mathcal{A}$ is stochastic. Then, for $\theta \in (0,1)$, the matrix $\mathcal{A}(\theta) = (1+\theta)\mathcal{I} - \theta\mathcal{A}$ is irreducible nonnegative stochastic, by Theorem 3.3, if $\lambda(\theta) \in \sigma(\mathcal{A}(\theta))$, $\omega(\theta) = \min a_{i\cdots i}(\theta)$, we can get

$$\bigl|\lambda(\theta) - \omega(\theta)\bigr| \leq 1 - \omega(\theta).$$

That is,

$$\bigl|1 + \theta - \theta\lambda - (1 + \theta - \theta \max a_{i\cdots i})\bigr| \leq 1 - (1 + \theta - \theta \max a_{i\cdots i}).$$

Then

$$|\Omega - \lambda| \leq \Omega - 1.$$

Transforming back to $\mathcal{A}$, we get

$$|\Omega - \lambda| \leq \Omega - \tau(\mathcal{A}).$$

Thus, we complete the proof. □

4 Ky Fan theorem for *M*-tensors

In this section we give the Ky Fan theorem for M-tensors. Denote by $\mathbb{Z}$ the set of m-order and n-dimensional real tensors whose off-diagonal entries are nonpositive.

Theorem 4.1 *Let $\mathcal{A}, \mathcal{B} \in \mathbb{Z}$, assume that $\mathcal{A}$ is an M-tensor and $\mathcal{B} \geq \mathcal{A}$. Then $\mathcal{B}$ is an M-tensor, and*

$$\tau(\mathcal{A}) \leq \tau(\mathcal{B}).$$

Proof If $x > 0$, from assume that $\mathcal{A}$ is an M-tensor and condition (D4) in [14], we know

$$\mathcal{A}x^{m-1} > 0.$$

Because $\mathcal{B} \geq \mathcal{A}$, we can get

$$\mathcal{B}x^{m-1} \geq \mathcal{A}x^{m-1} > 0,$$

then $\mathcal{B}$ is an M-tensor.

Let $a = \max_{1 \leq i \leq n} \mathcal{B}_{i \cdots i}$, from Theorem 3.1 and Corollary 3.2 in [13], assume that

$$\mathcal{B} = a\mathcal{I} - \mathcal{C}_{\mathcal{B}}, \qquad \mathcal{A} = a\mathcal{I} - \mathcal{C}_{\mathcal{A}},$$

where $\mathcal{C}_{\mathcal{A}}$, $\mathcal{C}_{\mathcal{B}}$ are nonnegative tensors.

Because $\mathcal{A}, \mathcal{B} \in \mathbb{Z}$ and $\mathcal{B} \geq \mathcal{A}$, then we can get

$$\mathcal{C}_{\mathcal{A}} \geq \mathcal{C}_{\mathcal{B}}.$$

From Lemma 3.5 in [12], we can get

$$\rho(\mathcal{C}_{\mathcal{A}}) \geq \rho(\mathcal{C}_{\mathcal{B}}).$$

Therefore,

$$\tau(\mathcal{A}) \leq \tau(\mathcal{B}).$$

Thus, we complete the proof. □

Theorem 4.2 *Let $\mathcal{A}$, $\mathcal{B}$ be of order m dimension n, suppose that $\mathcal{B}$ is an M-tensor and $|b_{i_1 \cdots i_m}| \geq |a_{i_1 \cdots i_m}|$ for all $i_1 \neq \cdots \neq i_m$. Then, for any eigenvalue λ of $\mathcal{A}$, there exists $i \in 1, \ldots, n$ such that $|\lambda - a_{i \cdots i}| \leq b_{i \cdots i} - \tau(\mathcal{B})$.*

Proof We first suppose that $\mathcal{B}$ is an M-tensor, $\tau(\mathcal{B})$ is an eigenvalue of $\mathcal{B}$ with a positive corresponding eigenvector v. Denote

$$W = \operatorname{diag}(v_1, \ldots, v_n),$$

where v_i is the ith component of v. Let

$$\mathcal{C} = \mathcal{A} \cdot W^{1-m} \overbrace{W \cdot \ldots \cdot W}^{[m-1]}$$

and let λ be an eigenvalue of $\mathcal{A}$ with x, a corresponding eigenvector, *i.e.*, $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$. Then, as in the proof of Theorem 4.1 in [12], we have

$$\mathcal{C}\left(W^{-1}x\right)^{m-1} = \lambda\left(W^{-1}x\right)^{m-1}.$$

By the definition of $\mathcal{C}$, we have $c_{i\cdots i} = a_{i\cdots i}$, $i = 1, \ldots, n$. Applying the first conclusion of Theorem 6 of [8], we can get

$$\begin{aligned} |\lambda - c_{i\cdots i}| &\le \sum_{\delta_{ii_2\cdots i_m}=0} |c_{ii_2\cdots i_m}| \\ &= v_i^{1-m} \sum |a_{ii_2\cdots i_m}| v_{i_2} \cdots v_{i_m} \\ &\le v_i^{1-m} \sum |b_{ii_2\cdots i_m}| v_{i_2} \cdots v_{i_m} \\ &= v_i^{1-m} \left(b_{i\cdots i} v^{m-1} - \sum_{i_1,\ldots,i_m=1} b_{ii_2\cdots i_m} v_{i_2} \cdots v_{i_m} \right) \\ &= b_{i\cdots i} - \tau(\mathcal{B}). \end{aligned} \tag{7}$$

Thus, we complete the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Acknowledgements
This research is supported by NSFC (61170311), Chinese Universities Specialized Research Fund for the Doctoral Program (20110185110020), Sichuan Province Sci. & Tech. Research Project (12ZC1802). The first author is supported by the Fundamental Research Funds for Central Universities.

Received: 3 January 2014 Accepted: 27 February 2014 Published: 13 Mar 2014

References

1. Chang, KC, Pearson, K, Zhang, T: On eigenvalue problems of real symmetric tensors. J. Math. Anal. Appl. **350**, 416-422 (2009)
2. Chang, KC, Qi, L, Zhou, G: Singular values of a real rectangular tensor. J. Math. Anal. Appl. **37**, 284-294 (2010)
3. Chang, KC, Pearson, K, Zhang, T: Primitivity, the convergence of the NZQ method, and the largest eigenvalue for nonnegative tensors. SIAM J. Matrix Anal. Appl. **32**, 806-819 (2011)
4. Qi, L: Eigenvalues and invariants of tensor. J. Math. Anal. Appl. **325**, 1363-1377 (2007)
5. Qi, L: Symmetric nonnegative tensors and copositive tensors. Linear Algebra Appl. **439**, 228-238 (2013)
6. Liu, Y, Zhou, G, Ibrahim, NF: An always convergent algorithm for the largest eigenvalue of an irreducible nonnegative tensor. J. Comput. Appl. Math. **235**, 286-292 (2010)
7. Ng, M, Qi, L Zhou, G: Finding the largest eigenvalue of a non-negative tensor. SIAM J. Matrix Anal. Appl. **31**, 1090-1099 (2009)
8. Qi, L: Eigenvalues of a real supersymmetric tensor. J. Symb. Comput. **40**, 1302-1324 (2005)
9. Lim, LH: Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, CAMSAP 05, pp. 129-132 (2005)
10. Chang, KC, Zhang, T, Pearson, K: Perron-Frobenius theorem for nonnegative tensors. Commun. Math. Sci. **6**, 507-520 (2008)
11. Yang, Y, Yang, Q: Further results for Perron-Frobenius theorem for nonnegative tensors. SIAM J. Matrix Anal. Appl. **31**, 2517-2530 (2010)

12. Yang, Y, Yang, Q: Further results for Perron-Frobenius theorem for nonnegative tensors. II. SIAM J. Matrix Anal. Appl. **32**, 1236-1250 (2011)
13. Zhang, L, Qi, L, Zhou, G *M*-tensors and the Positive Definiteness of a Multivariate Form. arXiv:1202.6431v1
14. Ding, W, Qi, L, Wei, Y: *M*-tensors and nonsingular *M*-tensors. Linear Algebra Appl. **439**, 3264-3278 (2013)
15. Li, C, Li, Y, Kong, X: New eigenvalue inclusion sets for tensors. Numer. Linear Algebra Appl. **21**, 39-50 (2014)

10.1186/1029-242X-2014-114
Cite this article as: He and Huang: Inequalities for *M*-tensors. *Journal of Inequalities and Applications* 2014, 2014:114

Submit your manuscript to a SpringerOpen journal and benefit from:

► Convenient online submission
► Rigorous peer review
► Immediate publication on acceptance
► Open access: articles freely available online
► High visibility within the field
► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com