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On stability and state feedback stabilization of singular linear matrix difference equations

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Abstract

In this article, we study the stability of a class of singular linear matrix difference equations whose coefficients are square constant matrices and the leading coefficient matrix is singular. Speciffically we analyze the stability, the asymptotic stability and the Lyapunov stability of the equilibrium states of an homogeneous singular linear discrete time system and we define the set of all equilibrium states. After we prove that if every equilibrium state of the homogeneous system is stable in the Lyapounov's sense, then all solutions of the non homogeneous system are continuously depending on the initial conditions and are bounded provided that the input vector is also bounded. Moreover, we consider the case where the equilibrium states of the system are not stable. For this case we provide necessary and sufficient conditions for stabilization.

Keywords: matrix difference equations, linear, discrete time system, stability, equilibrium state, pencil, singular

1 Introduction

Linear discrete time systems (or linear matrix difference equations), are systems in which the variables take their value at instantaneous time points. Discrete time systems differ from continuous time ones in that their signals are in the form of sampled data. With the development of the digital computer, the discrete time system theory plays an important role in control theory. Thus many authors have studied the stability of such systems, see [1-27]. In most cases these articles are referred to regular discrete time systems. In this article we study singular linear matrix difference equations. Thus we consider the singular discrete time system

$$FY_{k+1} = GY_k + V_k \tag{1}$$

with known initial conditions

$$Y_{k_0} \tag{2}$$

where $F, G \in \mathcal{M}(m \times m; \mathcal{F})$, (i.e. the algebra of square matrices with elements in the field $\mathcal{F}$) with $Y_k, V_k \in \mathcal{M}(m \times 1; \mathcal{F})$ and F is a singular matrix (detF = 0). For the sake of simplicity we set $\mathcal{M}_m = \mathcal{M}(m \times m; \mathcal{F})$ and $\mathcal{M}_{nm} = \mathcal{M}(n \times m; \mathcal{F})$. With $0_{m,n} \in \mathcal{M}_{mn}$ we will denote the zero matrix. For $V_k = 0_{m,1}$ we get the homogeneous system of (1)

$$FY_{k+1} = GY_k \tag{3}$$

Because of the singularity of the matrix F, in order to solve and to study these type of systems there are in the literature two methods. The first method is by using the theory of the Drazin inverse, see [4], and the second is by using matrix pencil theory and the Weierstrass canonical form which is a generalization of the Jordan canonical form. The advantage of the second method is that it gives a better understanding of the structure of the system and more deep, elegant results. In this article we will present a theory based on the matrix pencil of the system and we will show how the eigenvalues of the pencil are related with the stability of singular systems.

2 Mathematical backround

2.1 The matrix pencil

Matrix pencil theory has been used many times in articles for the study of linear discrete time systems with constant matrices, see for instance [9,14,21,27-33]. A matrix pencil is a family of matrices sF-G, parametrized by a complex numbers, see [14,21,23,27,34-36]. When G is square and F = I_m, where I_m is the identity matrix, the zeros of the function det(sF-G) are the eigenvalues of G. Consequently, the problem of finding the nontrivial solutions of the equation

$$sFX = GX \tag{4}$$

is called the generalized eigenvalue problem. Although the generalized eigenvalue problem looks like a simple generalization of the usual eigenvalue problem, it exhibits some important differences. In the first place, it is possible for det(sF-G) to be identically zero, independent of s. Second, it is possible for F to be singular, in which case the problem has infinite eigenvalues. To see this, write the generalized eigenvalue problem in the reciprocal form

$$FX = s^{-1}GX \tag{5}$$

If F is singular with a null vector X, then FX = $0_{m,1}$, so that X is an eigenvector of the reciprocal problem corresponding to eigenvalue s^{-1} = 0; i.e., $s = \infty$.

Definition 2.1.1. Given $G \in \mathcal{M}_{mn}$ and an indeterminate $s \in \mathcal{F}$, the matrix pencil sF-G is called regular when m = n and det(sF - G) ≠ 0. In any other case, the pencil will be called singular.

In this article, we consider the case that pencil is *regular*.

The class of sF-G is characterized by a uniquely defined element, known as complex Weierstrass canonical form, sF_w - Q_w, see [14,21,27,34-36], specified by the complete set of invariants of sF-G.

This is the set of *elementary divisors* (e.d.) obtained by factorizing the invariant polynomials into powers of homogeneous polynomials irreducible over field $\mathcal{F}$. In the case where sF-G is regular, we have e.d. of the following type:

- e.d. of the type $(s - a_j)^{p_j}$, *are called finite elementary divisors* (f.e.d.), where a_j is a finite eigenavalue of algebraic multiplicity p_j
- e.d. of the type $\hat{s}^q = \frac{1}{s^q}$ are called *infinite elementary divisors* (i.e.d.), where q the algebraic multiplicity of the infinite eigenvalues

We assume that $\sum_{i=1}^{\nu} p_j = p$ and p+q = m.

Definition 2.1.2. Let $B_1, B_2, \ldots, B_n$ be elements of $\mathcal{M}_n$. The direct sum of them denoted by $B_1 \oplus B_2 \oplus \ldots \oplus B_n$ is the blockdiag $[B_1\ B_2 \ldots B_n]$.

From the regularity of sF-G, there exist nonsingular matrices $P, Q \in \mathcal{M}_m$ such that

$$\begin{aligned} PFQ &= F_w = I_p \oplus H_q \\ PGQ &= G_w = J_p \oplus I_q \end{aligned} \tag{6}$$

Where sF_w - Q_w is the complex Weierstrass form of the regular pencil sF-G and is defined by

$$sF_w - Q_w := sI_p - J_p \oplus sH_q - I_q \tag{7}$$

where the first normal Jordan type element is uniquely defined by the set of the finite eigenvalues,

$$(s - a_1)^{p_1}, \ \ldots, \ (s - a_\nu)^{p_\nu}$$

of sF-G and has the form

$$sI_p - J_p := sI_{p_1} - J_{p_1}(a_1) \oplus \cdots \oplus sI_{p_\nu} - J_{p_\nu}(a_\nu) \tag{8}$$

The second uniquely defined block sH_q - I_q corresponds to the infinite eigenvalues

$$\hat{s}^{q_1}, \ldots, \hat{s}^{q_\sigma}, \sum_{j=1}^{\sigma} q_j = q$$

of sF-G and has the form

$$sH_q - I_q := sH_{q_1} - I_{q_1} \oplus \ldots \oplus sH_{q_\sigma} - I_{q_\sigma} \tag{9}$$

Thus, H_q is a nilpotent element of $\mathcal{M}_n$ with index $\tilde{q} = \max\{q_j : j = 1, 2, \ldots, \sigma\}$, where

$$H_q^{\tilde{q}} = 0_{q,q}$$

and $I_{p_j}, J_{p_j}(a_j), H_{q_j}$ are defined as

$$I_{p_j} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathcal{M}_{p_j},$$

$$J_{p_j}(a_j) = \begin{bmatrix} a_j & 1 & \cdots & 0 & 0 \\ 0 & a_j & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_j & 1 \\ 0 & 0 & \cdots & 0 & a_j \end{bmatrix} \in \mathcal{M}_{p_j} \tag{10}$$

$$H_{q_j} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathcal{M}_{q_j}.$$

For algorithms about the computations of the Jordan matrices, see [14,21,34-36].

2.2 The solution of an homogeneous singular linear discrete time system

In this subsection, we will obtain formulas of the solutions of homogeneous singular linear matrix difference equations.

Definition 2.2.1. Consider the system (1) with known initial conditions (2). Then the initial conditions are called consistent should they satisfy (3) and non-consistent should they not.

For the regular matrix pencil of system (3), there exist nonsingular matrices $P, Q \in \mathcal{M}_m$ as applied in (6), see subsection 2.1. Let

$$Q = \left[Q_p \, Q_q\right] \tag{11}$$

where $Q_p \in \mathcal{M}_{mp}$ is a matrix with columns the p linear independent (generalized) eigenvectors of the p finite eigenvalues of sF-G and $Q_p \in \mathcal{M}_{mq}$ is a matrix with columns the q linear independent (generalized) eigenvectors of the q infinite eigenvalues of sF-G.

Proposition 2.2.1. The initial conditions (2) of the system (3) are consistent if and only if

$$Y_{k_0} \in \text{colspan} Q_p \tag{12}$$

Proof. See [14,21,30,32]

Proposition 2.2.2. Consider the system (3) with initial conditions (2). Then the solution is unique if and only if the initial conditions are consistent. Moreover the analytic solution is given by

$$Y_k = Q_p J_p^{k-k_0} Z_{k_0}^p \tag{13}$$

where $Z_{k_0}^p$ is the unique solution of the algebraic system $Y_{k_0} = Q_p Z_{k_0}^p$.

Proof. See [14,21,28-33,37-39]

2.3 The solution of a non homogeneous singular linear discrete time system

Consider the singular discrete time system (1) with known initial conditions (2).

For the regular matrix pencil sF-G, there exist nonsingular matrices $P, Q \in \mathcal{M}_m$ as applied in (6), see also Section 2.1. Let

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \tag{14}$$

with $P_1 \in \mathcal{M}_{pm}$, $P_2 \in \mathcal{M}_{qm}$ and

$$Q = \left[Q_p \; Q_q\right]$$

where $Q_p \in \mathcal{M}_{mp}$ is a matrix with columns the p linear independent (generalized) eigenvectors of the p finite eigenvalues of sF-G and $Q_q \in \mathcal{M}_{mq}$ is a matrix with columns the q linear independent (generalized) eigenvectors of the q infinite eigenvalues of sF-G.

Proposition 2.3.1. Consider the system (1) with initial conditions (2). Then the solution is unique if and only if

$$Y_{k_0} \in \text{colspan}Q_p - Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V_{k_0+i} \tag{15}$$

Moreover the analytic solution is given from the formula

$$Y_k = Q_p \left(J_p^{k-k_0} Z_{k_0}^p + \sum_{i=k_0}^{k-1} J_p^{k-i-1} P_1 V_i \right) - Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V_{k+i} \tag{16}$$

where $Z_{k_0}^p$ is the unique solution of the algebraic system

$$Y_{k_0} = Q_p Z_{k_0}^p - Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V_{k_0+i}$$

Proof. See [14,21,28-33,37-39]

3 Stability of equilibrium state(s) of homogeneous singular discrete time systems

Definition 3.1. For any system of the form (1), with a constant input vector V_k = V, Y_* is an equilibrium state if it does not change under the initial condition, i.e.: Y_* is an equilibrium state if and only if $Y_{k_0} = Y_*$ implies that $Y_k = Y_*$ for all k ≥ k_0+1.

The set of equilibrium states for a given singular linear system in the form of (3) is given by the following Proposition.

Proposition 3.1. Consider the system (3). Then if 1 is not an eigenvalue of the matrix pencil sF-G then

$$Y_* = 0_{m,1} \tag{17}$$

is the unique equilibrium state of the system (3). If 1 is a finite eigenvalue of the matrix pencil sF-G, then the set E of the equilibrium points of the system (3) is the vector space defined by

$$E = N_r(F - G) \cap \text{colspan}Q_p \tag{18}$$

where N_r is the right null space of the matrix F-G, Q_p is a matrix with columns the p linear independent (generalized) eigenvectors of the p finite eigenvalues of the matrix pencil sF-G or from Proposition 2.2.1 the set of the consistent initial conditions of (3).

Proof. If Y_* is an equilibrium state of system (3), then this implies that for

$$Y_{k_0} = Y_*$$

we have

$$Y_* = Y_k = Y_{k+1}$$

If 1 is not an eigenvalue of the matrix pencil sF-G then det(F-G)≠0 and from the system (3) we have

$$FY_* = GY_*$$

or

$$(F - G)Y_* = 0_{m,1}$$

Then the above algebraic system has the unique solution

$$Y_* = 0_{m,1}$$

which is the unique equilibrium state of the system (3). If 1 is a finite eigenvalue of the matrix pencil sF-G then det(F-G) = 0. If Y_* is an equilibrium state of system (3), then this implies that for

$$Y_{k_0} = Y_*$$

we have

$$Y_* = Y_k = Y_{k+1}$$

This requires that Y_* must be a consistent initial condition which from Proposition 2.2.1 is equal to

$$Y_* \in \text{colspan}Q_p$$

Moreover from system (3) we have

$$FY_* = GY_*$$

or

$$(F - G)Y_* = 0_{m,1}$$

or

$$Y_* \in N_r(F - G)$$

So

$$Y_* \subseteq N_r(F - G) \cap \text{ colspan}Q_p$$

or

$$E \subseteq N_r(F - G) \cap \text{ colspan}Q_p \tag{19}$$

Let now $Y_* \in N_r(F - G) \cap \text{colspan}Q_p$ then we can consider

$$Y_{k_0} = Y_*$$

as a consistent initial condition and

$$(F - G)Y_* = 0_{m1}$$

or

$$FY_* = GY_*$$

where Y_* is solution of system (3) and combined with $Y_{k_0} = Y_*$ we have $Y_* \in$ E or

$$N_r(F - G) \cap \text{ colspan}Q_p \subseteq E$$

From (19) and (20) we arrive at (18).

Definition 3.2 [25]. Let $X \in \mathcal{M}_{r1}$ and $||X|| \in \mathcal{F}$ be norm of vector X. Then if $||A|| = \max \left\{ \frac{||AX||}{||X||} : X \in \mathcal{M}_{r1}, X \neq 0_{r,r} \right\}$

$$||A|| = \max \left\{ \frac{||AX||}{||X||} : X \in \mathcal{M}_{r1}, X \neq 0_{r,r} \right\} \tag{21}$$

Definition 3.3 [25]. Let $A = [a_{ij}]_{i,\ j\ =\ 1,2,\dots,\ r}$ be a square matrix with $A \in \mathcal{M}_{rr}$ and let $X = [x_i]_{i\ =\ 1,2,\dots,\ r}$ be a vector with $X \in \mathcal{M}_{r1}$, then

$$\begin{aligned} ||A||_1 &= \max_{1 \le j \le r} \sum_{i=1}^{r} |a_{ij}| \\ ||X||_1 &= \sum_{i=1}^{r} |x_i| \end{aligned} \tag{22}$$

is by definition the 1-norm for matrices and the 1-morm for vectors respectively, which is simply the maximum absolute column sum of the matrix and the absolute column sum of the vector respectively. Furthermore

$$\begin{aligned} ||A||_\infty &= \max_{1 \le i \le r} \sum_{j=1}^{r} |a_{ij}| \\ ||X||_\infty &= \max_{1 \le i \le r} |x_i| \end{aligned} \tag{23}$$

is by definition the ∞-norm for matrices and the ∞-norm for vectors respectively, which is simply the maximum absolute row sum of the matrix and the maximum absolute row of the vector.

Definition 3.4 [7]. An equilibrium state $Y_* \in$ E of the system (3) is stable in the sense of Lyapounov if, for every δ >0, there exists $\epsilon > 0$, such that the trajectories starting in

$$||Y_{k_0} - Y_*|| \le \delta \tag{24}$$

do not leave

$$||Y - Y_*|| \le \epsilon \tag{25}$$

as k increases indefinitevely.

Definition 3.5 [7]. An equilibrium state $Y_* \in$ E of the system (3) is asymptotically stable if it is stable in the sense of Lyapounov and if every solution starting within (24) converges without leaving (25) to Y_* as k increases indefinitely, i.e.

$$\lim_{k \to \infty} Y_k = Y_* \tag{26}$$

Definition 3.6. An equilibrium state $Y_* \in$ E of the system (3) is asymptotically stable in the large if, asymptotic stability holds for every equilibrium state of the system. In this case the system (3) is asymptotically stable in the large. Consequently, a necessary condition for asymptotic stability in the large is that there exists a unique equilibrium state since the limit $\lim_{k \to \infty} Y_k$ is unique. For system (3) this unique equilibrium state is $Y_* = 0_{m,1}$.

Theorem 3.1. Consider the system (3) and its solution (13). Then an equilibrium state $Y_* \in$ E of the singular discrete time system (3) is stable in the sense of Lyapounov if there exist a constant c $\in$ (0, +∞), such that $\left\| J_p^{k-k_0} \right\| \leq c < +\infty$, for all k ≥ k_0.

Proof. The solution of the system (3) is given from (13),

$$Y_k = Q_p J_p^{k-k_0} Z_{k_0}^p$$

where $Z_{k_0}^p$ is the solution of the algebraic system

$$Y_{k_0} = Q_p Z_{k_0}^p$$

Since the columns of the matrix Q_p are linear independent, the matrix is left invertible. Thus we can define its left inverse matrix $Q_p^{-1} \in \mathcal{M}_{pm}$ such that

$$Q_p^{-1} Q_p = I_p$$

Then

$$Y_k = Q_p J_p^{k-k_0} Q_p^{-1} Y_{k_0}$$

We assume that there exist a constant c $\in$ (0, +∞) such that $\left\| J_p^{k-k_0} \right\| \leq c < +\infty$, for all $k > k_0$. Furthermore let an equilibrium state $Y_* \in E$. Then

$$Y_* = Q_p J_p^{k-k_0} Q_p^{-1} Y_*$$

and easy we obtain

$$Y_k - Y_* = Q_p J_p^{k-k_0} Q_p^{-1} (Y_{k_0} - Y_*)$$

or

$$Y_k - Y_* = \left[Q_p 0_{m,q} \right] \begin{bmatrix} J_p^{k-k_0} & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} \begin{bmatrix} Q_p^{-1} \\ 0_{q,m} \end{bmatrix} (Y_{k_0} - Y_*) \tag{27}$$

If we set $\|Q_p\| = \left\| \left[Q_p 0_{m,q} \right] \right\|$ and $\|Q_p^{-1}\| = \left\| \begin{bmatrix} Q_p^{-1} \\ 0_{q,m} \end{bmatrix} \right\|$. Then by taking norms for every k≥k_0 in (27) we have

$$||Y_k - Y_*|| \leq ||Q_p|| \left\| J_p^{k-k_0} \right\| \left\| Q_p^{-1} \right\| ||Y_{k_0} - Y_*|| \tag{28}$$

Hence for any $\epsilon > 0$, if we chose $\delta(\epsilon) = \frac{\epsilon}{\|Q_p\| \, \|Q_p^{-1}\| c}$, then for

$$||Y_{k_0} - Y_*|| \leq \delta(\epsilon)$$

implies that for every $\epsilon > 0$

$$||Y_k - Y_*|| \leq ||Q_p|| \left\| J_p^{k-k_0} \right\| \left\| Q_p^{-1} \right\| ||Y_{k_0} - Y_*|| \leq ||Q_p|| c \left\| Q_p^{-1} \right\| \frac{\epsilon}{||Q_p|| \, \|Q_p^{-1}\| \, c} \leq \epsilon$$

or

$$||Y_k - Y_*|| \leq \epsilon \tag{29}$$

Theorem 3.2. The system (3) is asymptotically stable in the large, if and only if, all the finite eigenvalues of the matrix pencil sF-G lie within the open disc,

$$|s| < 1$$

Proof. The solution of the system (3) is

$$Y_k = Q_p J_p^{k-k_0} Z_{k_0}^p$$

Let a_j be a finite eigenavalue of the matrix pencil sF-G with algebraic multiplicity p_j. Then the Jordan matrix $J_p^{k-k_0}$ can be written as

$$J_p^{k-k_0} = \text{blockdiag}\left[J_{p_1}^{k-k_0}(a_1) J_{p_2}^{k-k_0}(a_2) \dots J_{p_\nu}^{k-k_0}(a_\nu)\right]$$

with $J_p^{k-k_0} \in \mathcal{M}_{p_j}$ be a Jordan block. Every element of this matrix has the specific form

$$(k-k_0)^{p_j} a_j^{k-k_0}$$

The sequence

$$(k-k_0)^{p_j} |a_j^{k-k_0}|$$

can be written as

$$(k-k_0)^{p_j} e^{(k-k_0) ln|a_j|}$$

The system (3) has the unique equilibrium state $Y_* = 0_{m,1}$ when for every j, $a_j \neq 1$, (see Proposition 3.1) and then the system is asymptotically stable in the large, when

$$\lim_{k \to \infty} Y_k = Y_*$$

Thus this holds if and only if

$$\ln|a_j| < 0$$

or

$$|a_j| < 1$$

Then for $k \to +\infty$

$$(k-k_0)^{p_j} e^{(k-k_0)|\ln a_j|} \to 0$$

or

$$(k-k_0)^{p_j} |a_j|^{(k-k_0)} \to 0$$

or for every k≥k_0

$$J_p^{k-k_0} \to 0_{p,p}$$

Then for every initial condition Y_{k_0}

$$\lim_{k\to\infty} Y_k = 0_{m,1}$$

Corollary 3.1. Let r(sF-G) = $\max_{1\le j\le \nu} |a_j|$ be the spectral radius of the finite eigenvalues of the matrix pencil sF-G. Then the system (3) is asymptotically stable in the large, if and only if

$$r(sF-G) < 1$$

3.1 Lyapounov theorem on uniform stability

Definition 3.1.1 [23]. The singular linear discrete time system (3) is called uniformly stable if there exists a finite positive constant c such that for any k_0 and Y_{k_0} the corresponding solution satisfies

$$||Y_k||_2 \le c||Y_{k_0}||_2$$

Where $||\cdot||_2$ is the Euclidean norm.

It can be shown for regular linear discrete time systems, see [7,23], that if a positive scalar function $W(Y_k)$ can be found such that its forward difference $\Delta W\,(Y_k)$, where

$$\Delta W(Y_k) = W(Y_{k+1}) - W(Y_k)$$

taken along the trajectory is always negative, then as time increases, $W(Y_k)$ takes smaller and smaller values and finally shrinks to zero, and therefore Y_k also shrinks to zero. This implies the asymptotic stability of the origin of the state space. Lyapounov's main stability Theorem, provides a sufficient condition for asymptotic stability. This Theorem states that if there exists a scalar function $W(Y_k)$ satisfying the conditions, $W(Y_k)$ is posistive definite and $\Delta W(Y_k)$ is negative definite, then the equilibrium state at the origin is uniformly asymptotically stable.

We consider the singular discrete time system described by (3). We shall investigate the stability of this state by using this method of Lyapounov. Let us choose as a possible Lyapounov function

$$W(Y_k) = Y_k^* F^* T F Y_k$$

where $()^*$ is the Hermitian tensor and T is a positive Hermitian (or a positive definite real symmetric) matrix. Then

$$\Delta W(Y_k) = Y_{k+1}^* F^* T F Y_{k+1} - Y_k^* F^* T F Y_k$$

or

$$\Delta W(Y_k) = (FY_{k+1})^* T F Y_{k+1} - Y_k^* F^* T F Y_k$$

or

$$\Delta W(Y_k) = (GY_k)^* T G Y_k - Y_k^* F^* T F Y_k$$

or

$$\Delta W(Y_k) = Y_k^* G^* T G Y_k - Y_k^* F^* T F Y_k$$

or

$$\Delta W(Y_k) = Y_k^* (G^* T G - F^* T F) Y_k$$

Since $W(Y_k)$ is chosen to be positive definite, we require $\Delta W(Y_k)$ be negative definite. Therefore,

$$\Delta W(Y_k) = -Y_k^* S Y_k$$

where

$$S = -(G^* T G - F^* T F)$$

must be positive definite. Note that a positive definite T is a necessary and sufficient condition.

Theorem 3.1.1. Consider the singular linear discrete time system (3) with

$$n_1 I_m \leq F \leq n_2 I_m$$

A necessary and sufficient condition for the system (3) to be uniformly stable is that, given any positive definite Hermitian (or any positive definite real symmetric) matrix S, there exists a positive definite Hermitian (or any positive definite real symmetric) matrix T with

$$m_1 I_m \leq T \leq m_2 I_m$$

such that the matrix

$$S = -(G^* T G - F^* T F)$$

is positive definite. Where n_1, n_2, m_1 and m_2 are finite positive constants.

Proof. Suppose T satisfies the stated requirements. Given a consistent initial condition (2) and the corresponding unique solution of the system (3) from (13), we have

$$(Y_k)^* F^* T F Y_k - (Y_{k_0})^* F^* T F Y_{k_0} = \sum_{j=k_0}^{k-1} \left[(Y_{j+1})^* F^* T F Y_{j+1} - (Y_j)^* F^* T F Y_j \right]$$

or

$$(Y_k)^* F^* T F Y_k - (Y_{k_0})^* F^* T F Y_{k_0} = \sum_{j=k_0}^{k-1} \left[(Y_j)^* G^* T G Y_j - (Y_j)^* F^* T F Y_j \right]$$

or

$$(Y_k)^* F^* T F Y_k - (Y_{k_0})^* F^* T F Y_{k_0} = \sum_{j=k_0}^{k-1} (Y_j)^* \left[G^* T G - F^* T F \right] Y_j$$

or

$$(Y_k)^* F^* T F Y_k - (Y_{k_0})^* F^* T F Y_{k_0} = -\sum_{j=k_0}^{k-1} (Y_j)^* S Y_j$$

where S is positive definite and thus we have

$$(Y_k)^* F^* T F Y_k - (Y_{k_0})^* F^* T F Y_{k_0} \leq 0$$

Furthermore

$$(Y_k)^* F^* T F Y_k \leq (Y_{k_0})^* F^* T F Y_{k_0}$$

or

$$(n_1)^2 m_1 ||Y_k||_2 \leq (n_2)^2 m_2 ||Y_{k_0}||_2$$

Therefore

$$||Y_k||_2 \leq \frac{(n_2)^2 m_2}{(n_1)^2 m_1} ||Y_{k_0}||_2$$

And thus the system (3) is uniformly stable by Definition 3.1.1.

Example 3.1.1. Consider the system (3) and let

$$F = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$G = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

Let us choose

$$-(G^* T G - F^* T F) = I_2$$

If the matrix T is found to be positive definite, then the system is uniformly stable. Let $T = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Consequently,

$$T = \begin{bmatrix} \frac{1}{3} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} \end{bmatrix}$$

By applying Sylvester's criterion for the positive definiteness of matrix T, we find T is positive definite. Hence, the system is uniformly stable.

4 Stability of non homogeneous singular matrix difference equations

The definition of an equilibrium state of a non homogeneous system in the form of (1) is given from definition 3.1.

Proposition 4.1. Consider the system (1) with a constant input vector V_k = V. Then if 1 is not an eigenvalue of the matrix pencil sF-G

$$Y_* = (F - G)^{-1} V$$

is the unique equilibrium state of the system (1). If 1 is a finite eigenvalue of the matrix pencil sF-G, then the set $\hat{E}$ of the equilibrium points of the system (1) is the vector space defined by

$$\hat{E} = N_r(F - G) \cap (\text{colspan}Q_p - Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V)$$

where N_r is the right null space of the matrix F-G and Q_p, Q_q, P_2 are matrices defined in (11), (14).

Proof. The proof is similar to the proof of Proposition 3.1. Note that from Proposition 2.3.1, the system (1) has a unique solution if and only if the given initial conidtion (2) lies inside the set

$$Y_{k_0} \in \text{colspan}Q_p - Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V$$

Thus from Proposition 4.1, the unique equilibrium state for a system in the form of (1) with a constant input vector V_k = V is $Y_* = (F - G)^{-1}V$, when det(F-G) ≠ 0.

Theorem 4.1. Consider the system (1) with a constant input vector V_k = V and a consistent initial condition (2). Then the equilibrium state is asymptotically stable in the large, if and only if, all the finite eigenvalues of the matrix pencil sF-G lie within the open disc,

$$|s| < 1$$

Proof. The general solution of the linear system

$$FY_{k+1} = GY_k + V$$

is the sum of a partial solution and the solution of the homogeneous system (3). Since an equilibrium state Y_* can be assumed as a partial solution, the general solution of this system is

$$Y_k = Q_p J_p^{k-k_0} Z_{k_0}^p + Y_*$$

Given a consistent initial condition Y_{k_0} and since the columns of the matrix Q_p are linear independent, the matrix is left invertible. Thus we can define its left inverse matrix $Q_p^{-1} \in \mathcal{M}_{pm}$ and the general solution can be written in the form of

$$Y_k = Q_p J_p^{k-k_0} Q_p^{-1}(Y_{k_0} - Y_*) + Y_*$$

Let a_j be a finite eigenavalue of the matrix pencil sF-G with algebraic multiplicity p_j. Then the Jordan matrix $J_p^{k-k_0}$ can be written as

$$J_p^{k-k_0} = \text{blockdiag}\left[J_{p_1}^{k-k_0}(a_1) J_{p_2}^{k-k_0}(a_2) \ldots J_{p_\nu}^{k-k_0}(a_\nu)\right]$$

with $J_p^{k-k_0} \in \mathcal{M}_{p_j}$ be a Jordan block. Every element of this matrix has the specific form

$$(k - k_0)^{p_j} a_j^{k-k_0}$$

The sequence

$$(k-k_0)^{p_j}|a_j^{k-k_0}|$$

can be written as

$$(k-k_0)^{p_j}e^{(k-k_0)ln|a_j|}$$

The system has a unique equilibrium state when for every j, $a_j \neq 1$, because then det (F-G) ≠ 0 and the unique equilibrium state is $Y_* = (F - G)^{-1}V$. In this case the system is asymptotically stable in the large, when

$$\lim_{k\to\infty} Y_k = Y_*$$

Thus this holds if and only if

$$\ln|a_j| < 0$$

or

$$|a_j| < 1$$

Then for $k \to +\infty$

$$(k-k_0)^{p_j}e^{(k-k_0)|\ln a_j|} \to 0$$

or

$$(k-k_0)^{p_j}|a_j|^{(k-k_0)} \to 0$$

or for every k≥k_0

$$J_p^{k-k_0} \to 0_{p,p}$$

From (30)

$$Y_k - Y_* = \begin{bmatrix} Q_p 0_{m,q} \end{bmatrix} \begin{bmatrix} J_p^{k-k_0} & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} \begin{bmatrix} Q_p^{-1} \\ 0_{q,m} \end{bmatrix} (Y_{k_0} - Y_*) \tag{31}$$

If we set $\|Q_p\| = \|\begin{bmatrix} Q_p 0_{m,q} \end{bmatrix}\|$ and $\|Q_p^{-1}\| = \left\|\begin{bmatrix} Q_p^{-1} \\ 0_{q,m} \end{bmatrix}\right\|$, then by taking norms for every k≥k_0 in (31), for every consistent initial condition Y_{k_0} we have

$$||Y_k - Y_*|| = \left\|Q_p J_p^{k-k_0} Q_p^{-1}(Y_{k_0} - Y_*)\right\| || \leq ||Q_p|| \left\|J_p^{k-k_0}\right\| \|Q_p^{-1}\| ||Y_{k_0} - Y_*|| \to 0$$

and thus

$$\lim_{k\to\infty} Y_k = Y_*$$

or

$$\lim_{k\to\infty} Y_k = (F-G)^{-1}V$$

Knowing from Proposition 2.3.1 the closed formula that provides the unique solution of the singular system (1), we can prove the following Theorem.

Theorem 4.2. The unique solution of the non homogeneous finite ($k_0 \le k \le k_N$) discrete time system with a bounded input vector V_k (1) is bounded if every equilibrium state of the homogeneous system (3) is stable in the sense of Lyapounov.

Proof. The solution of system (1) is given from the formula (16),

$$Y_k = Q_p\left(J_p^{k-k_0} Z_{k_0}^p + \sum_{i=k_0}^{k-1} J_p^{k-i-1} P_1 V_i\right) - Q_q \sum_{i=0}^{q_*-1} H_q^i P_2 V_{k+i}.$$

or

$$\begin{aligned} Y_k = \left[Q_p 0_{m,q}\right]\left(\begin{bmatrix} J_p^{k-k_0} & 0_{p,q} \\ 0_{q,p} & 0_{q,q}\end{bmatrix}\begin{bmatrix} Z_{k_0}^p \\ 0_{q,m}\end{bmatrix} + \sum_{i=k_0}^{k-1}\begin{bmatrix} J_p^{k-i-1} & 0_{p,q} \\ 0_{q,p} & 0_{q,q}\end{bmatrix}\begin{bmatrix} P_1 \\ 0_{q,m}\end{bmatrix} V_i\right)- \\ -\left[Q_q 0_{m,p}\right]\sum_{i=0}^{q_*-1}\begin{bmatrix} H_q^i & 0_{q,p} \\ 0_{p,q} & 0_{p,p}\end{bmatrix}\begin{bmatrix} P_2 \\ 0_{p,m}\end{bmatrix} V_{k+i}. \end{aligned} \tag{32}$$

If we set $\|Q_p\| = \left\|\left[Q_p 0_{m,q}\right]\right\|$, $\|Q_p^{-1}\| = \left\|\begin{bmatrix} Q_p^{-1} \\ 0_{q,m}\end{bmatrix}\right\|$, $\|P_1\| = \left\|\begin{bmatrix} P_1 \\ 0_{q,m}\end{bmatrix}\right\|$ and $\|P_2\| = \left\|\begin{bmatrix} P_2 \\ 0_{q,m}\end{bmatrix}\right\|$, then by taking norms in (32)

$$||Y_k|| \le ||Q_p|| \left\|J_p^{k-k_0}\right\| \left\|Z_{k_0}^p\right\| + ||Q_p|| \sum_{i=k_0}^{k-1}\left\|J_p^{k-i-1}\right\| ||P_1|| \, ||V_i|| - ||Q_q|| \sum_{i=0}^{q_*-1}\left\|H_q^i\right\| ||P_2|| \, ||V_{k+i}|| \tag{33}$$

If every equilibrium state of the system (3) is stable in the sense of Lyapounov, the matrix $J_p^{k-k_0}$ is bounded,

$$||J_p^{k-k_0}|| \le c_1 \tag{34}$$

and moreover for a finite discrete time system, $k_0 \le k \le k_N$, we have

$$\sum_{i=k_0}^{k-1} ||J_p^{k-i-1}|| \le (k_N - 1 - k_0)c_1 \tag{35}$$

If the input vector V_k is bounded, we have

$$||V_k|| \le c_2 \tag{36}$$

By applying (34), (35) and (36) into (33) we have

$$||Y_k|| \le ||Q_p|| c_1 \left\|Z_{k_0}^p\right\| + ||Q_p|| k_N c_1 ||P_1|| c_2 - ||Q_q|| c_3 ||P_2|| c_2$$

where

$$c_3 = \sum_{i=0}^{q_*-1} ||H_q^i||$$

and thus we have proved that every solution of a finite discrete time system in the form of (1) is bounded.

Theorem 4.3. Let the system (3) be asymptotically stable in the large. Then after a δ perturbation in the set of consistent initial conditions of the non homogeneous discrete time system (1), the unique solution changes by an amount depending on δ.

Proof. The solution of system (1) is given from the formula (16),

$$Y_k = Q_p\left(J_p^{k-k_0}Z_{k_0}^p + \sum_{i=k_0}^{k-1} J_p^{k-i-1}P_1V_i\right) - Q_q\sum_{i=0}^{q_*-1} H_q^iP_2V_{k+i}.$$

where $Z_{k_0}^p$ is the solution of the algebraic system

$$Y_{k_0} = Q_pZ_{k_0}^p - Q_q\sum_{i=0}^{q_*-1} H_q^iP_2V_{k+i}$$

Since the columns of the matrix Q_p are linear independent, the matrix is left invertible. Thus we can define its left inverse matrix $Q_p^{-1} \in \mathcal{M}_{pm}$ such that

$$Q_p^{-1}Q_p = I_p$$

Then

$$Z_{k_0}^p = Q_p^{-1}Y_{k_0} + Q_p^{-1}Q_q\sum_{i=0}^{q_*-1} H_q^iP_2V_{k+i}$$

and the solution can be written as

$$Y_k = Q_p\left(J_p^{k-k_0}Q_p^{-1}Y_k + Q_p^{-1}Q_q\sum_{i=0}^{q_*-1} H_q^iP_2V_{k+i} + \sum_{i=k_0}^{k-1} J_p^{k-i-1}P_1V_i\right) - Q_q\sum_{i=0}^{q_*-1} H_q^iP_2V_{k+i}$$

If we perturb the initial conditions of the system accordingly

$$||Y_{k_0} - \tilde{Y}_{k_0}|| \leq \delta$$

then the solution of the system with initial conditions $\tilde{Y}_{k_0}$ changes to

$$\tilde{Y}_k = Q_p\left(J_p^{k-k_0}Q_p^{-1}\tilde{Y}_k + Q_p^{-1}Q_q\sum_{i=0}^{q_*-1} H_q^iP_2V_{k+i} + \sum_{i=k_0}^{k-1} J_p^{k-i-1}P_1V_i\right) - Q_q\sum_{i=0}^{q_*-1} H_q^iP_2V_{k+i}$$

and substracting $\tilde{Y}_k$ from Y_k, we obtain

$$Y_k - \tilde{Y}_k = Q_pJ_p^{k-k_0}Q_p^{-1}(Y_{k_0} - \tilde{Y}_{k_0})$$

or

$$Y_k - \tilde{Y}_k = \left[Q_p 0_{m,q}\right]\begin{bmatrix} J_p^{k-k_0} & 0_{p,q} \\ 0_{q,p} & 0_{q,q}\end{bmatrix}\begin{bmatrix} Q_p^{-1} \\ 0_{q,m}\end{bmatrix}(Y_{k_0} - \tilde{Y}_{k_0}) \tag{37}$$

If we set $\|Q_p\| = \left\|\left[Q_p 0_{m,q}\right]\right\|$ and $\|Q_p^{-1}\| = \left\|\begin{bmatrix} Q_p^{-1} \\ 0_{q,m}\end{bmatrix}\right\|$. Then by taking norms for every k≥k_0 in (37) we have

$$\left\|Y_k - \tilde{Y}_k\right\| \;\leq\; ||Q_p|| \;\left\|J_p^{k-k_0}\right\| \left\|Q_p^{-1}\right\| \left\|Y_{k_0} - \tilde{Y}_{k_0}\right\| \tag{38}$$

If every equilibrium state of the system (3) is stable in the sense of Lyapounov, the matrix $J_p^{k-k_0}$ is bounded,

$$\left\|J_p^{k-k_0}\right\| \leq c_1$$

Then from (38) we have

$$\left\|Y_k - \tilde{Y}_k\right\| \leq c_1 c_2 c_3 \delta$$

where $c_1 = \left\|Q_p\right\|, c_2 = \left\|Q_p^{-1}\right\|$. Therefore if we chose $\epsilon = \epsilon(\delta) = c_1 c_2 c_3 \delta$, we obtain,

$$\left\|Y_k - \tilde{Y}_k\right\| \leq \epsilon$$

5 State feedback stabilization

In this section we will study the case where the system (3) has non stable equilibrium states. We state the following question. Under what conditions is it possible to apply to the initial unstable system (3) an external input vector

$$V_k = BU_k \tag{39}$$

where $\mathrm{B} \in \mathcal{M}_{mr}$, $U_k \in \mathcal{M}_{r1}$ and a state feedback law

$$U_k = \tilde{G}_1 Y_k \tag{40}$$

such that the new system has stable equilibrium states. If it is possible, the system (3) will be called stabilizable. So, the initial system (3) takes the form, first

$$FY_{k+1} = GY_k + BU_k \tag{41}$$

and after applying the feedback law (40)

$$FY_{k+1} = \tilde{G}Y_k \tag{42}$$

where $\tilde{G} = G + B\tilde{G}_1$ and $\tilde{G}_1 \in \mathcal{M}_{rm}$ is the feedback gain.

Proposition 5.1. Consider the singular system (3) and the corresponding singular system of the form (41). We also suppose that some of the finite eigenvalues of the matrix pencil sF-G does not lie within the open disc $|s| <1$. Then this system becomes stable by a feedback law of the form (40) if

$$\mathrm{rank}\,[sF - G,\; B] = m \tag{43}$$

Proof. Since the system (3) has finite eigenvalues of the matrix pencil sF-G that don't lie within the open disc $|s| <1$, there exists non stable equilibrium state(s). We consider the system (41) and we want to apply an appropriate feedback law in the form (40), such that all finite eigenvalues of the matrix pencil $\mathrm{sF} - \tilde{G}$ of the system (41) lie within the open disc $|s| <1$. Thus

$$\det(sF - \tilde{G}) \neq 0$$

and

$$\mathrm{rank}(sF - \tilde{G}) = m \tag{44}$$

Thus we require

$$\mathrm{rank}[sF - G, B] = \mathrm{rank}[sF - G, B]\begin{bmatrix} I_m \\ -\tilde{G}_1 \end{bmatrix} = \mathrm{rank}(sF - G - B\tilde{G}_1) = \mathrm{rank}(sF - \tilde{G}) = m$$

or

$$\mathrm{rank}[sF - G, B] = m$$

Example 5.1. Consider the system (3) with

$$F = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} -2 & 0 & 2 \\ -1 & \frac{1}{3} & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{45}$$

Then

$$\det(sF - G) = (s + 2)\left(s + \frac{2}{3}\right)$$

the equilibrium state is not stable. We consider now the system (41) with

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{46}$$

Since for $s = 2,\ \frac{2}{3}$

$$\mathrm{rank}[sF - G, B] = 3$$

the system is stabilizable. We assume the feedback law

$$U_k = \left[\ \tfrac{3}{2}\ 0\ 0\ \right] Y_k$$

and by applying it in (41) we get the system (42), with

$$\tilde{G} = \begin{bmatrix} -\frac{1}{2} & 0 & 2 \\ -1 & \frac{1}{3} & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{47}$$

with

$$\det(sF - \tilde{G}) = \left(s + \frac{1}{2}\right)\left(s - \frac{2}{3}\right)$$

and since the matrix pencil doesn't have the finite eigenvalue 1 the system has the unique equilibrium state $0_{m,1}$ and moreover since all the finite eigenvalues are inside the unite circle, the unique equilibrium state is asymptotically stable in the large and

$$\lim_{k\to\infty} Y_k = 0_{m,1}$$

6 Conclusions

In this article, we studied the stability of a class of linear singular discrete time systems whose coefficients are square constant matrices and the leading matrix coefficient is singular. We presented a theory based on the matrix pencil of the system and we showed how the eigenvalues of the pencil are related with the stability of singular systems. We studied the stability of systems in the form of (3) and the behavior of the solution Y_k as k increases from k_0 to ∞. Furthermore we reformulated the Lyapounov Theorem for uniform stability to be applied in singular systems. After we considered the system (1) and proved that all solutions of the non homogeneous system are bounded provided that the homogeneous system (3) is asymptotically stable in the large. Moreover we provided properties to avoid a "blow up" in the system when having small perturbations in the initial conditions. Finally for the case of not stable equilibrium states we gave necessary and sufficient conditions for state feedback stabilization. As a further extension of this article, we can discuss possible applications based on the presented approach, as is the very famous Leondief model, see [4], or the very important Leslie population growth model and backward population projection, see also [4], the Host-parasitoid Models in physics, see [38] or the distribution of heat through a long rod or bar as suggested in [24]. Furthermore another interesting case for further studies is the case of systems with a singular pencil. This is the case where the constant matrices of the system are not square or they are square with an identically zero matrix pencil. For all these, there is some research in progress.

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