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Ishikawa-hybrid proximal point algorithm for NSVI system

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Abstract

A nonlinear set-valued inclusions system framework for an Ishikawa-hybrid proximal point algorithm is developed and studied using the notion of an (A, η)-accretive mapping. Convergence analysis for the algorithm of solving the nonlinear set-valued inclusions system and existence analysis of solution for the system are explored along with some results on the resolvent operator corresponding to the (A, η)-accretive mapping in a Banach space. The result that the sequence generated by the algorithm converges linearly to a solution of the system with the convergence rate $\|\boldsymbol{\Psi}\|$ is proved.

MSC: 49J40; 47H06

Keywords: nonlinear set-valued inclusions system; (A, η)-accretive mapping; resolvent operator; Ishikawa-hybrid proximal point algorithm; convergence rate

1 Introduction

The nonlinear set-valued inclusions system, which was introduced and studied by Hassouni and Moudafi [1], is a useful and important extension of the variational inequality and variational inclusions system. In recent years, various variational inclusions systems and nonlinear set-valued inclusions systems have been intensively studied. For example, Kassay and Kolumbán [2], Chen, Deng and Tan [3], Yan, Fang and Huang [4], Fang, Huang and Thompson [5], Jin [6], Verma [7], Li, Xu and Jin [8], Kang, Cho and Liu [9] *et al.* introduced and studied various set-valued variational inclusions systems. For the past few years, many existence results and iterative algorithms for various variational inclusions systems have been studied. For details, please see [1–28] and the references therein.

Example 1.1 In 2001, Chen, Deng and Tan [3] have studied the problem associated with the following system of variational inequalities, which is finding $(x, y) \in H \times H$ (H, Hilbert space) such that

$$\begin{cases} \langle \rho T(y) + x - y, w - x \rangle \geq \rho\varphi_1(x) - \rho\varphi_1(w), \\ \langle tS(x) + y - x, w - y \rangle \geq t\varphi_2(y) - \rho\varphi_2(w) \quad (\forall w \in H), \end{cases} \tag{1}$$

where $\varphi_i : H \to R \cup \infty$ is a proper, convex, lower semicontinuous functional and $\partial\varphi_i(\cdot)$ denotes the subdifferential operator of φ_i $(i = 1, 2)$.

Example 1.2 Let X be a real q-uniformly smooth Banach space, and $S, T, M_1, M_2 : X \to X$ be four single-valued mappings. Find $x, y \in X$ such that

$$\begin{cases} 0 \in x - y + \rho T(y) + \rho M_1(x), \\ 0 \in y - x + tS(x) + tM_2(y), \end{cases} \tag{2}$$

which is studied by Jin in [6].

Inspired and motivated by Examples 1.1-1.2 and recent research work in this field (see [7, 8]), in this paper, we will introduce and discuss the problem associated with the following class of new nonlinear set-valued inclusions systems (NSVI Systems), which is finding $(x, y) \in X \times X$ for any $f, g : X \to X$ such that $z \in S(x)$, $w \in T(y)$, and

$$\begin{cases} f(x) \in F(z, y) + M(y), \\ g(y) \in G(w, x) + N(x), \end{cases} \tag{3}$$

where X is a real q-uniformly smooth Banach space, $A, B : X \to X$, $\eta_1, \eta_2 : X \times X \to X$, and $F, G : X \times X \to X$ are single-valued mappings; $M : X \to 2^X$ is a set-valued (A, η_1)-accretive mapping and $N : X \to 2^X$ is a set-valued (B, η_2)-accretive mapping, and $S, T : X \to CB(X)$ are two set-valued mappings.

If $f(x) = -x$, $g(y) = -y$, $F(w, y) = \rho T(y) - y$, $G(y, x) = tS(x) - x$, $N = tM_2$ and $M = \rho M_1(\cdot)$, then the problem (3) reduces to Example 1.1. If $M_1 = M_2 = \partial\varphi$, $\varphi_i : H \to R \cup \infty$ is a proper, convex, lower semicontinuous functional and $\partial\varphi_i(\cdot)$ denotes the subdifferential operator of φ_i ($X = H$, Hilbert space, and $i = 1, 2$), then the problem (3) changes to Example 1.2.

If X is a real q-uniformly smooth Banach space, and $G(\cdot, \cdot) = N(y, g(x))$, $f(u) = u$ and $S(u) = Q(u)(u \in X)$, then the problem (3) reduces to the problem associated with the following variational inclusions:

For any $u \in X$, find $x \in X$ and $y = Q(x)$ such that

$$u \in N\big(y, g(x)\big) + M(y), \tag{4}$$

which is developed by Li in 2010 [8].

The main purpose of this paper is to introduce and study a generalized nonlinear set-valued inclusions system framework for an Ishikawa-hybrid proximal point algorithm using the notion of (A, η)-accretive due to Lan-Cho-Verma [10] in a Banach space, to analyse convergence for the algorithm of solving the system and existence of a solution for the system and to prove the result that sequence $\{(x^n, y^n)\}_{n=0}^{\infty}$ generated by the algorithm converges linearly to a solution of the nonlinear set-valued inclusions system with the convergence rate $\|\boldsymbol{\Psi}\|$.

2 Preliminaries

Let X be a real q-uniformly smooth Banach space with a dual space X^*, $\langle\cdot,\cdot\rangle$ be the dual pair between X and X^*, 2^X denote the family of all the nonempty subsets of X, and $CB(X)$ denote the family of all nonempty closed bounded subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. Let us recall the following results and concepts.

Definition 2.1 A single-valued mapping $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \le \tau \|x - y\|, \quad \forall x, y \in X.$$

Definition 2.2 A single-valued mapping $A : X \to X$ is said to be

(i) accretive if

$$\langle A(x_1) - A(x_2), J_q(x_1 - x_2) \rangle \ge 0, \quad \forall x_1, x_2 \in X;$$

(ii) strictly accretive if A is accretive and $\langle A(x_1) - A(x_2), J_q(x_1 - x_2) \rangle = 0$ if and only if $x_1 = x_2$, $\forall x_1, x_2 \in X$;

(iii) r-strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle A(x_1) - A(x_2), J_q(\eta(x_1, x_2)) \rangle \ge r \|x_1 - x_2\|^q, \quad \forall x_1, x_2 \in X;$$

(iv) γ-Lipschitz continuous if there exists a constant $\gamma > 0$ such that

$$\|A(x_1) - A(x_2)\| \le \gamma \|x_1 - x_2\|, \quad \forall x_1, x_2 \in X;$$

(v) Let $f : X \to X$ be a single-valued mapping. A is said to be (σ, φ)-relaxed cocoercive with respect to f if for any $x_1, x_2 \in X$, there exist two constants $\sigma, \varphi > 0$ such that

$$\langle A(x_1) - A(x_2), J_q(f(x_1) - f(x_2)) \rangle \ge -\sigma \|A(x_1) - A(x_2)\|^q + \varphi \|x_1 - x_2\|^q.$$

Definition 2.3 A set-valued mapping $S : X \to CB(X)$ is said to be

(i) D-Lipschitz continuous if there exists a constant $\alpha > 0$ such that

$$D(S(x), S(y)) \le \alpha \|x - y\|, \quad \forall x, y \in X,$$

where $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$.

(ii) β-strongly η-accretive if there exists a constant $\beta > 0$ such that

$$\langle u_1 - u_2, J_q(\eta(x, y)) \rangle \ge \beta \|x - y\|^q, \quad \forall x, y \in X, u_1 \in S(x), u_2 \in S(y).$$

Definition 2.4 Let $A : X \to X$ and $\eta : X \times X \to X$ be single-valued mappings. A set-valued mapping $M : X \to 2^X$ is said to be

(i) accretive if

$$\langle u_1 - u_2, J_q(x, y) \rangle \ge 0, \quad \forall x, y \in X, u_1 \in M(x), u_2 \in M(y);$$

(ii) η-accretive if

$$\langle u_1 - u_2, J_q(\eta(x,y))\rangle \geq 0, \quad \forall x, y \in X, u_1 \in M(x), u_2 \in M(y);$$

(iii) m-relaxed η-accretive, if there exists a constant $m > 0$ such that

$$\langle u_1 - u_2, J_q(\eta(x,y))\rangle \geq -m\|x-y\|^q, \quad \forall x, y \in X, u_1 \in M(x), u_2 \in M(y);$$

(iv) A-accretive if M is accretive and $(A + \rho M)(X) = X$ for all $\rho > 0$;
(v) (A,η)-accretive if M is m-relaxed η-accretive and $(A + \rho M)(X) = X$ for every $\rho > 0$.

Based on [10], we can define the resolvent operator $R^{A,\eta}_{\rho,M}$ as follows.

Lemma 2.5 ([10]) *Let $\eta : X \times X \to X$ be a τ-Lipschitz continuous mapping, $A : X \to X$ be an r-strongly η-accretive mapping, and $M : X \to 2^X$ be a set-valued (A,η)-accretive mapping. Then the generalized resolvent operator $R^{A,\eta}_{\rho,M} : X \to X$ is $\tau^{q-1}/(r - m\rho)$-Lipschitz continuous*; *that is*,

$$\|R^{A,\eta}_{\rho,M}(x) - R^{A,\eta}_{\rho,M}(y)\| \leq \frac{\tau^{q-1}}{r - m\rho}\|x - y\| \quad \text{for all } x, y \in X,$$

where $\rho \in (0, r/m)$, $q > 1$.

Remark 2.6 The (A,η)-accretive mappings are more general than (H,η)-monotone mappings, A-monotone operators and η-subdifferential operators in a Banach space or a Hilbert space, and the resolvent operators associated with (A,η)-accretive mappings include as special cases the corresponding resolvent operators associated with them, respectively [3–6, 9, 25].

In the study of characteristic inequalities in q-uniformly smooth Banach spaces X, Xu [14] proved the following result.

Lemma 2.7 ([14]) *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Lemma 2.8 ([8]) *Let $a, b, c > 0$ be real, for any real $q \geq 1$, if $a^q \leq b^q + c^q$, then*

$$a \leq b + c.$$

3 Existence theorem of solutions

Let us study the existence theorem of solutions for the inclusions system (3).

Theorem 3.1 *Let X be a Banach space, $f, g : X \to X$ be two single-valued mappings, $F : X \times X \to X$ be a (μ_1, ν_1)-Lipschitz continuous mapping and $G : X \times X \to X$ be a (μ_2, ν_2)-Lipschitz continuous mapping, $\eta_i : X \times X \to X$ be a τ_i-Lipschitz continuous mapping $(i = 1, 2)$, $A : X \to X$ be an r_1-strongly η_1-accretive mapping, $B : X \to X$ be an r_2-strongly η_2-accretive mapping, $M : X \to 2^X$ be a set-valued (A, η_1)-accretive mapping and $N : X \to 2^X$ be a set-valued (B, η_2)-accretive mapping. Then the following statements are mutually* *equivalent*:

(i) *An element* (x,y) *is a solution of the problem* (3);

(ii) *For* $(x,y) \in X \times X$, $z \in S(x)$ *and* $w \in T(y)$, *the following relations hold*:

$$\begin{cases} x = R_{\rho_1,M}^{A,\eta_1}(A(x) + \rho_1 f(x) - \rho_1 F(z,y)), \\ y = R_{\rho_2,N}^{B,\eta_2}(B(y) + \rho_2 g(y) - \rho_2 G(w,x)), \end{cases} \tag{5}$$

where $\rho_i > 0$ *is a constant* $(i = 1,2)$;

(iii) *For* $(x,y) \in X \times X$, $z \in S(x)$, $w \in T(y)$, *and any* $1 > \lambda > 0$, *the following relations hold*:

$$\begin{cases} x = (1-\lambda)x + \lambda R_{\rho_1,M}^{A,\eta_1}(A(x) + \rho_1 f(x) - \rho_1 F(z,y)), \\ y = (1-\lambda)y + \lambda R_{\rho_2,N}^{B,\eta_2}(B(y) + \rho_2 g(y) - \rho_2 G(w,x)), \end{cases} \tag{6}$$

where $\rho_i > 0$ *is a constant* $(i = 1,2)$;

Proof This directly follows from the definition of $R_{\rho_1,M}^{A,\eta_1}$, $R_{\rho_2,N}^{B,\eta_2}$, and the problem (3) for $i = 1,2$. □

Theorem 3.2 *Let* X *be a* q*-uniformly smooth Banach space. Let* $f,g : X \to X$ *be two single-valued* κ_1 *or* κ_2*-Lipschitz continuous mappings, respectively,* $\eta_i : X \times X \to X$ *be a single-valued* τ_i*-Lipschitz continuous mapping* $(i = 1,2)$, $F,G : X \times X \to X$ *be two single-valued* (μ_1,ν_1) *or* (μ_2,ν_2)*-Lipschitz continuous mappings, respectively. Let* $A : X \to X$ *be single-valued* r_1*-strongly* η_1*-accretive,* ω_1*-Lipschitz continuous,* (σ_1,φ_1)*-relaxed cocoercive with respect to* f, *and* $B : X \to X$ *be single-valued* r_2*-strongly* η_2*-accretive,* ω_2*-Lipschitz continuous,* (σ_2,φ_2)*-relaxed cocoercive with respect to* g. *Let* $S,T : X \to X$ *be two set-valued* γ_1 *or* γ_2*-Lipschitz continuous mappings, respectively. If* $M : X \to 2^X$ *is a set-valued* (A,η_1)*-accretive mapping and* $N : X \to 2^X$ *is a set-valued* (B,η_2)*-accretive mapping, and the following condition holds*:

$$\begin{cases} \tau^q(\rho_1\mu_1\gamma_1 + l_1) < \tau(r_1 - m_1\rho_1), & \tau^q(\rho_2\mu_2\gamma_2 + l_2) < \tau(r_2 - m_2\rho_2), \\ l_1 = \sqrt[q]{\omega_1^q + c_q\rho_1^q\kappa_1^q + q\sigma_1\omega_1^q - q\varphi_1}, & l_2 = \sqrt[q]{\omega_2^q + c_q\rho_2^q\kappa_2^q + q\sigma_2\omega_2^q - q\varphi_2}, \end{cases} \tag{7}$$

where $c_q > 0$ *is the same as in Lemma* 2.7 *and* $\rho_i \in (0, \frac{r_i}{m_i})$ $(i = 1,2)$, *then the problem* (3) *has a solution* $x^*, y^* \in X$, $z^* \in S(x^*)$, $w^* \in T(y^*)$.

Proof Define two mappings $Q_1, Q_2 : X \to X$ as follows:

$$\begin{cases} Q_1(x) = (1-\lambda)x + \lambda R_{\rho_1,M}^{A,\eta_1}(A(x) + \rho_1 f(x) - \rho_1 F(z,y)), \\ Q_2(y) = (1-\lambda)y + \lambda R_{\rho_2,N}^{B,\eta_2}(B(y) + \rho_2 g(y) - \rho_2 G(w,x)) \\ \quad (\forall x,y \in X, z \in S(x), w \in T(y)). \end{cases} \tag{8}$$

For elements $x_1, x_2, y_1, y_2 \in X$, if letting

$$\Omega_i = A(x_i) - \rho_1 f(x_i) - \rho_1 F(z_i, y_i) \quad (i = 1,2),$$

then by (8), Lemma 2.5 and Lemma 2.7, we have

$$\begin{aligned} \|Q_1(x_1) - Q_1(x_2)\| &= \|(1-\lambda)x_1 + \lambda R^{A,\eta_1}_{\rho_1,M}(\Omega_1) - (1-\lambda)x_2 - \lambda R^{A,\eta_1}_{\rho_1,M}(\Omega_2)\| \\ &\le (1-\lambda)\|x_1 - x_2\| + \lambda \|R^{A,\eta_1}_{\rho_1,M}(\Omega_1) - R^{A,\eta_1}_{\rho_1,M}(\Omega_2)\| \\ &\le (1-\lambda)\|x_1 - x_2\| + \lambda \frac{\tau^{q-1}}{r_1 - m_1\rho_1}\big[\rho_1\big(\|F(z_2,y_2) - F(z_1,y_1)\|\big) \\ &\quad + \|A(x_1) - A(x_2) - \rho_1\big(f(x_1) - f(x_2)\big)\|\big], \end{aligned} \tag{9}$$

and by (μ_1, ν_1)-Lipschitz continuity of $F(\cdot,\cdot)$ and γ_1-Lipschitz continuity of S, we obtain

$$\begin{aligned} \|F(z_2,y_2) - F(z_1,y_1)\| &\le \mu_1\|z_2 - z_1\| + \nu_1\|y_2 - y_1\| \\ &\le \mu_1\gamma_1\|x_2 - x_1\| + \nu_1\|y_2 - y_1\|. \end{aligned} \tag{10}$$

Since A is ω_1-Lipschitz continuous and (σ_1, φ_1)-relaxed cocoercive with respect to f, and f is κ_1-Lipschitz continuous so that for $z_1 \in S(x_1)$, $z_2 \in S(x_2)$, we have

$$\begin{aligned} &\|A(x_1) - A(x_2) - \rho_1\big(f(x_1) - f(x_2)\big)\|^q \\ &\quad\le \|A(x_1) - A(x_2)\|^q + c_q\rho_1^q\|f(x_1) - f(x_2)\|^q \\ &\qquad - q\big\langle A(x_1) - A(x_2), J_q\big(f(x_1) - f(x_2)\big)\big\rangle \\ &\quad\le \big(\omega_1^q + c_q\rho_1^q\kappa_1^q + q\sigma_1\omega_1^q - q\varphi_1\big)\|x_2 - x_1\|^q. \end{aligned} \tag{11}$$

Combining (9), (10) and (11), we can get

$$\begin{aligned} &\|Q_1(x_1) - Q_1(x_2)\| \\ &\quad\le (1-\lambda)\|x_1 - x_2\| + \lambda\frac{\tau^{q-1}}{r_1 - m_1\rho_1}\big[\rho_1\big(\mu_1\gamma_1\|x_2 - x_1\| + \nu_1\|y_2 - y_1\|\big) \\ &\qquad + \sqrt[q]{\omega_1^q + c_q\rho_1^q\kappa_1^q + q\sigma_1\omega_1^q - q\varphi_1}\big(\omega_1^q + c_q\rho_1^q\kappa_1^q + q\sigma_1\omega_1^q - q\varphi_1\big)\|x_2 - x_1\|\big] \\ &\quad\le \big((1-\lambda) + \lambda\theta_1\big)\|x_1 - x_2\| + \lambda\frac{\tau^{q-1}}{r_1 - m_1\rho_1}\nu_1\|y_2 - y_1\|, \end{aligned} \tag{12}$$

where

$$\theta_1 = \frac{\tau^{q-1}}{r_1 - m_1\rho_1}\big(\rho_1\mu_1\gamma_1 + \sqrt[q]{\omega_1^q + c_q\rho_1^q\kappa_1^q + q\sigma_1\omega_1^q - q\varphi_1}\big).$$

For elements $x_1, x_2, y_1, y_2 \in X$, $z_i \in S(x_i)$, $y_i \in T(y_i)$ $(i = 1,2)$, if letting

$$\Theta_i = B(y_i) + \rho_2 g(y_i) - \rho_2 G(w_i, x_i) \quad (i = 1,2),$$

then by using the same method as the one used above,

$$\begin{aligned} \|Q_2(y_1) - Q_2(y_2)\| &= \|(1-\lambda)y_1 + \lambda R^{B,\eta_2}_{\rho_2,N}(\Theta_1) - (1-\lambda)y_2 - \lambda R^{B,\eta_2}_{\rho_2,N}(\Theta_2)\| \\ &\le (1-\lambda)\|y_1 - y_2\| + \lambda\|R^{B,\eta_2}_{\rho_2,N}(\Theta_1) - R^{B,\eta_2}_{\rho_2,N}(\Theta_2)\| \end{aligned}$$

$$\begin{aligned}&\le (1-\lambda)\|y_1-y_2\|+\lambda\frac{\tau^{q-1}}{r_2-m_2\rho_2}\big(\rho_2\big\|G(w_2,x_2)-G(w_1,x_1)\big\|\\&\quad+\big\|B(y_1)-B(y_2)-\rho_2\big(g(y_1)-g(y_2)\big)\big\|\big)\\&\le \lambda\frac{\tau^{q-1}}{r_2-m_2\rho_2}\rho_2\nu_2\|x_2-x_1\|+\big[(1-\lambda)+\lambda\theta_2\big]\|y_2-y_1\|\end{aligned}\tag{13}$$

hold, where

$$\theta_2=\frac{\tau^{q-1}}{r_2-m_2\rho_2}\big(\rho_2\mu_2\gamma_2+\sqrt[q]{\omega_2^q+c_q\rho_2^q\kappa_2^q+q\sigma_2\omega_2^q-q\varphi_2}\big).$$

If setting

$$\begin{aligned}&\Gamma_{11}=\theta_1,\qquad \Gamma_{12}=\frac{\tau^{q-1}}{r_1-m_1\rho_1}\rho_1\nu_1,\\&\Gamma_{21}=\frac{\tau^{q-1}}{r_2-m_2\rho_2}\rho_2\nu_2,\qquad \Gamma_{22}=\theta_2,\end{aligned}\tag{14}$$

$\vec{a}=(\|Q_1(x_1)-Q_1(x_2)\|,\|Q_2(y_1)-Q_2(y_2)\|)^T$ and $\vec{b}=(\|x_1-x_2\|,\|y_1-y_2\|)^T$, then from (12), (13) and (14), we have $\vec{a}\le(1-\lambda)\mathbf{E}+\lambda\boldsymbol{\Psi}\vec{b}$, where

$$\mathbf{E}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \boldsymbol{\Psi}=\begin{pmatrix}\Gamma_{11}&\Gamma_{12}\\\Gamma_{21}&\Gamma_{22}\end{pmatrix},\quad 0<\lambda<1,\tag{15}$$

where $\boldsymbol{\Psi}$ is called the matrix for nonlinear set-valued inclusions system. By using [16], we have

$$\|\vec{a}\|\le(1-\lambda)+\lambda\|\boldsymbol{\Psi}\|\|\vec{b}\|.\tag{16}$$

Letting

$$\|\boldsymbol{\Psi}\|=\max\{\Gamma_{11},\Gamma_{12},\Gamma_{21},\Gamma_{22}\}.$$

It follows from (16), the assumption of the condition (7) and $S(x),T(y)\in CB(X)$ that $0<\|\boldsymbol{\Psi}\|<1$, $(1-\lambda)+\lambda\|\boldsymbol{\Psi}\|<1$, and there exist $x^*,y^*\in X$ and $z^*\in S(x^*)$, $w^*\in T(y^*)$ such that

$$\begin{cases}Q_1(x^*)=x^*,\\Q_2(y^*)=y^*.\end{cases}$$

Therefore, the following relations hold for Theorem 3.1(ii)-(iii):

$$\begin{cases}x^*=R^{A,\eta_1}_{\rho_1,M}(A(x^*)+\rho_1f(x^*)-\rho_1F(z^*,y^*)),\\y^*=R^{B,\eta_2}_{\rho_2,N}(B(y^*)+\rho_2g(y^*)-\rho_2G(w^*,x^*)),\end{cases}\tag{17}$$

where $\rho_i>0$ is a constant ($i=1,2$). Thus, by Theorem 3.1, we know that (x^*,y^*,z^*,w^*) is a solution of the problem (3). This completes the proof. □

4 Ishikawa-hybrid proximal algorithm

In 2008, Verma developed a hybrid version of the Eckstein-Bertsekas [11] proximal point algorithm, introduced the algorithm based on the (A,η)-maximal monotonicity framework [7] and studied convergence of the algorithm, and so did Li, Xu and Jin in [12]. Based on Theorem 3.1, we develop an Ishikawa-hybrid proximal point algorithm for finding an iterative sequence solving the problem (3) as follows.

Algorithm 4.1 Let X be a q-uniformly smooth Banach space. Let $f,g:X\to X$ be two single-valued κ_1 or κ_2-Lipschitz continuous mappings, respectively, $\eta_i:X\times X\to X$ be a single-valued τ_i-Lipschitz continuous mapping $(i=1,2)$, $F,G:X\times X\to X$ be two single-valued (μ_1,ν_1) or (μ_2,ν_2)-Lipschitz continuous mappings, respectively. Let $A:X\to X$ be single-valued r_1-strongly η_1-accretive, ω_1-Lipschitz continuous, (σ_1,φ_1)-relaxed cocoercive with respect to f, and $B:X\to X$ be single-valued r_2-strongly η_2-accretive, ω_2-Lipschitz continuous, (σ_2,φ_2)-relaxed cocoercive with respect to g. Let $S,T:X\to X$ be two set-valued γ_1 or γ_2-Lipschitz continuous mappings, respectively, $M:X\to 2^X$ be a set-valued (A,η_1)-accretive mapping and $N:X\to 2^X$ be a set-valued (B,η_2)-accretive mapping. Suppose that $\{\alpha_i^n\}_{n=0}^\infty$, $\{\beta_i^n\}_{n=0}^\infty$, $\{\xi_i^n\}_{n=0}^\infty$, $\{\zeta_i^n\}_{n=0}^\infty$ and $\{\rho_i^n\}_{n=0}^\infty$ $(i=1,2)$ are ten nonnegative sequences such that

$$\lim_{n\to\infty}\xi_i^n=0,\qquad \lim_{n\to\infty}\zeta_i^n=0,\qquad \alpha=\limsup_{n\to\infty}\alpha_i^n<1,$$
$$\beta=\limsup_{n\to\infty}\beta_i^n<1,\qquad \rho_i^n\uparrow\rho_i<\frac{r_i}{m_i}\quad(i=1,2),$$

then we can get $x^1,y^1\in X$ and $z^1\in S(x^1)$, $w^1\in T(y^1)$ as follows.

Step 1: For arbitrarily chosen initial points $x^0\in X$, $y^0\in X$, we choose suitable $z^0\in S(x^0)$, $w^0\in T(y^0)$, setting

$$\begin{cases} u^0=(1-\alpha_1^0)x^0+\alpha_1^0e_1^0,\\ x^1=(1-\beta_1^0)x^0+\beta_1^0d_1^0,\end{cases}$$

where e_1^0, d_1^0 satisfy

$$\begin{cases} \|e_1^0-R_{\rho_1^0,M}^{A,\eta_1}(A(x^0)+\rho_1^0f(x^0)+\rho_1^0F(z^0,y^0))\|\\ \quad\le\xi_1^0\|e_1^0-x^0\|\quad(z^0\in S(x^0)),\\ \|d_1^0-R_{\rho_1^0,M}^{A,\eta_1}(A(u^0)+\rho_1^0f(u^0)+\rho_1^0F(z_1^0,y^0))\|\\ \quad\le\zeta_1^0\|d_1^0-u^0\|\quad(z_1^0\in S(u^0)),\end{cases}$$

and

$$\begin{cases} v^0=(1-\alpha_2^0)y^0+\alpha_2^0e_2^0,\\ y^1=(1-\beta_2^0)y^0+\beta_2^0d_2^0,\end{cases}$$

where e_2^0, d_2^0 satisfy

$$\begin{cases} \|e_2^0 - R^{B,\eta_2}_{\rho_2^0,N}(B(y^0) + \rho_2^0 g(y^0) - \rho_2^0 G(w^0, x^0))\| \\ \quad \le \xi_2^0 \|e_2^0 - y^0\| \quad (w^0 \in T(y^0), \\ \|d_2^0 - R^{B,\eta_2}_{\rho_2^0,N}(B(v^0) + \rho_2^0 g(v^0) - \rho_2^0 G(w_2^0, x^0))\| \\ \quad \le \zeta_2^0 \|d_2^0 - v^0\| \quad (w_2^0 \in T(v^0)). \end{cases}$$

By using Nadler [15], we can choose suitable $z^1 \in S(x^1)$, $w^1 \in T(y^1)$ such that

$$\begin{cases} \|z^0 - z^1\| \le (1 + \frac{1}{1})D(S(x^0), S(x^1)), \\ \|w^0 - w^1\| \le (1 + \frac{1}{1})D(T(y^0), T(y^1)), \\ \|z_1^0 - z_1^1\| \le (1 + \frac{1}{1})D(S(U^0), S(U^1)), \\ \|w_2^0 - w_2^1\| \le (1 + \frac{1}{1})D(T(V^0), T(V^1)). \end{cases}$$

Therefore, we obtain $x^1, y^1 \in X$ and $z^1 \in S(x^1)$, $w^1 \in T(y^1)$ and give the next step for generating sequences $\{x^n\}_{n=2}^\infty$, $\{y^n\}_{n=2}^\infty$, $\{z^n\}_{n=2}^\infty$ and $\{w^n\}_{n=2}^\infty$.

Step 2: From $x^1, y^1 \in X$ and $z^1 \in S(x^1)$, $w^1 \in T(y^1)$, the sequences $\{x^n\}_{n=2}^\infty$, $\{y^n\}_{n=2}^\infty$, $\{z^n\}_{n=2}^\infty$ and $\{w^n\}_{n=2}^\infty$ are generated by the iterative procedure

$$\begin{cases} u^n = (1 - \alpha_1^n)x^n + \alpha_1^n e_1^n, \\ x^{n+1} = (1 - \beta_1^n)x^n + \beta_1^n d_1^n, \\ \|e_1^n - R^{A,\eta_1}_{\rho_1^n,M}(A(x^n) + \rho_1^n f(x^n) + \rho_1^n F(z^n, y^n))\| \\ \quad \le \xi_1^n \|e_1^n - x^n\| \quad (z^n \in S(x^n)), \\ \|d_1^n - R^{A,\eta_1}_{\rho_1^n,M}(A(u^n) + \rho_1^n f(u^n) - \rho_1^n F(z_1^n, y^n))\| \\ \quad \le \zeta_1^n \|d_1^n - u^n\| \quad (z_1^n \in S(u^n)), \end{cases} \tag{18}$$

and

$$\begin{cases} v^n = (1 - \alpha_2^n)y^n + \alpha_2^n e_2^n, \\ y^{n+1} = (1 - \beta_2^n)y^n + \beta_2^n d_2^n, \\ \|e_2^n - R^{B,\eta_2}_{\rho_2^n,N}(B(y^n) + \rho_2^n g(y^n) - \rho_2^n G(w^n, x^n))\| \\ \quad \le \xi_2^n \|e_2^n - y^n\| \quad (w^n \in T(y^n)), \\ \|d_2^n - R^{B,\eta_2}_{\rho_2^n,N}(B(v^n) + \rho_2^n g(v^n) - \rho_2^n G(w_2^n, x^n))\| \\ \quad \le \zeta_2^n \|d_2^n - v^n\| \quad (w_2^n \in T(v^n)). \end{cases} \tag{19}$$

By using Nadler [15], we can choose suitable $z^{n+1} \in S(x^{n+1})$, $w^{n+1} \in T(y^{n+1})$ such that

$$\begin{cases} \|z^n - z^{n+1}\| \le (1 + \frac{1}{1+n})D(S(x^n), S(x^{n+1})), \\ \|w^n - w^{n+1}\| \le (1 + \frac{1}{1+n})D(T(y^n), T(y^{n+1})), \end{cases} \tag{20}$$

for $n = 0, 1, 2, \ldots$.

Remark 4.2 If we choose some suitable operators A, B, η_1, η_1, F, G, S, T, M, N, f, g and a space X, then Algorithm 4.1 can degenerate to a number of known algorithms for solving the system of variational inequalities and variational inclusions (see [2–6, 8–10, 25]).

5 Convergence of Ishikawa-hybrid proximal Algorithm 4.1

In this section, we prove that $\{(x^n, y^n, z^n, w^n)\}_{n=0}^{\infty}$ generated by Ishikawa-hybrid proximal Algorithm 4.1 converges linearly to a solution (x^*, y^*, z^*, w^*) of the problem (3) as the convergence rate $\|\Psi\|$.

Theorem 5.1 *Let X be a q-uniformly smooth Banach space. Let $f, g : X \to X$ be two single-valued κ_1 or κ_2-Lipschitz continuous mappings, respectively, $\eta_i : X \times X \to X$ be a single-valued τ_i-Lipschitz continuous mapping $(i = 1, 2)$, $F, G : X \times X \to X$ be two single-two valued (μ_1, ν_1) or (μ_2, ν_2)-Lipschitz continuous mappings, respectively. Let $A : X \to X$ be a single-valued r_1-strongly η_1-accretive and ω_1-Lipschitz continuous mapping, and let $B : X \to X$ be a single-valued r_2-strongly η_2-accretive and ω_2-Lipschitz continuous mapping. Let $S, T : X \to X$ be two set-valued γ_1 or γ_2-Lipschitz continuous mappings, respectively, A be (σ_1, φ_1)-relaxed cocoercive with respect to f and B be (σ_2, φ_2)-relaxed cocoercive with respect to g. Suppose that $M : X \to 2^X$ is a set-valued (A, η_1)-accretive mapping and $N : X \to 2^X$ is a set-valued (B, η_2)-accretive mapping, and the following conditions hold*:

$$\begin{cases} \max\{\rho_1\mu_1\gamma_1 + l_1, \alpha_1\beta_1\nu_1\theta_1(1 + \rho_1 - \beta_1\rho_1), \\ \quad \beta_2\rho_2\nu_2(1 + 2\alpha_2\theta_2), (\rho_2\mu_2\gamma_2 + l_2)\} < \tau^{1-q}(r_1 - m_1\rho_1), \\ 2\frac{\beta_1(1-\alpha_1)}{\alpha_1+\beta_1}\theta_1 + \frac{\alpha_1\beta_1}{\alpha_1+\beta_1}\theta_1^2 < 1 - \theta_1, \\ (1 - \beta_2 + \beta_2\theta_2) + 3(1 - \alpha_2 + \alpha_2\theta_2)\beta_2\theta_2 + \beta_2\theta_2 < 1, \\ l_1 = \sqrt[q]{\omega_1^q + c_q\rho_1^q\kappa_1^q + q\sigma_1\omega_1^q - q\varphi_1}, \qquad l_2 = \sqrt[q]{\omega_2^q + c_q\rho_2^q\kappa_2^q + q\sigma_2\omega_2^q - q\varphi_2}, \\ \theta_1 = \frac{\tau^{q-1}}{r_1 - m_1\rho_1}(\rho_1\mu_1\gamma_1 + l_1), \qquad \theta_2 = \frac{\tau^{q-1}}{r_2 - m_2\rho_2}(\rho_2\mu_2\gamma_2 + l_2), \end{cases} \tag{21}$$

and eight nonnegative sequences $\{\alpha_i^n\}_{n=0}^{\infty}$, $\{\beta_i^n\}_{n=0}^{\infty}$, $\{\xi_i^n\}_{n=0}^{\infty}$, $\{\zeta_i^n\}_{n=0}^{\infty}$ and $\{\rho_i^n\}_{n=0}^{\infty}$ $(i = 1, 2)$ satisfy the following conditions:

$$\lim_{n\to\infty} \xi_i^n = \lim_{n\to\infty} \zeta_i^n = 0, \qquad \alpha_i = \limsup_{n\to\infty} \alpha_i^n < 1, \tag{22}$$

$$\beta_i = \limsup_{n\to\infty} \beta_i^n < 1, \qquad \rho_i^n \uparrow \rho_i < \frac{r_i}{m_i}. \tag{23}$$

Then the problem (3) *has a solution $(x^*, y^*, z^*, w^*) z^* \in S(x^*)$, $w^* \in T(y^*)$, and the sequence $\{x^n, y^n\}_{n=0}^{\infty}$ generated by Ishikawa-hybrid proximal Algorithm* 4.1 *converges linearly to a solution (x^*, y^*) of the problem* (3) *as the convergence rate*

$$\begin{aligned} \|\Psi\| = \max\biggl\{ & 1 - (\alpha_1 + \beta_1) + (\alpha_1 + \beta_1)\theta_1 + (2 - 2\alpha_1 + \alpha_1\theta_1)\beta_1\theta_1, \\ & \alpha_1\beta_1\nu_1\theta_1\frac{\tau^{q-1}}{r_1 - m_1\rho_1}(1 + \rho_1 - \beta_1\rho_1), \\ & 1 - \beta_2 + 2\beta_2\theta_2(2 - \alpha_2 + \alpha_2\theta_2) + \beta_2\theta_2(1 - \alpha_2 + \alpha_2\theta_2), \\ & (1 - \beta_2 + \beta_2\theta_2) + 3(1 - \alpha_2 + \alpha_2\theta_2)\beta_2\theta_2 + \beta_2\theta_2 \biggr\}, \end{aligned} \tag{24}$$

where $c_q > 0$ is the same as in Lemma 2.5, $\rho_i \in (0, \frac{r_i}{m_i})$ $(i = 1, 2)$.

Proof Let (x^*, y^*, z^*, w^*) ($z^* \in S(x^*)$, $w^* \in T(y^*)$) be the solution of the problem (3), then for any $\lambda > 0$,

$$\begin{cases} x^* = (1-\lambda)x^* + \lambda R^{A,\eta_1}_{\rho_1,M}(A(x^*) + \rho_1 f(x^*) - \rho_1 F(z^*, y^*)), \\ y^* = (1-\lambda)y^* + \lambda R^{B,\eta_2}_{\rho_2,N}(B(y^*) + \rho_2 g(y^*) - \rho_2 G(w^*, x^*)). \end{cases} \tag{25}$$

For $n \geq 0$, we write

$$\begin{cases} s_1^n = (1-\alpha_1^n)x^n + \alpha_1^n R^{A,\eta_1}_{\rho_1^n,M}(A(x^n) + \rho_1^n f(x^n) + \rho_1^n F(z^n, y^n)), \\ t_1^{n+1} = (1-\beta_1^n)x^n + \beta_1^n R^{A,\eta_1}_{\rho_1^n,M}(A(s_1^n) + \rho_1^n f(s_1^n) + \rho_1^n F(z_2^n, y^n)) \\ \quad (z_2^n \in S(s_1^n)). \end{cases} \tag{26}$$

It follows from the hypotheses of the mappings A, f, F, S, M, η_1 and $R^{A,\eta_1}_{\rho_1^n,M}$ in Algorithm 4.1 that

$$\begin{aligned} \|s_1^n - x^*\| &\leq \|(1-\alpha_1^n)x^n + \alpha_1^n R^{A,\eta_1}_{\rho_1^n,M}(A(x^n) + \rho_1^n f(x^n) + \rho_1^n F(z^n, y^n)) \\ &\quad - (1-\alpha_1^n)x^* - \alpha_1^n R^{A,\eta_1}_{\rho_1^n,M}(A(x^*) + \rho_1^n f(x^*) - \rho_1^n F(z^*, y^*))\| \\ &\leq [(1-\alpha_1^n) + \alpha_1^n\theta_1]\|x^n - x^*\| + \alpha_1^n \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\rho_1^n \nu_1\|y^n - y^*\|, \\ \|u^n - x^*\| &\leq \|(1-\alpha_1^n)x^n + \alpha_1^n e_1^n \\ &\quad - (1-\alpha_1^n)x^* - \alpha_1^n R^{A,\eta_1}_{\rho_1^n,M}(A(x^*) + \rho_1^n f(x^*) - \rho_1^n F(z^*, y^*))\| \\ &\leq (1-\alpha_1^n)\|x^n - x^*\| + \alpha_1^n\|e_1^n - R^{A,\eta_1}_{\rho_1^n,M}(A(x^*) + \rho_1^n f(x^*) - \rho_1^n F(z^*, y^*))\| \\ &\leq (1-\alpha_1^n)\|x^n - x^*\| + \alpha_1^n\|e_1^n - R^{A,\eta_1}_{\rho_1^n,M}(A(x^n) + \rho_1^n f(x^n) - \rho_1^n F(z^n, y^n))\| \\ &\quad + \alpha_1^n\|R^{A,\eta_1}_{\rho_1^n,M}(A(x^n) + \rho_1^n f(x^n) - \rho_1^n F(z^n, y^n)) \\ &\quad - R^{A,\eta_1}_{\rho_1^n,M}(A(x^*) + \rho_1^n f(x^*) - \rho_1^n F(z^*, y^*))\| \\ &\leq (1-\alpha_1^n)\|x^n - x^*\| + \alpha_1^n\xi_1^n\|e_1^n - x^n\| \\ &\quad + \alpha_1^n \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}[\rho_1^n\mu_1\gamma_1\|x^n - x^*\| + \nu_1\|y^n - y^*\| \\ &\quad + \sqrt[q]{\omega_1^q + c_q(\rho_1^n)^q\kappa_1^q + q\sigma_1\omega_1^q - q\varphi_1}\|x^n - x^*\|] \\ &\leq (1-\alpha_1^n)\|x^n - x^*\| + \xi_1^n\|u^n - x^*\| + \xi_1^n\|x^n - x^*\| \\ &\quad + \alpha_1^n \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}[\rho_1^n\mu_1\gamma_1\|x^n - x^*\| + \nu_1\|y^n - y^*\| \\ &\quad + \sqrt[q]{\omega_1^q + c_q(\rho_1^n)^q\kappa_1^q + q\sigma_1\omega_1^q - q\varphi_1}\|x^n - x^*\|] \\ &\leq (1-\alpha_1^n + \xi_1^n + \alpha_1^n\theta_1)\|x^n - x^*\| + \xi_1^n\|u^n - x^*\| \\ &\quad + \alpha_1^n \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\nu_1\|y^n - y^*\|; \end{aligned}$$

that is,

$$\begin{aligned} \|s_1^n - x^*\| &\le \big[(1-\alpha_1^n) + \alpha_1^n\theta_1(n)\big]\|x^n - x^*\| + \alpha_1^n \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\rho_1^n\nu_1\|y^n - y^*\|, \\ \|u^n - x^*\| &\le \frac{1}{1-\xi_1^n}\big(1 - \alpha_1^n + \xi_1^n + \alpha_1^n\theta_1(n)\big)\|x^n - x^*\| \\ &\quad + \frac{1}{1-\xi_1^n}\alpha_1^n\frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\nu_1\|y^n - y^*\|, \end{aligned} \tag{27}$$

where $\theta_1(n) = \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}(\rho_1^n\mu_1\gamma_1 + \sqrt[q]{\omega_1^q + c_q(\rho_1^n)^q\kappa_1^q + q\sigma_1\omega_1^q - q\varphi_1})$, $z^n \in S(x^n)$ and $x^* \in S(x^*)$.

From (24)-(27) and (13), we have

$$\begin{aligned} \|t_1^{n+1} - x^*\| &\le (1-\beta_1^n)\|x^n - x^*\| \\ &\quad + \beta_1^n\big\|R^{A,\eta_1}_{\rho_1^n,M}\big(A(x^*) + \rho_1^n f(x^*) - \rho_1^n F(z^*, y^*)\big) \\ &\quad - R^{A,\eta_1}_{\rho_1^n,M}\big(A(s_1^n) + \rho_1^n f(s_1^n) + \rho_1^n F(z_2^n, y^n)\big)\big\| \\ &\le \bigg((1-\beta_1^n) + \beta_1^n\frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\big(\rho_1^n\mu_1\gamma_1 \\ &\quad + \sqrt[q]{\omega_1^q + c_q(\rho_1^n)^q\kappa_1^q + q\sigma_1\omega_1^q - q\varphi_1}\big)\bigg)\|s_1^n - x^*\| \\ &\quad + \beta_1^n\frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\nu_1\|y^n - y^*\| \\ &\le \big((1-\beta_1^n) + \beta_1^n\theta_1(n)\big)\|s_1^n - x^*\| + \beta_1^n\frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\nu_1\|y^n - y^*\|. \end{aligned} \tag{28}$$

By Algorithm 4.1, $x^{n+1} - x^n = \beta_1^n(d_1^n - x^n)$ and $u^n - x^n = \alpha_1^n(e_1^n - x^n)$, we have

$$\begin{aligned} \|x^{n+1} - t_1^{n+1}\| &\le \big\|(1-\beta_1^n)x^n + \beta_1^n d_1^n - (1-\beta_1^n)x^n \\ &\quad - \beta_1^n R^{A,\eta_1}_{\rho_1^n,M}\big(A(s_1^n) + \rho_1^n f(s_1^n) - \rho_1^n F(z_2^n, y^n)\big)\big\| \\ &\le \beta_1^n\big\|d_1^n - R^{A,\eta_1}_{\rho_1^n,M}\big(A(s_1^n) + \rho_1^n f(s_1^n) - \rho_1^n F(z_2^n, y^n)\big)\big\| \\ &\le \beta_1^n\big(\big\|d_1^n - R^{A,\eta_1}_{\rho_1^n,M}\big(A(u^n) + \rho_1^n f(u^n) - \rho_1^n F(z_1^n, y^n)\big)\big\| \\ &\quad + \big\|R^{A,\eta_1}_{\rho_1^n,M}\big(A(u^n) + \rho_1^n f(u^n) - \rho_1^n F(z_1^n, y^n)\big) \\ &\quad - R^{A,\eta_1}_{\rho_1^n,M}\big(A(s_1^n) + \rho_1^n f(s_1^n) - \rho_1^n F(z_2^n, y^n)\big)\big\|\big) \\ &\le \beta_1^n\zeta_1^n\|d_1^n - u^n\| \\ &\quad + \beta_1^n\big\|R^{A,\eta_1}_{\rho_1^n,M}\big(A(u^n) + \rho_1^n f(u^n) - \rho_1^n F(z_1^n, y^n)\big) \\ &\quad - R^{A,\eta_1}_{\rho_1^n,M}\big(A(s_1^n) + \rho_1^n f(s_1^n) - \rho_1^n F(z_2^n, y^n)\big)\big\| \\ &\le \beta_1^n\zeta_1^n\|d_1^n - u^n\| \\ &\quad + \beta_1^n\frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\big[\rho_1^n\mu_1\gamma_1\|s_1^n - u^n\| + \nu_1\|y^n - y^n\| \\ &\quad + \sqrt[q]{\omega_1^q + c_q(\rho_1^n)^q\kappa_1^q + q\sigma_1\omega_1^q - q\varphi_1}\|s_1^n - u^n\|\big] \end{aligned}$$

$$
\begin{aligned}
&\le \zeta_1^n \|x^{n+1} - x^n\| + \beta_1^n \zeta_1^n \|u^n - x^n\| \\
&\quad + \beta_1^n \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n} \big[\rho_1^n \mu_1 \gamma_1 \|s_1^n - u^n\| + \nu_1 \|y^n - y^n\| \\
&\quad + \sqrt[q]{\omega_1^q + c_q (\rho_1^n)^q \kappa_1^q + q\sigma_1 \omega_1^q - q\varphi_1} \|s_1^n - u^n\|\big] \\
&\quad + \beta_1^n \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n} \big[\rho_1^n \mu_1 \gamma_1 + \sqrt[q]{\omega_1^q + c_q (\rho_1^n)^q \kappa_1^q + q\sigma_1 \omega_1^q - q\varphi_1}\big] \|s_1^n - u^n\| \\
&\le \zeta_1^n \|x^{n+1} - x^*\| + \zeta_1^n \|x^n - x^*\| + \beta_1^n \zeta_1^n \|u^n - x^n\| \\
&\quad + \beta_1^n \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n} \big[\rho_1^n \mu_1 \gamma_1 + \sqrt[q]{\omega_1^q + c_q (\rho_1^n)^q \kappa_1^q + q\sigma_1 \omega_1^q - q\varphi_1}\big] \\
&\quad \times \big(\|s_1^n - x^*\| + \|x^* - x^n\| + \|x^n - u^n\|\big) \\
&\le \zeta_1^n \|x^{n+1} - x^*\| + \big(\zeta_1^n + \beta_1^n \theta_1(n)\big) \|x^n - x^*\| \\
&\quad + \beta_1^n \big(\zeta_1^n + \theta_1(n)\big)\big(\|u^n - x^*\| + \|x^n - x^*\|\big) + \beta_1^n \theta_1 \|s_1^n - x^*\| \\
&\le \zeta_1^n \|x^{n+1} - x^*\| + \big(\zeta_1^n + 2\beta_1^n \theta_1(n) + \beta_1^n \zeta_1^n\big) \|x^n - x^*\| \\
&\quad + \beta_1^n \big(\zeta_1^n + \theta_1(n)\big) \|u^n - x^*\| + \beta_1^n \theta_1(n) \|s_1^n - x^*\|. \qquad (29)
\end{aligned}
$$

It follows from (26)-(29) that

$$
\begin{aligned}
\|x^{n+1} - x^*\| &\le \|x^{n+1} - t_1^{n+1}\| + \|t_1^{n+1} - x^*\| \\
&\le \zeta_1^n \|x^{n+1} - x^*\| + \bigg[\zeta_1^n + 2\beta_1^n \theta_1(n) + \beta_1^n \zeta_1^n \\
&\quad + \beta_1^n \big(\zeta_1^n + \theta_1(n)\big) \frac{1}{1 - \xi_1^n} \big(1 - \alpha_1^n + \xi_1^n + \alpha_1^n \theta_1(n)\big) \\
&\quad + \big(1 - \beta_1^n\big)\big(1 - \alpha_1^n + \alpha_1^n \theta_1(n)\big)\bigg] \|x^n - x^*\| \\
&\quad + \bigg[\beta_1^n \big(\zeta_1^n + \theta_1(n)\big) \frac{1}{1 - \xi_1^n} \alpha_1^n \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n} \nu_1 \\
&\quad + \big(1 - \beta_1^n\big) \beta_1^n \alpha_1^n \theta_1(n) \rho_1^n \nu_1 \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\bigg] \|y^n - y^*\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\|x^{n+1} - x^*\| &\le \frac{1}{1 - \zeta_1^n} \bigg[\zeta_1^n + 2\beta_1^n \theta_1(n) + \beta_1^n \zeta_1^n \\
&\quad + \beta_1^n \big(\zeta_1^n + \theta_1(n)\big) \frac{1}{1 - \xi_1^n} \big(1 - \alpha_1^n + \xi_1^n + \alpha_1^n \theta_1(n)\big) \\
&\quad + \big(1 - \beta_1^n\big)\big(1 - \alpha_1^n + \alpha_1^n \theta_1(n)\big)\bigg] \|x^n - x^*\| \\
&\quad + \frac{1}{1 - \zeta_1^n} \bigg[\beta_1^n \big(\zeta_1^n + \theta_1(n)\big) \frac{1}{1 - \xi_1^n} \alpha_1^n \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n} \nu_1 \\
&\quad + \big(1 - \beta_1^n\big) \beta_1^n \alpha_1^n \theta_1(n) \rho_1^n \nu_1 \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\bigg] \|y^n - y^*\|, \qquad (30)
\end{aligned}
$$

where

$$\theta_1(n) = \frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\left(\rho_1^n\mu_1\gamma_1 + \sqrt[q]{\omega_1^q + c_q\left(\rho_1^n\right)^q\kappa_1^q + q\sigma_1\omega_1^q - q\varphi_1}\right).$$

For $n \geq 0$, we write

$$\begin{cases} s_2^n = (1-\alpha_2^n)y^n + \alpha_2^n R^{B,\eta_2}_{\rho_2^n,N}(B(y^n) + \rho_2^n g(y^n) - \rho_2^n G(w^n, x^n)) \\ \quad (w^n \in T(y^n)), \\ t_2^{n+1} = (1-\beta_2^n)y^n + \beta_2^n R^{B,\eta_2}_{\rho_2^n,N}(B(s_2^n) + \rho_2^n g(s_2^n) - \rho_2^n G(w_2^n, x^n)) \\ \quad (w_2^n \in T(s_2^n)). \end{cases} \tag{31}$$

By using the hypotheses of the mappings B, g, G, T, N, η_2 and $R^{B,\eta_2}_{\rho_2^n,N}$ in Theorem 5.1, and the same method as the one above, we can get

$$\begin{aligned} \|s_2^n - y^*\| &\leq \big\|(1-\alpha_2^n)y^n + \alpha_2^n R^{B,\eta_2}_{\rho_2^n,N}\big(B(y^n) + \rho_2^n g(y^n) - \rho_2^n G(w^n, x^n)\big) \\ &\qquad - (1-\alpha_2^n)y^* - \alpha_2^n R^{B,\eta_2}_{\rho_2^n,N}\big(B(y^*) + \rho_2 g(y^*) - \rho_2 G(w^*, x^*)\big)\big)\big\| \\ &\leq \alpha_2^n \frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\rho_2^n\nu_2\|x^n - x^*\| + \big[(1-\alpha_2^n) + \alpha_2^n\theta_2(n)\big]\|y^n - y^*\|, \\ \|v^n - y^*\| &\leq \big\|(1-\alpha_2^n)y^n + \alpha_2^n e_2^n - (1-\beta_2^n)y^* \\ &\qquad - \beta_2^n R^{B,\eta_2}_{\rho_2^n,N}\big(B(y^*) + \rho_2^n g(y^*) - \rho_2^n G(w^*, x^*)\big)\big\| \\ &\leq (1-\alpha_2^n)\|y^n - y^*\| + \alpha_2^n\big\|e_2^n - R^{B,\eta_2}_{\rho_2^n,N}\big(B(y^n) + \rho_2^n g(y^n) - \rho_2^n G(w^n, x^n)\big)\big\| \\ &\qquad + \alpha_2^n\big\|R^{B,\eta_2}_{\rho_2^n,N}\big(B(y^n) + \rho_2^n g(y^n) - \rho_2^n G(w^n, x^n)\big) \\ &\qquad - R^{B,\eta_2}_{\rho_2^n,N}\big(B(y^*) + \rho_2^n g(y^*) - \rho_2^n G(w^*, x^*)\big)\big\| \\ &\leq (1-\alpha_2^n)\|y^n - y^*\| + \alpha_2^n\xi_2^n\|e_2^n - y^n\| \\ &\qquad + \alpha_2^n \frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\rho_2^n\nu_2\|x^n - x^*\| + \alpha_2^n\theta_2(n)\|y^n - y^*\| \\ &\leq (2 - \alpha_2^n + \alpha_2^n\theta_2)\|y^n - y^*\| + \xi_2^n\|v^n - y^*\| + \alpha_2^n\frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\rho_2^n\nu_2\|x^n - x^*\|; \end{aligned}$$

that is,

$$\begin{aligned} \|s_2^n - y^*\| &\leq \alpha_2^n\frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\rho_2^n\nu_2\|x^n - x^*\| + \big(1 - \alpha_2^n + \alpha_2^n\theta_2(n)\big)\|y^n - y^*\|, \\ \|v^n - y^*\| &\leq \frac{1}{1-\xi_2^n}\big(2 - \alpha_2^n + \alpha_2^n\theta_2(n)\big)\|y^n - y^*\| \\ &\qquad + \frac{1}{1-\xi_2^n}\alpha_2^n\rho_2^n\nu_2\frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\|x^n - x^*\|, \end{aligned} \tag{32}$$

where $\alpha_2^n(e_2^n - y^n) = v^n - y^n$, and $\theta_2(n) = \frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}(\rho_2^n\mu_2\gamma_2 + \sqrt[q]{\omega_2^q + c_q(\rho_2^n)^q\kappa_2^q + q\sigma_2\omega_2^q - q\varphi_2})$.

Moreover, we have

$$\begin{aligned}
\|t_2^{n+1} - y^*\| &\le \big\|(1-\beta_2^n)y^n + \beta_2^n d_2^n - (1-\beta_2^n)y^* \\
&\quad - \beta_2^n R_{\rho_2^n,N}^{B,\eta_2}\big(B(y^*) + \rho_2^n g(y^*) - \rho_2^n G(w^*,x^*)\big)\big\| \\
&\le (1-\beta_2^n)\|y^n - y^*\| + \beta_2^n \big\|d_2^n - R_{\rho_2^n,N}^{B,\eta_2}\big(B(v^n) + \rho_2^n g(v^n) - \rho_2^n G(w_2^n,x^n)\big)\big\| \\
&\quad + \beta_2^n \big\|R_{\rho_2^n,N}^{B,\eta_2}\big(B(v^n) + \rho_2^n g(v^n) - \rho_2^n G(w_2^n,x^n)\big) \\
&\quad - R_{\rho_2^n,N}^{B,\eta_2}\big(B(y^*) + \rho_2^n g(y^*) - \rho_2^n G(w^*,x^*)\big)\big\| \\
&\le (1-\beta_2^n)\|y^n - y^*\| + \zeta_2^n\beta_2^n\|d_2^n - y^n\| + \beta_2^n\zeta_2^n\|y^n - s_2^n\| \\
&\quad + \beta_2^n \frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\rho_2^n\nu_2\|x^n - x^*\| + \beta_2^n\theta_2(n)\|v^n - y^*\| \\
&\le (1-\beta_2^n + \zeta_2^n + \beta_2^n\zeta_2^n)\|y^n - y^*\| + \beta_2^n\frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\rho_2^n\nu_2\|x^n - x^*\| \\
&\quad + \zeta_2^n\|y^{n+1} - y^*\| + \beta_2^n\zeta_2^n\|s_2^n - y^*\| + \beta_2^n\theta_2(n)\|v^n - y^*\|,
\end{aligned}$$

for (19) $y^{n+1} - y^n = \beta_2^n(d_2^n - y^n)$.

It follows from (26) that

$$\begin{aligned}
\|y^{n+1} - t_2^{n+1}\| &\le \big\|(1-\beta_2^n)y^n + \beta_2^n d_2^n - (1-\beta_2^n)y^n \\
&\quad - \beta_2^n R_{\rho_2^n,N}^{B,\eta_2}\big(B(s_2^n) + \rho_2^n g(s_2^n) - \rho_2^n G(w_2^n,x^n)\big)\big\| \\
&\le \beta_2^n\big\|d_2^n - R_{\rho_2^n,N}^{B,\eta_2}\big(B(s_2^n) + \rho_2^n g(s_2^n) - \rho_2^n G(w_2^n,x^n)\big)\big\| \\
&\le \beta_2^n\zeta_2^n\|d_2^n - v^n\| + \beta_2^n\frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\rho_2^n\nu_2\|x^n - x^n\| + \beta_2^n\theta_2(n)\|s_2^n - v^n\| \\
&\le \zeta_2^n\big(\|y^{n+1} - y^*\| + \|y^n - y^*\|\big) + \zeta_2^n\beta_2^n\big(\|y^n - y^*\| + \|v^n - y^*\|\big) \\
&\quad + \beta_2^n\theta_2(n)\big(\|s_2^n - y^*\| + \|y^* - v^n\|\big) \\
&\le \zeta_2^n\|y^{n+1} - y^*\| + (\zeta_2^n + \zeta_2^n\beta_2^n)\|y^n - y^*\| \\
&\quad + \big(\zeta_2^n\beta_2^n + \beta_2^n\theta_2(n)\big)\|v^n - y^*\| + \beta_2^n\theta_2(n)\|s_2^n - y^*\|.
\end{aligned} \tag{33}$$

Combining (30), (31), (32), (33) and (19), we have

$$\begin{aligned}
\|y^{n+1} - y^*\| &\le \frac{1}{1-2\zeta_2^n}\bigg[(1-\beta_2^n + 2\beta_2^n\zeta_2^n + 2\zeta_2^n)\|y^n - y^*\| \\
&\quad + \beta_2^n\frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\rho_2^n\nu_2\|x^n - x^*\| + \big(2\beta_2^n\theta_2(n) + \zeta_2^n\beta_2^n\big)\|v^n - y^*\| \\
&\quad + \beta_2^n\big(\zeta_2^n + \theta_2(n)\big)\|s_2^n - y^*\|\bigg] \\
&\le \frac{1}{1-2\zeta_2^n}\bigg(1 - \beta_2^n + 2\beta_2^n\zeta_2^n + 2\zeta_2^n \\
&\quad + \big(2\beta_2^n\theta_2(n) + \zeta_2^n\beta_2^n\big)\frac{1}{1-\xi_2^n}\big(2 - \alpha_2^n + \alpha_2^n\theta_2(n)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \beta_2^n\big(\zeta_2^n + \theta_2(n)\big)\big[\big(1-\alpha_2^n\big) + \alpha_2^n\theta_2(n)\big]\Big)\big\|y^n - y^*\big\| \\
&+ \frac{1}{1-2\zeta_2^n}\bigg(\beta_2^n\frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\rho_2^n\nu_2 + \beta_2^n\big(\zeta_2^n + \theta_2(n)\big)\alpha_2^n\frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\rho_2^n\nu_2 \\
&+ \big(2\beta_2^n\theta_2(n) + \zeta_2^n\beta_2^n\big)\frac{1}{1-\xi_2^n}\alpha_2^n\rho_2^n\nu_2\frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\bigg)\big\|x^n - x^*\big\|. \qquad (34)
\end{aligned}
$$

By using (22) and (23), let

$$
\begin{aligned}
a_{11} &= \limsup_{n\to\infty}\frac{1}{1-\zeta_1^n}\bigg[\zeta_1^n + 2\beta_1^n\theta_1(n) + \beta_1^n\zeta_1^n \\
&\quad + \beta_1^n\big(\zeta_1^n + \theta_1(n)\big)\frac{1}{1-\xi_1^n}\big(1-\alpha_1^n + \xi_1^n + \alpha_1^n\theta_1(n)\big) \\
&\quad + \big(1-\beta_1^n\big)\big(1-\alpha_1^n + \alpha_1^n\theta_1(n)\big)\bigg] \\
&= 1 - (\alpha_1 + \beta_1) + (\alpha_1+\beta_1)\theta_1(n) + \big(2 - 2\alpha_1 + \alpha_1\theta_1(n)\big)\beta_1\theta_1(n), \\
a_{12} &= \limsup_{n\to\infty}\frac{1}{1-\zeta_1^n}\bigg[\beta_1^n\big(\zeta_1^n + \theta_1(n)\big)\frac{1}{1-\xi_1^n}\alpha_1^n\frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\nu_1 \\
&\quad + \big(1-\beta_1^n\big)\beta_1^n\alpha_1^n\theta_1(n)\rho_1^n\nu_1\frac{\tau^{q-1}}{r_1 - m_1\rho_1^n}\bigg] \\
&= \alpha_1\beta_1\nu_1\theta_1\frac{\tau^{q-1}}{r_1 - m_1\rho_1}(1 + \rho_1 - \beta_1\rho_1), \\
a_{21} &= \limsup_{n\to\infty}\frac{1}{1-2\zeta_2^n}\bigg(\beta_2^n\frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\rho_2^n\nu_2 + \beta_2^n\big(\zeta_2^n + \theta_2(n)\big)\alpha_2^n\frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\rho_2^n\nu_2 \qquad (35) \\
&\quad + \big(2\beta_2^n\theta_2(n) + \zeta_2^n\beta_2^n\big)\frac{1}{1-\xi_2^n}\alpha_2^n\rho_2^n\nu_2\frac{\tau^{q-1}}{r_2 - m_2\rho_2^n}\bigg) \\
&= \beta_2\rho_2\nu_2(1 + 2\alpha_2\theta_2)\frac{\tau^{q-1}}{r_2 - m_2\rho_2}, \\
a_{22} &= \limsup_{n\to\infty}\frac{1}{1-2\zeta_2^n}\Big(1 - \beta_2^n + 2\beta_2^n\zeta_2^n + 2\zeta_2^n \\
&\quad + \big(2\beta_2^n\theta_2(n) + \zeta_2^n\beta_2^n\big)\frac{1}{1-\xi_2^n}\big(2 - \alpha_2^n + \alpha_2^n\theta_2\big) \\
&\quad + \beta_2^n\big(\zeta_2^n + \theta_2(n)\big)\big[\big(1-\alpha_2^n\big) + \alpha_2^n\theta_2(n)\big]\Big) \\
&= 1 - \beta_2 + 2\beta_2\theta_2(2 - \alpha_2 + \alpha_2\theta_2) + \beta_2\theta_2(1 - \alpha_2 + \alpha_2\theta_2) \\
&= (1 - \beta_2 + \beta_2\theta_2) + 3(1 - \alpha_2 + \alpha_2\theta_2)\beta_2\theta_2 + \beta_2\theta_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_1 &= \limsup_{n\to\infty}\theta_1(n) = \frac{\tau^{q-1}}{r_1 - m_1\rho_1}(\rho_1\mu_1\gamma_1 + l_1), \\
\theta_2 &= \limsup_{n\to\infty}\theta_2(n) = \frac{\tau^{q-1}}{r_2 - m_2\rho_2}(\rho_2\mu_2\gamma_2 + l_2).
\end{aligned}
$$

Let $\vec{A} = (\|x^{n+1} - x^*\|, \|y^{n+1} - y^*\|)^T$ and $\vec{B} = (\|x^n - x^*\|, \|y^n - y^*\|)^T$, then from (33), (34) and (35), we have $\vec{a} \leq \mathbf{\Psi}\vec{b}$, where

$$\mathbf{\Psi} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{36}$$

which is called the matrix for a nonlinear set-valued inclusions system involving (A,η)-accretive mappings. By using [16], we have

$$\|\vec{A}\| \leq \|\mathbf{\Psi}\|\|\vec{B}\|. \tag{37}$$

Let

$$\|\mathbf{\Psi}\| = \max\{a_{11}, a_{12}, a_{21}, a_{22}\}.$$

It follows from (21)-(23), Theorem 3.1 and [15] that $0 < \|\mathbf{\Psi}\| < 1$ and there exist $x^*, y^* \in X$ and $z^* \in S(x^*)$, $w^* \in T(y^*)$ [17] such that

$$\begin{cases} Q_1(x^*) = x^*, \\ Q_2(y^*) = y^*; \end{cases}$$

and the sequence $\{x^n, y^n\}_{n=0}^{\infty}$ generated by Ishikawa-hybrid proximal Algorithm 4.1 converges linearly to a solution (x^*, y^*) of the problem (3) as the convergence rate

$$\begin{aligned} \|\mathbf{\Psi}\| = \max\Bigg\{ & 1 - (\alpha_1 + \beta_1) + (\alpha_1 + \beta_1)\theta_1 + (2 - 2\alpha_1 + \alpha_1\theta_1)\beta_1\theta_1, \\ & \alpha_1\beta_1\nu_1\theta_1\frac{\tau^{q-1}}{r_1 - m_1\rho_1}(1 + \rho_1 - \beta_1\rho_1), \\ & 1 - \beta_2 + 2\beta_2\theta_2(2 - \alpha_2 + \alpha_2\theta_2) + \beta_2\theta_2(1 - \alpha_2 + \alpha_2\theta_2), \\ & (1 - \beta_2 + \beta_2\theta_2) + 3(1 - \alpha_2 + \alpha_2\theta_2)\beta_2\theta_2 + \beta_2\theta_2\Bigg\}, \end{aligned}$$

where $c_q > 0$ is the same as in Lemma 2.7, $\rho_i \in (0, \frac{r_i}{m_i})$ $(i = 1, 2)$. This completes the proof. □

Remark 5.2 For a suitable choice of the mappings A, B, η_i, F, G, M, N, S, T, f, g and X, we can obtain several known results in [7, 9, 11, 12] as special cases of Theorem 5.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HGL presided over and drafted the manuscript, MQ participated in the revisions of the manuscript.

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