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Well-posedness for parametric generalized vector quasivariational inequality problems of the Minty type

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Abstract
In this paper, we introduce the concepts of well-posedness, and of well-posedness in the generalized sense for parametric generalized vector quasivariational inequality problems of the Minty type. The necessary and sufficient conditions for the various kinds of well-posedness of these problems are obtained. Our results are different from some main results in the literature and extend them.
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1 Introduction and preliminaries

A vector variational inequality in a finite-dimensional Euclidean space was introduced first by Giannessi [1]. Later, this problem has been extended and studied by many authors in abstract spaces; see [2–6]. Moreover, vector variational inequality problems have many important applications in vector optimization problems [7–9], vector equilibria problems [10, 11], and variational relation problems [12, 13].

The concept of well-posedness for unconstrained scalar optimization problems was first introduced and studied by Tykhonov [8], which has become known as Tykhonov well-posedness. In 1966, Levitin and Polyak [14] introduced the concept of well-posedness for constrained scalar optimization problems. With the development of the theory about optimization problems, the concept of well-posedness has been generalized to several related problems, as vector optimization problems, see [15–20], variational inequality problems, see [15, 21–23], equilibria problems, see [24–33] and the references therein. Recently, Fang and Huang [22] studied the well-posedness for a vector variational inequality of the Minty type and the Stampacchia type. Very recently, Lalitha and Bhatia [23] also studied a quasi-variational inequality problem of the Minty type, and the well-posedness for this problem was obtained.

Motivated and inspired by the work mentioned, in this paper, we also study the parametric generalized vector quasivariational inequality problems. However, we only study the well-posedness for generalized vector quasivariational inequality problems of the Minty type. The well-posedness for generalized vector quasivariational inequality problems of the Stampacchia type is the same as the Minty type. Let X, Y, Γ, Λ be metric spaces and

$C \subset Y$ be a closed, convex, and pointed cone with $\operatorname{int} C \neq \emptyset$. The cone C induces a partial ordering in Y defined by

$$y < x \quad \Leftrightarrow \quad y - x \in -\operatorname{int} C, \quad \forall x, y \in Y,$$

$$y \not< x \quad \Leftrightarrow \quad y - x \notin -\operatorname{int} C, \quad \forall x, y \in Y,$$

where $\operatorname{int} C$ denotes the interior of C.

Let $L(X,Y)$ be the space of all linear continuous operators from X into Y, and $A \subset X$ be a nonempty subset. Let $K_1 : A \times \Gamma \to 2^A$, $K_2 : A \times \Gamma \to 2^A$, and $T : A \times \Gamma \to 2^{L(X,Y)}$ be set-valued mappings. Let $Q : L(X,Y) \to L(X,Y)$, $\eta : A \times A \times \Lambda \to A$ be continuous single-valued mappings. We denote by $\langle z, x \rangle$ the value of a linear operator $z \in L(X;Y)$ at $x \in X$, and we always assume that $\langle \cdot, \cdot \rangle$ is continuous.

Now we adopt the following notations (see [10, 12, 13]). For subsets M and N under consideration we adopt the notations

$$(u,v) \text{ w } M \times N \quad \text{means } \forall u \in M, \exists v \in N,$$

$$(u,v) \text{ m } M \times N \quad \text{means } \exists v \in N, \forall u \in M,$$

$$(u,v) \text{ s } M \times N \quad \text{means } \forall u \in M, \forall v \in N,$$

$$(u,v)\ \bar{w}\ M \times N \quad \text{means } \exists u \in M, \forall v \in N \text{ and similarly for } \bar{m}, \bar{s}.$$

where w, m, and s are used for weak, middle, and strong, respectively, kinds of considered problems. Let $\alpha \in \{\text{w}, \text{m}, \text{s}\}$, $\bar{\alpha} \in \{\bar{w}, \bar{m}, \bar{s}\}$, and, for $\gamma \in \Gamma$, $\lambda \in \Lambda$. We consider the following parametric generalized vector quasivariational inequality problems of the Minty type (in short: ($\text{MQVIP}^{\gamma\lambda}$)).

($\text{MQVIP}^{\gamma\lambda}$) Find $\bar{x} \in K_1(\bar{x}, \gamma)$ such that $(y,z) \alpha K_2(\bar{x}, \gamma) \times T(y, \gamma)$ satisfies

$$\big\langle Q(z), \eta(y, \bar{x}, \lambda) \big\rangle \not< 0.$$

Denote by (MQVIP) the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$. For each $\gamma \in \Gamma$, $\lambda \in \Lambda$, and let $E(\gamma) := \{x \in A : x \in K_1(x, \gamma)\}$. We denote by $\Psi_\alpha(\gamma, \lambda)$ the solution sets of ($\text{MQVIP}^{\gamma\lambda}$).

Throughout the article, we assume that $\Psi_\alpha(\gamma, \lambda) \neq \emptyset$ for each (γ, λ) in the neighborhoods $(\gamma_0, \lambda_0) \in \Gamma \times \Lambda$.

Next, we recall some basic definitions and some of their properties.

Definition 1.1 ([34, 35]) Let X and Z be two topological vector spaces and let $G : X \to 2^Z$ be a multifunction.

(i) G is said to be *lower semicontinuous* (*lsc*) *at* x_0 if $G(x_0) \cap U \neq \emptyset$ for each open set $U \subseteq Z$ implies the existence of a neighborhood V of x_0 such that $G(x) \cap U \neq \emptyset$, $\forall x \in V$.

(ii) G is said to be *upper semicontinuous* (*usc*) *at* x_0 if for each open set $U \supseteq G(x_0)$, there is a neighborhood V of x_0 such that $U \supseteq G(x)$, $\forall x \in V$.

(iii) G is said to be *closed at* x_0 if for each net $\{(x_n, y_n)\} \in \operatorname{graph} G := \{(x,y) | y \in G(x)\}$, $(x_n, y_n) \to (x_0, y_0)$, it follows that $(x_0, y_0) \in \operatorname{graph} G$.

Lemma 1.2 ([34, 35]) *Let X and Z be two topological vector spaces and $G : X \to 2^Z$ be a multifunction.*

(i) *If Z is compact and G is closed at x_0, then G is usc at x_0.*
(ii) *If G is usc at x_0 and $G(x_0)$ is closed, then G is closed at x_0.*

The structure of this article is as follows. In the remaining part of this section, we recall definitions for later use. In Section 2, we introduce concepts of well-posedness, and well-posedness in the generalized sense for parametric generalized vector quasivariational inequality problems of the Minty type. Moreover, the necessary and sufficient conditions for the various kinds of well-posedness of these problems are obtained.

2 Main results

Definition 2.1 Let $\{(\gamma_n, \lambda_n)\} \subseteq \Gamma \times \Lambda$ converges to (γ_0, λ_0). A sequence $\{x_n\} \subseteq A$ is said to be an approximating sequence for (MQVIP) corresponding to $\{(\gamma_n, \lambda_n)\}$, if

(i) $x_n \in K_1(x_n, \lambda_n)$, $\forall n$;
(ii) there exists a sequence $\{\varepsilon_n\} \in \operatorname{int} C$ that converges to 0 such that

$$(y,z)\alpha K_2(x_n,\gamma_n) \times T(y,\gamma_n) \quad \text{satisfies} \quad \langle Q(z), \eta(y,x_n,\lambda_n)\rangle + \varepsilon_n \not< 0.$$

Definition 2.2 The problem (MQVIP) is said to be well-posed at (γ_0, λ_0) if

(i) the problem (MQVIP) has a unique solution x_0, *i.e.*, $\Psi_\alpha(\gamma_0, \lambda_0) = \{x_0\}$;
(ii) for any sequence $\{(\gamma_n, \lambda_n)\} \subseteq \Gamma \times \Lambda$ converges to (γ_0, λ_0), every approximating sequence $\{x_n\}$ for (MQVIP) corresponding to $\{(\gamma_n, \lambda_n)\}$ converges to x_0.

Definition 2.3 The problem (MQVIP) is said to be well-posed in the generalized sense at (γ_0, λ_0) if

(i) the solution set $\Psi_\alpha(\gamma_0, \lambda_0)$ of (MQVIP) is nonempty;
(ii) for any sequence $\{(\gamma_n, \lambda_n)\} \subseteq \Gamma \times \Lambda$ that converges to (γ_0, λ_0), every approximating sequence $\{x_n\}$ for (MQVIP) corresponding to $\{(\gamma_n, \lambda_n)\}$ has a subsequence which converges to some point of $\Psi_\alpha(\gamma_0, \lambda_0)$.

For $\gamma \in \Gamma$, $\lambda \in \Lambda$, and $\varepsilon \in \operatorname{int} C$, we denote the approximate solution set of (MQVIP) by $\Omega(\gamma, \lambda, \varepsilon)$:

$$\Omega(\gamma,\lambda,\varepsilon) := \bigl\{x \in K_1(x,\gamma) | (y,z)\alpha K_2(x,\gamma) \times T(y,\gamma) : \langle Q(z), \eta(y,x,\lambda)\rangle + \varepsilon \not< 0\bigr\}.$$

Remark 2.4

(i) In the special case, where $A = B$, $X = Y$, $\Gamma = \Lambda$, $K_1(x,\gamma) = K_2(x,\gamma) = A$, $\eta(y,x,\lambda) = y - x$, and Q is an identity map, let $T : A \times \Gamma \to L(X,Y)$ be a single-valued mapping, then the problem (MQVIP$^{\gamma\lambda}$) reduces to the problem (MVVI$^{\lambda}$) studied in [22].
(ii) In the special case as in Remark 2.4(i), then Definitions 2.1, 2.2, and 2.3 reduce to Definitions 2.2, 2.5, and 2.6, respectively, of Fang and Huang in [22].
(iii) Well-posedness for vector problems has been defined in different ways. In this paper, we denote $\varepsilon \in \operatorname{int} C$ instead of ϵe, with ϵ being positive numbers and $e \in \operatorname{int} C$, *i.e.*, only a fixed direction e is allowed (see [24, 28]).

Remark 2.5 ([36]) Let X and Z be two metric spaces and $G : X \to 2^Z$ be a multifunction. If $G(x_0)$ is compact, then G is usc at x_0 if and only if for any sequence $\{x_n\}$ that converges to x_0 and for any sequence $\{y_n\} \subseteq G(x_n)$, there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to some $y_0 \in G(x_0)$. If, in addition, $G(x_0) = \{y_0\}$ is a singleton, then the above limit point y must be y_0 and the whole $\{y_n\}$ converges to y_0.

The following theorem gives sufficient conditions for the well-posedness and the well-posedness in the generalized sense for (MQVIP).

Theorem 2.6 *Assume for problem* (MQVIP) *that*

(i) *E is usc at γ_0 and $E(\gamma_0)$ is a compact set*;
(ii) *in $K_1(A, \Gamma) \times \{\gamma_0\}$, K_2 is lsc*;
(iii) *in $K_2(K_1(A, \Gamma), \Gamma) \times \{\gamma_0\}$, T is usc and compact-valued if* $\alpha = \mathrm{w}$ (*or* $\alpha = \mathrm{m}$), *and lsc if* $\alpha = \mathrm{s}$.

Then (MQVIP) *is well-posed in the generalized sense at (γ_0, λ_0). Moreover, if $\Psi_\alpha(\gamma_0, \lambda_0)$ is a singleton, then this problem is well-posed at (γ_0, λ_0).*

Proof Since $\alpha = \{\mathrm{w}, \mathrm{m}, \mathrm{s}\}$, we have in fact three cases. However, the proof techniques are similar. We consider only the case $\alpha = \mathrm{s}$. We first prove that Ω_s is upper semicontinuous at $(\gamma_0, \lambda_0, 0)$. Indeed, we suppose to the contrary the existence of an open subset V of $\Omega_s(\gamma_0, \lambda_0, 0)$ such that for all $\{(\gamma_n, \lambda_n, \varepsilon_n)\} \subseteq \Gamma \times \Lambda \times C$ it converges to $\{(\gamma_0, \lambda_0, 0)\}$, that is, $x_n \in \Omega_s(\gamma_n, \lambda_n, \varepsilon_n)$, $x_n \notin V$, for all n. Since E is usc and is compact-valued at γ_0, we can assume that x_n tends to x_0 for some $x_0 \in E(\gamma_0)$. If $x_0 \notin \Omega_s(\gamma_0, \lambda_0, 0) = \Psi(\gamma_0, \lambda_0)$, $\exists y_0 \in K_2(x_0, \gamma_0)$, $\exists z_0 \in T(y_0, \gamma_0)$ such that

$$\langle Q(z_0), \eta(y_0, x_0, \lambda_0)\rangle < 0.$$

By the lower semicontinuity of K_2, T at (x_0, γ_0) and (y_0, γ_0), $\forall y_0 \in K_2(x_0, \gamma_0)$, $\forall z_0 \in T(y_0, \gamma_0)$ there exists $y_n \in K_2(x_n, \gamma_n)$, $z_n \in T(y_n, \gamma_n)$ such that $y_n \to y_0$, $z_n \to z_0$. Since $x_n \in \Omega(\gamma_n, \lambda_n, \varepsilon_n)$, we have

$$\langle Q(z_n), \eta(y_n, x_n, \lambda_n)\rangle + \varepsilon_n \not< 0. \tag{2.1}$$

Let $\mathrm{id} : C \to C$ be an identity map, by the continuity of η, Q, and $\langle \cdot, \cdot\rangle$, it follows that $\langle \cdot, \cdot\rangle + \mathrm{id}$ is continuous (where id is continuous). So (2.1) implies

$$\langle Q(z_0), \eta(y_0, x_0, \lambda_0)\rangle \not< 0,$$

which is impossible. Hence, x_0 belongs to $\Omega_s(\gamma_0, \lambda_0, 0) \subseteq V$, which is again a contradiction, since $x_n \notin V$, for all n. Therefore, Ω_s is usc at $(\gamma_0, \lambda_0, 0)$.

Now we prove that $\Omega_s(\gamma_0, \lambda_0, 0)$ is compact, by checking its closedness. Indeed, let $x_n \in \Omega_s(\gamma_0, \lambda_0, 0)$, $x_n \to x_0$. This proof is similar to above and so we have $x_0 \in \Omega_s(\gamma_0, \lambda_0, 0)$ and hence $\Omega_s(\gamma_0, \lambda_0, 0)$ is compact. By Remark 2.5, we complete the proof. □

The following example shows that the upper semicontinuity and compactness of E are essential.

Example 2.7 Let $A = B = X = Y = \mathbb{R}$, $\Gamma = \Lambda = [0,1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, Q be an identity map, $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x,\gamma) = \left(-\gamma - \frac{1}{2}, \gamma\right],$$
$$\eta(y,x,\gamma) = \left\{\gamma^2 + \gamma + 2 + \varepsilon\right\},$$
$$T(y,\gamma) = \left\{\frac{1}{2^{\gamma+2}}\right\},$$
$$K_2(x,\gamma) = \left[0, 2^{\gamma^2+1}\right].$$

Then we have $E(0) = (-\frac{1}{2}, 0]$ and $E(\gamma) = (-\gamma - \frac{1}{2}, \gamma]$, $\forall \gamma \in (0,1]$. We show that assumptions (ii) and (iii) of Theorem 2.6 are fulfilled. But the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma,\lambda) \in \Gamma \times \Lambda\}$ is not well-posed in the generalized sense at $(0,0)$. The reason is that E is not usc at 0 and $E(0)$ is not compact. In fact

$$\Omega_\alpha(\gamma,\lambda,\varepsilon) = \begin{cases} (-\frac{1}{2}, 0], & \text{if } \gamma = 0, \\ (-\gamma - \frac{1}{2}, \gamma], & \text{if } \gamma \in (0,1]. \end{cases}$$

The following example shows that the lower semicontinuity of K_2 is essential.

Example 2.8 Let $A = B = [-1,1]$, $X = Y = \mathbb{R}$, $\Gamma = \Lambda = [0,1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, Q be an identity map, $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x,\gamma) = [0,1],$$
$$\eta(y,x,\gamma) = \{x + y - \varepsilon\},$$
$$T(y,\gamma) = \{1\},$$
$$K_2(x,\gamma) = \begin{cases} \{-1,0,1\}, & \text{if } \gamma = 0, \\ \{0,1\}, & \text{otherwise}. \end{cases}$$

We have $E(\gamma) = [0,1]$, $\forall \gamma \in [0,1]$. Hence E is usc at 0 and $E(0)$ is compact and the conditions (ii) and (iii) of Theorem 2.6 are easily seen to be fulfilled. But the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma,\lambda) \in \Gamma \times \Lambda\}$ is not well-posed in the generalized sense at $(0,0)$. The reason is that K_2 is not lower semicontinuous at $(x,0)$. In fact

$$\Omega_\alpha(\gamma,\lambda,\varepsilon) = \begin{cases} \{1\}, & \text{if } \gamma = 0, \\ [0,1], & \text{if} \gamma \in (0,1]. \end{cases}$$

Theorem 2.9 *Assume for problem* (MQVIP) *the assumptions* (ii) *and* (iii) *as in Theorem* 2.6 *and replace* (i) *by* (i′):

(i′) *A is compact*, K_1 *is closed in* $A \times \{\gamma_0\}$.

Then (MQVIP) *is well-posed in the generalized sense at* (γ_0, λ_0). *Moreover, if* $\Psi_\alpha(\gamma_0, \lambda_0)$ *is a singleton, then this problem is well-posed at* (γ_0, λ_0).

Proof We omit the proof since the technique is similar to that for Theorem 2.6 with suitable modifications. □

The following example shows that the compactness of A cannot be dropped.

Example 2.10 Let $A = B = X = Y = (-\infty, +\infty)$, $\Gamma = \Lambda = [0, +\infty)$, $C = [0, +\infty)$, $\gamma_0 = 0$, H be the identity map, $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^B$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x, \gamma) = \begin{cases} \{0\}, & \text{if } \gamma = 0, \\ \{2^{1+\gamma}\}, & \text{if } \gamma \neq 0, \end{cases}$$

$$K_2(x, \gamma) = [0, 1],$$

$$\eta(y, x, \lambda) = \left\{ \frac{1}{2^{\gamma+\gamma^2}} - \varepsilon 3^{\cos^2(\gamma)+\gamma+1} \right\},$$

$$T(y, \gamma) = \left\{ \frac{1}{3^{\cos^2(\gamma)+\gamma+1}} \right\}.$$

We see that K_1 is closed at $(x, 0)$, the assumptions (ii) and (iii) of Theorem 2.9 are satisfied. But the family $\{(\mathrm{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is not well-posed in the generalized sense at $(0, 0)$. The reason is that A is not compact. In fact, $\Omega_\alpha(0, 0, 0) = \{0\}$ and $\Omega_\alpha(\gamma, \lambda, \varepsilon) = \{2^{1+\gamma}\}$, $\forall \gamma \in (0, +\infty)$.

The following example shows that the closedness of K_1 is essential.

Example 2.11 Let $A = B = X = Y = [-3, 3]$, $\Gamma = \Lambda = [0, 1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, H be an identity map, $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^B$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x, \gamma) = (-3\gamma, 3],$$

$$K_2(x, \gamma) = [0, 3],$$

$$\eta(y, x, \gamma) = \{x^2 - yx - \varepsilon\},$$

$$T(y, \gamma) = \{1\}.$$

We show that A is compact and the conditions (ii), (iii) of Theorem 2.9 are easily seen to be fulfilled. But the family $\{(\mathrm{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is not well-posed in the generalized sense at $(0, 0)$. The reason is that K_1 is not closed at $(x, 0)$. In fact,

$$\Omega_\alpha(\gamma, \lambda, \varepsilon) = \begin{cases} \{3\}, & \text{if } \gamma = 0, \\ \{0, 3\}, & \text{if } \gamma \neq 0. \end{cases}$$

The following example shows that all assumptions of Theorem 2.6 are satisfied.

Example 2.12 Let $X = Y = \mathbb{R}$, $A = B = [0, 3]$, $\Gamma = \Lambda = [0, 1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, H be an identity map, and let $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x,\gamma) = K_2(x,\gamma) = [0,1],$$
$$\eta(y,x,\gamma) = \gamma^2 + 2\gamma + 1 - \varepsilon,$$
$$T(y,\gamma) = \{1\}.$$

Then $E(\gamma) = [0,1]$, $\forall\gamma \in [0,1]$. We see that all assumptions of Theorem 2.9 are satisfied. So, the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma,\lambda) \in \Gamma \times \Lambda\}$ is well-posed in the generalized sense at $(0,0)$. In fact, $\Omega(\gamma,\lambda,\varepsilon) = [0,1]$, $\forall\gamma \in [0,1]$.

For $(\gamma,\lambda) \in \Gamma \times \Lambda$, $\varepsilon \in \operatorname{int} C$, and positive ξ, we define the following sets of approximate solutions of the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma,\lambda) \in \Gamma \times \Lambda\}$:

$$\begin{aligned}
&\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon)\\
&\quad = \bigcup_{\gamma\in B(\gamma_0,\xi),\lambda\in B(\lambda_0,\xi)} \Omega_\alpha(\gamma,\lambda,\varepsilon)\\
&\quad = \bigcup_{\gamma\in B(\gamma_0,\xi),\lambda\in B(\lambda_0,\xi)} \bigl\{x \in K_1(x,\gamma) | (y,z)\alpha K_2(x,\gamma) \times T(y,\gamma) : \langle Q(z), \eta(y,x,\lambda)\rangle + \varepsilon \not< 0\bigr\},
\end{aligned}$$

where $B(\gamma_0,\xi)$ and $B(\lambda_0,\xi)$ are the closed balls centered at γ_0 and λ_0 with radius ξ.

Observe that, for every $(\gamma,\lambda) \in \Gamma \times \Lambda$,

(i) $\Sigma_\alpha^{\gamma_0\lambda_0}(0,0) = \Omega_\alpha(\gamma_0,\lambda_0,0) = \Psi_\alpha(\gamma_0,\lambda_0)$;

(ii) $\Psi_\alpha(\gamma_0,\lambda_0) \subseteq \Omega_\alpha(\gamma_0,\lambda_0,\varepsilon) \subseteq \Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon)$.

Theorem 2.13 *Assume X is complete and the following conditions hold*:

(i) *K_1 is closed in $A \times \{\gamma_0\}$, and in $K_1(A,\Gamma) \times \{\gamma_0\}$, K_2 is lsc*;

(ii) *in $K_2(K_1(A,\Gamma),\Gamma) \times \{\gamma_0\}$, T is usc and compact-valued if $\alpha = \mathrm{w}$ (or $\alpha = \mathrm{m}$), and lsc if $\alpha = \mathrm{s}$.*

Then (MQVIP) *is well-posed at (γ_0,λ_0) if and only if*

$$\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon) \neq \emptyset, \quad \forall\xi > 0, \varepsilon \in \operatorname{int} C \quad \textit{and} \quad \operatorname{diam} \Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon) \to 0 \quad \textit{as } (\xi,\varepsilon) \to (0,0).$$

Proof Similar arguments can be applied to the three cases. We present only the proof for the case where $\alpha = \mathrm{s}$. If (MQVIP) is well-posed at (γ_0,λ_0), then (MQVIP) has a unique solution $x_0 \in \Psi_s(\gamma_0,\lambda_0)$ and hence $\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon) \neq \emptyset$, $\forall\xi > 0$, $\varepsilon \in \operatorname{int} C$ as $\Psi_s(\gamma_0,\lambda_0) \subseteq \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)$. If $\operatorname{diam} \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon) \nrightarrow 0$ as $(\xi,\varepsilon) \to (0,0)$, then there exist $q > 0$ and $\xi_n > 0$, $\varepsilon_n \in \operatorname{int} C$, such that $\varepsilon_n \to 0$, $\xi_n \to 0$, and

$$\operatorname{diam} \Sigma_s^{\gamma_0\lambda_0}(\xi_n,\varepsilon_n) > q > 0, \quad \forall n \in \mathbb{N}.$$

Then there exist $x_n^1, x_n^2 \in \Sigma_s^{\gamma_0\lambda_0}(\xi_n,\varepsilon_n)$ such that $d(x_n^1,x_n^2) > \frac{q}{2} > 0$. Hence there exist $\gamma_n^1, \gamma_n^2 \in B(\gamma_0,\xi_n)$, and $\lambda_n^1, \lambda_n^2 \in B(\lambda_0,\xi_n)$ such that $\forall y \in K_2(x_n^1,\gamma_n^1)$, $\forall z \in T(y,\gamma_n^1)$ satisfy

$$\bigl\langle Q(z), \eta\bigl(y,x_n^1,\lambda_n^1\bigr)\bigr\rangle + \varepsilon_n \not< 0,$$

and $\forall y \in K_2(x_n^2,\gamma_n^2)$, $\forall z \in T(y,\gamma_n^2)$ satisfy

$$\bigl\langle Q(z), \eta\bigl(y,x_n^2,\lambda_n^2\bigr)\bigr\rangle + \varepsilon_n \not< 0,$$

i.e., $\{x_n^1\}$ and $\{x_n^2\}$ are approximating sequences for (MQVIP) corresponding to $\{(\gamma_n^1, \lambda_n^1)\}$ and $\{(\gamma_n^2, \lambda_n^2)\}$, respectively. Hence, the sequences $\{x_n^1\}$ and $\{x_n^2\}$ converges to the unique solution x_0 of (MQVIP$^{\gamma_0\lambda_0}$), contradicting the fact that $d(x_n^1, x_n^2) > \frac{q}{2} > 0$, $\forall n \in \mathbb{N}$.

Conversely, let $\{\gamma_n\} \to \gamma_0$ and $\{\lambda_n\} \to \lambda_0$, and $\{x_n\}$ be approximating sequences for (MQVIP) corresponding to $\{\gamma_n\}$ and $\{\lambda_n\}$. Then there is $\{\varepsilon_n\} \to 0$ such that $\forall y \in K_2(x_n, \gamma_n)$, $\forall z \in T(y, \gamma_n)$ satisfying

$$\langle Q(z), \eta(y, x_n, \lambda_n)\rangle + \varepsilon_n \not< 0, \quad \forall n \in \mathbb{N}.$$

This yields $x_n \in \Sigma_s^{\gamma_0\lambda_0}(\xi_n, \varepsilon_n)$ with $\{\xi_n\} = \max\{d(\gamma_n, \gamma_0), d(\lambda_n, \lambda_0)\} \to 0$, as $n \to +\infty$. Since $\operatorname{diam} \Sigma_s^{\gamma_0\lambda_0}(\xi_n, \varepsilon_n) \to 0$ as $(\xi_n, \varepsilon_n) \to (0,0)$, it follows that $\{x_n\}$ is Cauchy and converges to a point x_0. By the closedness of K_1 at (x_0, γ_0), $x_0 \in K_1(x_0, \gamma_0)$.

Next, we verify that $x_0 \in \Psi_s(\gamma_0, \lambda_0)$. Using the same argument as for Theorem 2.6, we deduce that $x_0 \in \Psi_s(\gamma_0, \lambda_0)$.

Now we prove that (MQVIP$^{\gamma_0\lambda_0}$) has a unique solution. If $\Psi_s(\gamma_0, \lambda_0)$ has two distinct solutions x_1 and x_2, it is not hard to see that $x_1, x_2 \in \Sigma_s^{\gamma_0\lambda_0}(\xi, \varepsilon)$, $\forall \xi > 0$, $\varepsilon \in \operatorname{int} C$. It follows that

$$0 < d(x_1, x_2) \le \Sigma_s^{\gamma_0\lambda_0}(\xi, \varepsilon) \to 0,$$

which is impossible. Hence, (MQVIP) is well-posed at (γ_0, λ_0). □

The following example shows that the uniqueness of well-posed is essential.

Example 2.14 Let $X = Y = \mathbb{R}$, $A = B = [-1,1]$, $\Gamma = \Lambda = [0,1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, H be an identity map, and let $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x, \gamma) = K_2(x, \gamma) = [0,1],$$

$$\eta(y, x, \gamma) = \gamma + 1 - \varepsilon,$$

$$T(y, \gamma) = \{1\}.$$

We show that the conditions (i) and (ii) of Theorem 2.13 are easily seen to be fulfilled and the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is well-posed at $(0,0)$. But $\operatorname{diam} \Sigma_\alpha^{\gamma_0\lambda_0}(\xi, \varepsilon) = [0,1] \nrightarrow 0$ as $(\xi, \varepsilon) \to (0,0)$.

The following example shows that all assumptions of Theorem 2.13 are satisfied.

Example 2.15 Let $A = B = X = Y = \mathbb{R}$, $\Gamma = \Lambda = [0,1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, H be an identity map, and let $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x, \gamma) = [0, +\infty),$$

$$\eta(y, x, \gamma) = y - x + \gamma,$$

$$T(y, \gamma) = \{1\},$$

$$K_2(x, \gamma) = [0,1].$$

We show that the conditions (i) and (ii) of Theorem 2.13 are easily seen to be fulfilled and $\Psi_\alpha(0,0) = \{0\}$ and

$$\Omega_\alpha(\gamma,\lambda,\varepsilon) = \begin{cases} [0,\varepsilon], & \text{if } \gamma = 0, \\ [0,\gamma+\varepsilon], & \text{if } \gamma \in (0,1] \end{cases}$$

and $\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon) = [0,\varepsilon]$ and the family $\{(\mathrm{MQVIP}^{\gamma\lambda}) : (\gamma,\lambda) \in \Gamma \times \Lambda\}$ is well-posed at $(0,0)$, and $\operatorname{diam} \Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon) \to 0$ as $(\xi,\varepsilon) \to (0,0)$.

Next, we consider the following notions of measures of noncompactness.

Definition 2.16 ([37, 38]) Let X is complete. The Kuratowski measure of the set $A \subseteq X$ is defined by

$$\zeta(A) = \inf\left\{\vartheta > 0 \,\middle|\, A \subseteq \bigcup_{i=1}^{n} L_i, \operatorname{diam} L_i < \vartheta, i = 1,2,\ldots,n, \text{for some } n \in \mathbb{N}\right\}.$$

Definition 2.17 ([37, 38]) A, B be nonempty subsets of X. The Hausdorff metric $H(\cdot,\cdot)$ between A and B is defined by

$$H(A,B) = \max\{H^*(A,B), H^*(B,A)\},$$

where $H^*(A,B) = \sup_{a\in A} d(a,B)$ with $d(a,B) = \inf_{b\in B} \|a-b\|$.

By the definitions of ζ and H, we have

$$\zeta(A) \le \zeta(B) + 2H(A,B),$$

for every all bounded sets A and B.

Remark 2.18 ([39, 40]) The function ζ is a regular measure of noncompactness defined by $\zeta : 2^X \to [0,+\infty]$ that satisfies the following conditions:
- (i) $\zeta(D) = +\infty$ if and only if the set D is unbounded;
- (ii) $\zeta(D) = \zeta(\mathrm{cl}(D))$;
- (iii) from $\zeta(D) = 0$ it follows that D is a totally bounded set;
- (iv) from $P \subseteq Q$ it follows that $\zeta(P) \le \zeta(Q)$;
- (v) if X is a complete space, and if $\{B_n\}$ is a sequence of closed subsets of X such that $B_{n+1} \subseteq B_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \zeta(B_n) = 0$, then $M = \bigcap_{n\in N} B_n$ is a nonempty compact set and $\lim_{n\to+\infty} H(B_n, M) = 0$, where H is a Hausdorff metric.

Lemma 2.19 *Assume we have problem* (MQVIP). *Let* Γ, Λ *be finite dimensional and the following conditions hold*:
- (i) K_1 *is closed in* $A \times \{\gamma_0\}$, *and in* $K_1(A,\Gamma) \times \{\gamma_0\}$, K_2 *is lsc*;
- (ii) *in* $K_2(K_1(A,\Gamma),\Gamma) \times \{\gamma_0\}$, T *is usc and compact-valued if* $\alpha = \mathrm{w}$ (*or* $\alpha = \mathrm{m}$), *and lsc if* $\alpha = \mathrm{s}$.

Then $\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon)$ *is closed*, *for all* $\xi > 0$, $\varepsilon \in \operatorname{int} C$.

Proof Similar arguments can be applied in the three cases. We present only the proof for the case where $\alpha = \mathrm{s}$. We let $x_n \in \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)$ such that $x_n \to x$. Hence, for all $n \in \mathbb{N}$, there exist $\gamma_n \in B(\gamma_0,\xi)$ and $\lambda_n \in B(\lambda_0,\xi)$ and $\forall y \in K_2(x_n,\gamma_n)$, $\forall z \in T(y,\gamma_n)$ such that

$$\bigl\langle Q(z),\eta(y,x_n,\lambda_n)\bigr\rangle + \varepsilon \not< 0, \quad \forall n \in \mathbb{N}.$$

Since $B(\gamma_0,\xi)$ and $B(\lambda_0,\xi)$ are compact, we can assume that $\{\gamma_n\} \to \gamma \in B(\gamma_0,\xi)$ and $\{\lambda_n\} \to \lambda \in B(\lambda_0,\xi)$. By the closedness of K_1 at (x,γ), we find that $x \in K_1(x,\gamma)$. We show that $\forall y \in K_2(x,\gamma)$, $\forall z \in T(y,\gamma)$ such that

$$\bigl\langle Q(z),\eta(y,x,\lambda)\bigr\rangle + \varepsilon \not< 0,$$

i.e., $x \in \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)$. Indeed, if $x \notin \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)$, then $\exists y \in K_2(x,\gamma)$, $\exists z \in T(y,\gamma)$ such that

$$\bigl\langle Q(z),\eta(y,x,\lambda)\bigr\rangle + \varepsilon < 0.$$

By the lower semicontinuity of K_2 and T, there exist $y_n \in K_2(x_n,\gamma_n)$, $z_n \in T(y_n,\gamma_n)$ such that $\{y_n\} \to y$, $\{z_n\} \to z$, for all n. As $x_n \in \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)$ we have

$$\bigl\langle Q(z_n),\eta(y_n,x_n,\lambda_n)\bigr\rangle + \varepsilon \not< 0.$$

By the continuity of Q, η, $\langle\cdot,\cdot\rangle$ and id, it follows that $\langle\cdot,\cdot\rangle$ + id is continuous. So we have

$$\bigl\langle Q(z),\eta(y,x,\lambda)\bigr\rangle + \varepsilon \not< 0,$$

and we see a contradiction. Hence $x \in \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)$. Thus, $\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)$ is closed. □

Next, we provide sufficient conditions for the two sets to coincide.

Lemma 2.20 *Assume for problem* (MQVIP) *the following conditions to hold*:
(i) $K_1(x,\cdot)$ *is closed at* γ_0, *and* $K_2(x,\cdot)$ *is lsc at* γ_0;
(ii) *in* $K_2(K_1(A,\Gamma),\Gamma) \times \{\gamma_0\}$, T *is usc and compact-valued if* $\alpha = \mathrm{w}$ (*or* $\alpha = \mathrm{m}$), *and lsc if* $\alpha = \mathrm{s}$.
Then $\Psi_\alpha(\gamma_0,\lambda_0) = \bigcap_{\varepsilon\in \operatorname{int} C,\xi>0} \Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon)$, *for every* $(\gamma_0,\lambda_0) \in \Gamma \times \Lambda$.

Proof We present only the proof for the case where $\alpha = \mathrm{s}$. We first prove that $\bigcap_{\varepsilon\in \operatorname{int} C} \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon) = \Omega_s(\gamma_0,\lambda_0,\varepsilon)$. It is easy to see that $\bigcap_{\varepsilon\in \operatorname{int} C} \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon) \supseteq \Omega_s(\gamma_0,\lambda_0,\varepsilon)$. Thus, we only need to show that $\bigcap_{\varepsilon\in \operatorname{int} C} \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon) \subseteq \Omega_s(\gamma_0,\lambda_0,\varepsilon)$. Indeed, let $x \in \bigcap_{\varepsilon\in \operatorname{int} C} \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)$, there are $\gamma_n \in B(\gamma_0,\xi)$ and $\lambda_n \in B(\lambda_0,\xi)$ such that $\forall y \in K_2(x,\gamma_n)$, $\forall z \in T(y,\gamma_n)$ satisfying

$$\bigl\langle Q(z),\eta(y,x,\lambda_n)\bigr\rangle + \varepsilon \not< 0.$$

Since $x \in K_1(x,\gamma_n)$, $\gamma_n \to \gamma_0$ and K_1 is closed, we have $x \in K_1(x,\gamma_0)$. Now we verify that $x \in \Omega(\gamma_0,\lambda_0,\varepsilon)$. Indeed, for each $y \in K_2(x,\gamma_0)$, by the semicontinuity of $K_2(x,\cdot)$ at γ_0 and the semicontinuity of T at (y,γ_0), there exist $y_n \in K_2(x,\gamma_n)$ and $z_n \in T(y_n,\gamma_n)$ such that $\{y_n\} \to y$, $\{z_n\} \to z$. As $x \in \Omega_s(\gamma_n,\lambda_n,\varepsilon)$, we have

$$\bigl\langle Q(z_n),\eta(y_n,x,\lambda_n)\bigr\rangle + \varepsilon \not< 0.$$

By the continuity of Q, η, $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle$ + id, and $(y_n, z_n, \gamma_n, \lambda_n) \to (y, z, \gamma_0, \lambda_0)$. We have

$$\big\langle Q(z), \eta(y, x, \lambda_0)\big\rangle + \varepsilon \not< 0,$$

i.e.,

$$\bigcap_{\varepsilon\in\operatorname{int} C} \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon) \subseteq \Omega_s(\gamma_0,\lambda_0,\varepsilon).$$

Hence

$$\bigcap_{\varepsilon\in\operatorname{int} C} \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon) = \Omega_s(\gamma_0,\lambda_0,\varepsilon).$$

It is clear that

$$\Psi_s(\gamma_0,\lambda_0) = \bigcap_{\xi>0} \Omega_s(\gamma_0,\lambda_0,\varepsilon) = \bigcap_{\xi>0,\varepsilon\in\operatorname{int} C} \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon).$$

□

The following theorem shows the well-posedness in the generalized sense at (γ_0, λ_0) for (MQVIP) by using the Kuratowski measure ζ.

Theorem 2.21 *Let X be complete, Γ, Λ be finite dimensional and the following conditions hold*:

(i) *K_1 is closed in $A \times \{\gamma_0\}$, and in $K_1(A,\Gamma) \times \{\gamma_0\}$, K_2 is lsc;*

(ii) *in $K_2(K_1(A,\Gamma),\Gamma) \times \{\gamma_0\}$, T is usc and compact-valued if α = w (or α = m), and lsc if α = s.*

Then (MQVIP) *is well-posed in the generalized sense at (γ_0, λ_0) if and only if*

$$\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon) \neq \emptyset, \quad \forall\xi > 0, \varepsilon \in \operatorname{int} C \quad \textit{and} \quad \zeta\big(\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon)\big) \to 0 \quad \textit{as } (\xi,\varepsilon) \to (0,0).$$

Proof Similar arguments can be applied in the three cases. We present only the proof for the case where α = s. Now we suppose that (MQVIP) is well-posed in the generalized sense at (γ_0, λ_0). Let Ψ_s be a solution set of (MQVIP$^{\gamma\lambda}$) for all $(\gamma, \lambda) \in \Gamma \times \Lambda$. Then, from Theorem 2.6, we see that $\Psi_s(\gamma_0, \lambda_0)$ is a nonempty compact. Clearly $\Psi_s(\gamma_0, \lambda_0) \subseteq \Sigma_s^{\gamma_0\lambda_0}(\xi, \varepsilon)$, $\forall\xi > 0$, $\varepsilon \in \operatorname{int} C$. Now we show that

$$\zeta\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)\big) \to 0 \quad \text{as } (\xi,\varepsilon) \to (0,0).$$

Indeed, since $\Psi_s(\gamma_0, \lambda_0) \subseteq \Sigma_s^{\gamma_0\lambda_0}(\xi, \varepsilon)$, $\forall\xi > 0$, $\varepsilon \in \operatorname{int} C$. Using the concept of Hausdorff metric, we have

$$\begin{aligned} &H\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon), \Psi_s(\gamma_0,\lambda_0)\big) \\ &\quad = \max\big\{H^*\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon), \Psi_s(\gamma_0,\lambda_0)\big), H^*\big(\Psi_s(\gamma_0,\lambda_0), \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)\big)\big\} \\ &\quad = H^*\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon), \Psi_s(\gamma_0,\lambda_0)\big). \end{aligned}$$

Suppose that $\Psi_s(\gamma_0, \lambda_0) \subseteq \bigcup_{i=1}^n L_i$, $\operatorname{diam} L_i < \vartheta$, $i = 1, 2, \dots, n$, for some $n \in \mathbb{N}$.

We set $\Delta_i = \{t \in A | d(t, L_i) \leq H(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon), \Psi_s(\gamma_0,\lambda_0))\}$.

We claim that $\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon) \subseteq \bigcup_{i=1}^n \Delta_i$. Indeed, let $x \in \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)$. Then $d(x, \Psi_s(\gamma_0,\lambda_0)) \leq H(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon), \Psi_s(\gamma_0,\lambda_0))$. Since $\Psi_s(\gamma_0,\lambda_0) \subseteq \bigcup_{i=1}^n L_i$, we see that

$$d\left(x, \bigcup_{i=1}^n L_i\right) \leq H\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon), \Psi_s(\gamma_0,\lambda_0)\big).$$

Hence, there is k such that $d(x, L_k) \leq H(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon), \Psi_s(\gamma_0,\lambda_0))$, *i.e.*, $x \in \Delta_k$. So

$$\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon) \subseteq \bigcup_{i=1}^n \Delta_i.$$

Note further that

$$\begin{aligned}\operatorname{diam}\Delta_i &= \operatorname{diam} L_i + 2H\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon), \Psi_s(\gamma_0,\lambda_0)\big)\\ &\leq \vartheta + 2H\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon), \Psi_s(\gamma_0,\lambda_0)\big).\end{aligned}$$

Hence,

$$\zeta\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)\big) \leq 2H\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon), \Psi_s(\gamma_0,\lambda_0)\big) + \zeta\big(\Psi_s(\gamma_0,\lambda_0)\big).$$

Since $\Psi_s(\gamma_0,\lambda_0)$ is compact, $\zeta(\Psi_s(\gamma_0,\lambda_0)) = 0$, so we have

$$\zeta\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)\big) \leq 2H^*\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon), \Psi_s(\gamma_0,\lambda_0)\big).$$

Now we prove that

$$H^*\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)\big) \to 0 \quad \text{as } (\xi,\varepsilon) \to (0,0).$$

Suppose to the contrary that

$$H^*\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)\big) \nrightarrow 0 \quad \text{as } (\xi,\varepsilon) \to (0,0).$$

There are $\theta > 0$, $(\xi_n,\varepsilon_n) \to (0,0)$, and $x_n \in \Sigma_s^{\gamma_0\lambda_0}(\xi_n,\varepsilon_n)$ such that

$$d\big(x_n, \Psi_s(\gamma_0,\lambda_0)\big) \geq \theta > 0, \quad \forall n \in \mathbb{N}.$$

$\{x_n\}$ is an approximating sequence of (MQVIP). By the well-posedness in the generalized sense of (MQVIP) at (γ_0,λ_0), there is a subsequence $\{x_k\}$ of $\{x_n\}$ converging to some point of $\Psi_s(\gamma_0,\lambda_0)$, which is impossible as $d(x_n, \Psi_s(\gamma_0,\lambda_0)) \geq \theta > 0$, $\forall n \in \mathbb{N}$. Hence

$$\zeta\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)\big) \to 0 \quad \text{as } (\xi,\varepsilon) \to (0,0).$$

Conversely, $\zeta(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)) \to 0$ as $(\xi,\varepsilon) \to (0,0)$. By Lemma 2.19, we see that $\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)$ is closed, for all $\xi > 0$, $\varepsilon \in \operatorname{int} C$. By Lemma 2.20, we have

$$\Psi_s(\gamma_0,\lambda_0) = \bigcap_{\varepsilon \in \operatorname{int} C, \xi > 0} \Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon).$$

Since $\zeta(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)) \to 0$ as $(\xi,\varepsilon) \to (0,0)$, the regular measure properties of ζ imply that $\Psi_s(\gamma_0,\lambda_0)$ is compact and

$$H\big(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon),\Psi_s(\gamma_0,\lambda_0)\big) \to 0 \quad \text{as } (\xi,\varepsilon) \to (0,0).$$

Let $\{x_n\}$ be an approximating sequence for (MQVIP) corresponding to $\{(\gamma_n,\lambda_n)\}$, where $\{\gamma_n\} \to \gamma_0$ and $\{\lambda_n\} \to \lambda_0$. There is $\{\varepsilon_n\} \to 0$ such that $\forall y \in K_2(x_n,\gamma_n)$, $\forall z \in T(y,\gamma_n)$ satisfying

$$\big\langle Q(z),\eta(y,x_n,\lambda_n)\big\rangle + \varepsilon_n \not< 0, \quad \forall n \in \mathbb{N}.$$

This means that $x_n \in \Sigma_s^{\gamma_0\lambda_0}(\xi_n,\varepsilon_n)$ with $\xi_n := \max\{d(\gamma_0,\gamma_n),d(\lambda_0,\lambda_n)\}$. We see that

$$d\big(x_n,\Psi_s(\gamma_0,\lambda_0)\big) \leq H\big(\Sigma_s^{\gamma_0\lambda_0}(\xi_n,\varepsilon_n),\Psi_s(\gamma_0,\lambda_0)\big) \to 0 \quad \text{as } n \to +\infty.$$

Hence, there is $\bar{x}_n \in \Psi_s(\gamma_0,\lambda_0)$ such that

$$d(x_n,\bar{x}_n) \to 0 \quad \text{as } n \to +\infty.$$

By the compactness of $\Psi_s(\gamma_0,\lambda_0)$, there is a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ convergent to some point x_0 of $\Psi_s(\gamma_0,\lambda_0)$. Therefore, the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ tends to x_0. Hence, (MQVIP) is well-posed in the generalized sense at (γ_0,λ_0). □

Remark 2.22 In cases as in Remark 2.4(i), Theorems 3.3, 3.4, and 3.5-3.6 in [22] are particular cases of Theorems 2.13, 2.21, and 2.9, respectively. However, the assumptions and our proof methods are very different from Theorems 3.3, 3.4, and 3.5-3.6 in [22].

The following example shows that the closedness of K_1 in Theorem 2.21 cannot be dropped.

Example 2.23 Let $X = Y = \mathbb{R}$, $A = B = [-5,5]$, $\Gamma = \Lambda = [0,1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, H be an identity map, and let $K_1,K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x,\gamma) = [-5\gamma,5],$$
$$\eta(y,x,\gamma) = x(x-y),$$
$$T(y,\gamma) = \{1\},$$
$$K_2(x,\gamma) = [0,5].$$

We show that K_2 is lsc in $K_1(A,\Gamma) \times \Gamma$ and the condition (ii) of Theorem 2.21 is easily seen to be fulfilled and $\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon) \subseteq [-5,5]$. Hence, $\zeta(\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon)) \to 0$ as $(\xi,\varepsilon) \to (0,0)$. But the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma,\lambda) \in \Gamma \times \Lambda\}$ is not well-posed in the generalized sense at $(0,0)$. The reason is that K_1 is not closed at $(A,0)$. Indeed, we let $\gamma_n = x_n = \frac{1}{n} \to 0$, as $n \to \infty$ and $t_n = \frac{1}{n} \in K_1(x_n,\gamma_n) = (-\frac{5}{n},5]$, $\forall n \in \mathbb{N}$. It is clear that $\{t_n\}$ is convergent to $0 \notin K_1(0,0) = (0,5]$. In fact, $\Omega_\alpha(\gamma_0,\lambda_0,\varepsilon) = \Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon) = \{5\}$.

The following example shows that the lower semicontinuity of K_2 in Theorem 2.21 is essential.

Example 2.24 Let $X = Y = \mathbb{R}$, $A = B = [-2,2]$, $\Gamma = \Lambda = [0,1]$, $C = \mathbb{R}_+$, $\varepsilon \in \operatorname{int} C$, $\xi > 0$, $\gamma_0 = 0$, H be an identity map, and let $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x,\gamma) = [0,2],$$
$$\eta(y,x,\gamma) = x + y,$$
$$T(y,\gamma) = \{1\},$$
$$K_2(x,\gamma) = \begin{cases} \{-2,0,2\}, & \text{if } \gamma = 0, \\ \{0,2\}, & \text{otherwise.} \end{cases}$$

We show that K_2 is lsc in $K_1(A,\Gamma) \times \Gamma$ and the condition (ii) of Theorem 2.21 is easily seen to be fulfilled and $\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon) \subseteq [-2,2]$. Hence, $\zeta(\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon)) \to 0$ as $(\xi,\varepsilon) \to (0,0)$. But the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma,\lambda) \in \Gamma \times \Lambda\}$ is not well-posed in the generalized sense at $(0,0)$. The reason is that K_2 is not lower semicontinuous. In fact

$$\Omega_\alpha(\gamma,\lambda,\varepsilon) = \Sigma_\alpha^{\gamma\lambda}(\xi,\varepsilon) = \begin{cases} [2-\varepsilon,2] \cap [0,2], & \text{if } \gamma = 0, \\ [0,2], & \text{if } \gamma \in (0,1]. \end{cases}$$

The following example shows that all assumptions of Theorem 2.21 are fulfilled.

Example 2.25 Let $X = Y = \mathbb{R}$, $A = B = \Gamma = \Lambda = [0,2]$, $C = \mathbb{R}_+$, $\varepsilon \in \operatorname{int} C$, $\xi > 0$, $\gamma_0 = 0$, H be an identity map, and let $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x,\gamma) = K_2(x,\gamma) = [\gamma, \gamma + 2],$$
$$\eta(y,x,\gamma) = \{2^{\gamma^2+1} - \varepsilon\},$$
$$T(y,\gamma) = \left\{\frac{1}{2^{\gamma^2+1}}\right\}.$$

We show that the assumptions (i) and (ii) of Theorem 2.21 are easily seen to be fulfilled and

$$\Omega_\alpha(\gamma,\lambda,\varepsilon) = \begin{cases} [\gamma,\gamma+2], & \text{if } \gamma \in (0,1], \\ [0,2], & \text{if } \gamma = 0, \end{cases}$$

and $\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon) \subseteq [0,2]$. Hence, $\zeta(\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon)) \to 0$ as $(\xi,\varepsilon) \to (0,0)$, and the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma,\lambda) \in \Gamma \times \Lambda\}$ is well-posed in the generalized sense at $(0,0)$.

Competing interests
The author declares that they have no competing interests.

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