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Common solutions of equilibrium and fixed point problems

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Abstract
In this paper, common solutions of equilibrium and fixed point problems are investigated. Convergence theorems of common solutions are established in a uniformly smooth and strictly convex Banach space.
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1 Introduction and preliminaries

Let E be a real Banach space. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. E is said to be smooth iff $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. It is also said to be uniformly smooth iff the above limit is attained uniformly for $x, y \in U_E$. E is said to be strictly convex iff $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex iff $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$.

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that E is (uniformly) smooth if and only if E^* is (uniformly) convex.

In what follows, we use $\rightharpoonup$ and $\to$ to stand for the weak and strong convergence, respectively. Recall that E enjoys the Kadec-Klee property iff for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property.

Let E be a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \forall x, y \in E.$$

Observe that, in a Hilbert space H, the equality is reduced to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$. As we all know, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces.

In this connection, Alber [1] recently introduced a generalized projection operator Π_C in a Banach space E, which is an analogue of the metric projection P_C in Hilbert spaces. Recall that the generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem $\phi(\bar{x},x) = \min_{y\in C}\phi(y,x)$. Existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$; for more details, see [1] and the references therein. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of a function ϕ that

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x,y,z \in E, \tag{1.1}$$

and

$$\left(\|x\| - \|y\|\right)^2 \le \phi(x,y) \le \left(\|y\| + \|x\|\right)^2, \quad \forall x,y \in E. \tag{1.2}$$

Let C be a nonempty subset of E, and let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to stand for the fixed point set of T. T is said to be closed iff for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. T is said to be asymptotically regular on C iff for any bounded subset K of C,

$$\limsup_{n\to\infty}\left\{\left\| T^{n+1}x - T^n x\right\| : x \in K\right\} = 0.$$

Recall that a point p in C is said to be an asymptotic fixed point of T iff C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. T is said to be relatively nonexpansive iff

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi(p,Tx) \le \phi(p,x), \quad \forall x \in C, \forall p \in F(T).$$

T is said to be relatively asymptotically nonexpansive iff

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi\left(p,T^n x\right) \le (1+\mu_n)\phi(p,x), \quad \forall x \in C, \forall p \in F(T), \forall n \ge 1,$$

where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Remark 1.1 The class of relatively asymptotically nonexpansive mappings which is an extension of the class of relatively nonexpansive mappings was first considered in [2] and [3].

Recall that T is said to be quasi-ϕ-nonexpansive iff

$$F(T) \neq \emptyset, \qquad \phi(p,Tx) \le \phi(p,x), \quad \forall x \in C, \forall p \in F(T).$$

Recall that T is said to be asymptotically quasi-ϕ-nonexpansive iff there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$F(T) \neq \emptyset, \qquad \phi\left(p,T^n x\right) \le (1+\mu_n)\phi(p,x), \quad \forall x \in C, \forall p \in F(T), \forall n \ge 1.$$

Remark 1.2 The class of asymptotically quasi-ϕ-nonexpansive mappings, which is an extension of the class of quasi-ϕ-nonexpansive mappings, was considered in [4, 5]; see also [6].

Remark 1.3 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings do not require the restriction $F(T) = \widetilde{F}(T)$.

Remark 1.4 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are generalizations of the class of quasi-nonexpansive mappings and the class of asymptotically quasi-nonexpansive mappings in Banach spaces.

Recall that T is said to be generalized asymptotically quasi-ϕ-nonexpansive iff $F(T) \neq \emptyset$, and there exist two nonnegative sequences $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$, and $\{\xi_n\} \subset [0,\infty)$ with $\xi_n \to 0$ as $n \to \infty$ such that

$$\phi\big(p, T^n x\big) \le (1 + \mu_n)\phi(p,x) + \xi_n, \quad \forall x \in C, \forall p \in F(T), \forall n \ge 1.$$

Remark 1.5 The class of generalized asymptotically quasi-ϕ-nonexpansive mappings [7] is a generalization of the class of generalized asymptotically quasi-nonexpansive mappings in the framework of Banach spaces which was introduced by Agarwal *et al.* [8].

Let F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem. Find $p \in C$ such that $F(p,y) \ge 0$, $\forall y \in C$. We use $EP(F)$ to denote the solution set of the equilibrium problem. Given a mapping $Q : C \to E^*$, let

$$F(x,y) = \langle Qx, y - x\rangle, \quad \forall x, y \in C.$$

Then $p \in EP(F)$ if and only if p is a solution of the following variational inequality. Find p such that

$$\langle Qp, y - p\rangle \ge 0, \quad \forall y \in C.$$

Numerous problems in physics, optimization and economics reduce to finding a solution of the equilibrium problem; see [9–36] and the related references therein. In [25], Kim studied a sequence $\{x_n\}$ which is generated in the following manner:

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T^n x_n), \\ u_n \in C \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n) + (k_n - 1)M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $M_n = \sup\{\phi(z,x_n) : z \in \mathcal{F}\}$ for each $n \geq 1$, $\{\alpha_n\}$ is a real sequence in $[0,1]$, $\{r_n\}$ is a real sequence in $[a,\infty)$, where a is some positive real number. In a uniformly smooth and strictly convex Banach space, which also enjoys the Kadec-Klee property, the author obtained a strong convergence theorem; for more details, see [25] and the references therein.

In this paper, motivated by the above result, we consider the projection algorithm for treating solutions of the equilibrium problem and fixed points of generalized asymptotically quasi-ϕ-nonexpansive mappings. A strong convergence theorem is established in a Banach space. The results presented this paper mainly improve the corresponding results announced in Qin Cho and Kang [5] and Kim [25].

In order to prove our main results, we need the following lemmas.

Lemma 1.6 [36] *Let E be a smooth and uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0,2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|)$$

for all $x,y \in B_r = \{x \in E : \|x\| \leq r\}$ and $t \in [0,1]$.

Lemma 1.7 [1] *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0 \quad \forall y \in C.$$

Lemma 1.8 [1] *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y,x) \quad \forall y \in C.$$

Lemma 1.9 [5, 22] *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that $F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0$, $\forall y \in C$. Define a mapping $T_r : E \to C$ by*

$$S_r x = \left\{z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle, \forall y \in C\right\}.$$

Then the following conclusions hold:

(1) *S_r is a single-valued firmly nonexpansive-type mapping, i.e., for all $x,y \in E$,*

$$\langle S_r x - S_r y, JS_r x - JS_r y\rangle \leq \langle S_r x - S_r y, Jx - Jy\rangle;$$

(2) *$F(S_r) = EP(F)$ is closed and convex*;

(3) *S_r is quasi-ϕ-nonexpansive*;

(4) *$\phi(q, S_r x) + \phi(S_r x, x) \leq \phi(q,x)$, $\forall q \in F(S_r)$.*

Lemma 1.10 [7] *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and let C be a nonempty closed and convex subset of E. Let $T: C \to C$ be a generalized asymptotically quasi-ϕ-nonexpansive mapping. Then $F(T)$ is closed and convex.*

2 Main results

Theorem 2.1 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and let C be a nonempty closed and convex subset of E. Let Δ be an index set. Let F_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *for every $i \in \Delta$. Let $T: C \to C$ be a generalized asymptotically quasi-ϕ-nonexpansive mapping. Assume that T is closed asymptotically regular on C and $\Omega := F(T) \cap \bigcap_{i\in\Delta} EF(F_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E, \quad \textit{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \bigcap_{i\in\Delta} C_{1,i}, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n x_n), \\ u_{n,i} \in C \textit{ such that } F_i(u_{n,i}, y) + \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1,i} = \{z \in C_n : \phi(z, u_{n,i}) \leq \phi(z, x_n) + \mu_n M_n + \xi_n\}, \\ C_{n+1} = \bigcap_{i\in\Delta} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $M_n = \sup\{\phi(z, x_n) : z \in \Omega\}$, $\{\alpha_n\}$ is a real number sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\{r_{n,i}\}$ is a real number sequence in $[a_i, \infty)$, where $\{a_i\}$ is a positive real number sequence. Then the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection from E onto Ω.

Proof In view of Lemmas 1.9 and 1.10, we find that the common solution set Ω is closed and convex. Next, we show that C_n is closed and convex. It suffices to show, for any fixed but arbitrary $i \in \Delta$, that $C_{n,i}$ is closed and convex. This can be proved by induction on n. It is obvious that $C_{1,i} = C$ is closed and convex. Assume that $C_{j,i}$ is closed and convex for some $j \geq 1$. We next prove that $C_{j+1,i}$ is closed and convex for the same j. This completes the proof that C_n is closed and convex. It is clear that $C_{j+1,i}$ is closed. We only prove the convexity. Indeed, $\forall a, b \in C_{j+1,i}$, we see that $a, b \in C_{j,i}$, and

$$\phi(a, u_{j,i}) \leq \phi(a, x_j) + \mu_j M_j + \xi_j,$$

and

$$\phi(b, u_{j,i}) \leq \phi(b, x_j) + \mu_j M_j + \xi_j.$$

Notice that the two inequalities above are equivalent to the following inequalities, respectively:

$$2\langle a, Jx_j - Ju_{j,i}\rangle \leq \|x_j\|^2 - \|u_{j,i}\|^2 + \mu_j M_j + \xi_j,$$

and

$$2\langle b, Jx_j - Ju_{j,i}\rangle \leq \|x_j\|^2 - \|u_{j,i}\|^2 + \mu_j M_j + \xi_j.$$

These imply that

$$2\langle ta + (1-t)b, Jx_j - Ju_{j,i}\rangle \leq \|x_j\|^2 - \|u_{j,i}\|^2 + \mu_j M_j + \xi_j, \quad \forall t \in (0,1).$$

Since $C_{j,i}$ is convex, we see that $ta + (1-t)b \in C_{j,i}$. Notice that the above inequality is equivalent to

$$\phi\big(ta + (1-t)b, u_{j,i}\big) \leq \phi\big(ta + (1-t)b, x_j\big) + \mu_j M_j + \xi_j.$$

This proves that $C_{j+1,i}$ is convex. This completes that C_n is closed and convex.

Next, we prove that $\Omega \subset C_n$. It suffices to claim that $\Omega \subset C_{n,i}$ for every $i \in \Delta$. Note that $\Omega \subset C_{1,i} = C$. Suppose that $\Omega \subset C_{j,i}$ for some j and for every $i \in \Delta$. Then, for $\forall w \in \Omega \subset C_{j,i}$, we have

$$\begin{aligned}
&\phi(w, u_{j,i}) \\
&\quad = \phi(w, S_{r_{j,i}} y_j) \\
&\quad = \phi\big(w, J^{-1}\big(\alpha_j Jx_j + (1-\alpha_j)JT^j x_j\big)\big) \\
&\quad = \|w\|^2 - 2\langle w, \alpha_j Jx_j + (1-\alpha_j)JT^j x_j\rangle \\
&\qquad + \big\|\alpha_j Jx_j + (1-\alpha_j)JT^j x_j\big\|^2 \\
&\quad \leq \|w\|^2 - 2\alpha_j\langle w, Jx_j\rangle - 2(1-\alpha_j)\langle w, JT^j x_j\rangle + \alpha_j\|x_j\|^2 \\
&\qquad + (1-\alpha_j)\big\|T^j x_j\big\|^2 \\
&\quad = \alpha_j\phi(w, x_j) + (1-\alpha_j)\phi\big(w, T^j x_j\big) \\
&\quad \leq \alpha_j\phi(w, x_j) + (1-\alpha_j)(1+\mu_j)\phi(w, x_j) + \xi_j(1-\alpha_j) \\
&\quad \leq \phi(w, x_j) + \mu_j\phi(w, x_j) + \xi_j \\
&\quad \leq \phi(w, x_j) + \mu_j M_j + \xi_j.
\end{aligned}$$

This shows that $w \in C_{j+1,i}$. This implies that $\Omega \subset C_n$ for every $n \geq 1$.

On the other hand, it follows from Lemma 1.8 that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0), \quad \forall w \in \Omega \subset C_n.$$

This shows that the sequence $\phi(x_n, x_0)$ is bounded. In view of (1.2), we see that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may, without loss of generality, assume that $x_n \rightharpoonup p \in C_n$. Note that $\phi(x_n, x_0) \leq \phi(p, x_0)$. It follows that

$$\phi(p, x_0) \leq \liminf_{n\to\infty} \phi(x_n, x_0) \leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \phi(p, x_0).$$

This implies that

$$\lim_{n\to\infty}\phi(x_n,x_0)=\phi(p,x_0).$$

Hence, we have $\|x_n\|\to\|p\|$ as $n\to\infty$. In view of the Kadec-Klee property of E, we obtain that $x_n\to p$ as $n\to\infty$.

Next, we show that $p\in F(T)$. By the construction of C_n, we have that $C_{n+1}\subset C_n$ and $x_{n+1}=\Pi_{C_{n+1}}x_0\in C_n$. It follows that

$$\begin{aligned}\phi(x_{n+1},x_n)&=\phi(x_{n+1},\Pi_{C_n}x_0)\\&\le\phi(x_{n+1},x_0)-\phi(\Pi_{C_n}x_0,x_0)\\&=\phi(x_{n+1},x_0)-\phi(x_n,x_0).\end{aligned}$$

Letting $n\to\infty$, we obtain that $\phi(x_{n+1},x_n)\to 0$. In view of $x_{n+1}\in C_{n+1}$, we see that

$$\phi(x_{n+1},u_{n,i})\le\phi(x_{n+1},x_n)+\mu_nM_n+\xi_n.$$

It follows that

$$\lim_{n\to\infty}\phi(x_{n+1},u_{n,i})=0.$$

From (1.2), we see that $\lim_{n\to\infty}\|u_{n,i}\|=\|p\|$. It follows that $\lim_{n\to\infty}\|Ju_{n,i}\|=\|Jp\|$. This implies that $\{Ju_{n,i}\}$ is bounded. Note that E is reflexive and E^* is also reflexive. We may assume that $Ju_{n,i}\rightharpoonup x^{*,i}\in E^*$. In view of the reflexivity of E, we see that $J(E)=E^*$. This shows that there exists an $x^i\in E$ such that $Jx^i=x^{*,i}$. It follows that

$$\begin{aligned}\phi(x_{n+1},u_{n,i})&=\|x_{n+1}\|^2-2\langle x_{n+1},Ju_n\rangle+\|u_n\|^2\\&=\|x_{n+1}\|^2-2\langle x_{n+1},Ju_n\rangle+\|Ju_n\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields that

$$\begin{aligned}0&\ge\|p\|^2-2\langle p,x^{*,i}\rangle+\|x^{*,i}\|^2\\&=\|p\|^2-2\langle p,Jx^i\rangle+\|Jx^i\|^2\\&=\|p\|^2-2\langle p,Jx^i\rangle+\|x^i\|^2\\&=\phi(p,x^i).\end{aligned}$$

That is, $p=x^i$, which in turn implies that $x^{*,i}=Jp$. It follows that $Ju_{n,i}\rightharpoonup Jp\in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain that $Ju_{n,i}-Jp\to 0$ as $n\to\infty$. Note that $J^{-1}:E^*\to E$ is demi-continuous. It follows that $u_{n,i}\rightharpoonup p$. Since E enjoys the Kadec-Klee property, we obtain that $u_{n,i}\to p$ as $n\to\infty$. Note that

$$\|x_n-u_{n,i}\|\le\|x_n-p\|+\|p-u_{n,i}\|.$$

It follows that

$$\lim_{n\to\infty} \|x_n - u_{n,i}\| = 0. \tag{2.1}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_{n,i}\| = 0. \tag{2.2}$$

Let $r = \sup_{n\geq 0}\{\|x_n\|, \|T^n x_n\|\}$. Since E is uniformly smooth, we know that E^* is uniformly convex. In view of Lemma 1.6, we see that

$$\begin{aligned}
&\phi(w, u_{n,i}) \\
&\quad = \phi(w, S_{r_{n,i}} y_n) \\
&\quad = \phi\bigl(w, J^{-1}\bigl(\alpha_n Jx_n + (1-\alpha_j)JT^n x_n\bigr)\bigr) \\
&\quad = \|w\|^2 - 2\bigl\langle w, \alpha_n Jx_n + (1-\alpha_n)JT^n x_n\bigr\rangle \\
&\qquad + \bigl\|\alpha_n Jx_n + (1-\alpha_n)JT^n x_n\bigr\|^2 \\
&\quad \leq \|w\|^2 - 2\alpha_n\langle w, Jx_n\rangle - 2(1-\alpha_n)\bigl\langle w, JT^n x_n\bigr\rangle + \alpha_n\|x_n\|^2 \\
&\qquad + (1-\alpha_n)\bigl\|T^n x_n\bigr\|^2 - \alpha_n(1-\alpha_n)g\bigl(\bigl\|Jx_n - JT^n x_n\bigr\|\bigr) \\
&\quad = \alpha_n\phi(w, x_n) + (1-\alpha_n)\phi\bigl(w, T^n x_n\bigr) - \alpha_n(1-\alpha_n)g\bigl(\bigl\|Jx_n - JT^n x_n\bigr\|\bigr) \\
&\quad \leq \alpha_n\phi(w, x_n) + (1-\alpha_n)(1+\mu_n)\phi(w, x_n) + \xi_n(1-\alpha_n) \\
&\qquad - \alpha_n(1-\alpha_n)g\bigl(\bigl\|Jx_n - JT^n x_n\bigr\|\bigr) \\
&\quad \leq \phi(w, x_n) + \mu_n\phi(w, x_n) + \xi_n - \alpha_n(1-\alpha_n)g\bigl(\bigl\|Jx_n - JT^n x_n\bigr\|\bigr) \\
&\quad \leq \phi(w, x_n) + \mu_n M_n + \xi_n - \alpha_n(1-\alpha_n)g\bigl(\bigl\|Jx_n - JT^n x_n\bigr\|\bigr).
\end{aligned}$$

It follows that

$$\alpha_n(1-\alpha_n)g\bigl(\bigl\|Jx_n - JT^n x_n\bigr\|\bigr) \leq \phi(w, x_n) - \phi(w, u_{n,i}) + \mu_n M_n + \xi_n.$$

Notice that

$$\begin{aligned}
\phi(w, x_n) - \phi(w, u_{n,i}) &= \|x_n\|^2 - \|u_{n,i}\|^2 - 2\langle w, Jx_n - Ju_{n,i}\rangle \\
&\leq \|x_n - u_{n,i}\|\bigl(\|x_n\| + \|u_{n,i}\|\bigr) + 2\|w\|\|Jx_n - Ju_{n,i}\|.
\end{aligned}$$

It follows from (2.1) and (2.2) that $\phi(w, x_n) - \phi(w, u_{n,i}) \to 0$ as $n \to \infty$. In view of $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we see that $\lim_{n\to\infty} g(\|Jx_n - JT^n x_n\|) = 0$. It follows from the property of g that

$$\lim_{n\to\infty} \bigl\|Jx_n - JT^n x_n\bigr\| = 0. \tag{2.3}$$

Since $x_n \to p$ as $n \to \infty$ and $J : E \to E^*$ is demi-continuous, we obtain that $Jx_n \rightharpoonup Jp \in E^*$. Note that

$$\bigl|\|Jx_n\| - \|Jp\|\bigr| = \bigl|\|x_n\| - \|p\|\bigr| \leq \|x_n - p\|.$$

This implies that $\|Jx_n\| \to \|Jp\|$ as $n \to \infty$. Since E^* enjoys the Kadec-Klee property, we see that

$$\lim_{n\to\infty} \|Jx_n - Jp\| = 0. \tag{2.4}$$

Notice that

$$\|JT^n x_n - Jp\| \le \|JT^n x_n - Jx_n\| + \|Jx_n - Jp\|.$$

It follows from (2.3) and (2.4) that

$$\lim_{n\to\infty} \|JT^n x_n - Jp\| = 0. \tag{2.5}$$

Note that $J^{-1} : E^* \to E$ is demi-continuous. It follows that $T^n x_n \rightharpoonup p$. On the other hand, we have

$$\big| \|T^n x_n\| - \|p\| \big| = \big| \|JT^n x_n\| - \|Jp\| \big| \le \|JT^n x_n - Jp\|.$$

In view of (2.5), we obtain that $\|T^n x_n\| \to \|p\|$ as $n \to \infty$. Since E enjoys the Kadec-Klee property, we obtain that

$$\lim_{n\to\infty} \|T^n x_n - p\| = 0. \tag{2.6}$$

Note that

$$\|T^{n+1} x_n - p\| \le \|T^{n+1} x_n - T^n x_n\| + \|T^n x_n - p\|.$$

It follows from the asymptotic regularity of T and (2.6) that

$$\lim_{n\to\infty} \|T^{n+1} x_n - p\| = 0.$$

That is, $TT^n x_n - p \to 0$ as $n \to \infty$. It follows from the closedness of T that $Tp = p$.

Next, we show that $p \in \bigcap_{i\in\Delta} EF(F_i)$. Notice that $\phi(w, y_n) \le \phi(w, x_n) + \mu_n M_n + \xi_n$. In view of $u_{n,i} = S_{r_{n,i}} y_n$, we find from Lemma 1.8 that

$$\begin{aligned} \phi(u_{n,i}, y_n) &= \phi(S_{r_{n,i}} y_n, y_n) \\ &\le \phi(w, y_n) - \phi(w, S_{r_{n,i}} y_n) \\ &\le \phi(w, x_n) - \phi(w, S_{r_{n,i}} y_n) + \mu_n M_n + \xi_n \\ &= \phi(w, x_n) - \phi(w, u_{n,i}) + \mu_n M_n + \xi_n. \end{aligned}$$

This in turn implies that

$$\lim_{n\to\infty} \phi(u_{n,i}, y_n) = 0.$$

It follows from (1.2) that $\|u_{n,i}\| - \|y_n\| \to 0$ as $n \to \infty$. In view of $u_{n,i} \to p$ as $n \to \infty$, we arrive at $\lim_{n\to\infty} \|y_n\| = \|p\|$. It follows that $\lim_{n\to\infty} \|Jy_n\| = \|Jp\|$. Since E^* is reflexive, we may assume that $Jy_n \rightharpoonup f^* \in E^*$. In view of $J(E) = E^*$, we see that there exists $f \in E$ such that $Jf = f^*$. It follows that

$$\phi(u_{n,i}, y_n) = \|u_{n,i}\|^2 - 2\langle u_{n,i}, Jy_n\rangle + \|Jy_n\|^2.$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields that

$$\begin{aligned} 0 &\geq \|p\|^2 - 2\langle p, f^*\rangle + \|f^*\|^2 \\ &= \|p\|^2 - 2\langle p, Jf\rangle + \|Jf\|^2 \\ &= \|p\|^2 - 2\langle p, Jf\rangle + \|f\|^2 \\ &= \phi(p, f). \end{aligned}$$

That is, $p = f$, which in turn implies that $f^* = Jp$. It follows that $Jy_n \rightharpoonup Jp \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain that $Jy_n - Jp \to 0$ as $n \to \infty$. Note that $J^{-1} : E^* \to E$ is demi-continuous. It follows that $y_n \rightharpoonup p$. Since E enjoys the Kadec-Klee property, we obtain that $y_n \to p$ as $n \to \infty$. Notice that $\|u_{n,i} - y_n\| \leq \|u_{n,i} - p\| + \|p - y_n\|$. It follows that

$$\lim_{n\to\infty} \|u_{n,i} - y_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Ju_{n,i} - Jy_n\| = 0.$$

From the assumption $r_{n,i} \geq a_i$, we see that

$$\lim_{n\to\infty} \frac{\|Ju_{n,i} - Jy_n\|}{r_{n,i}} = 0.$$

Notice that

$$F_i(u_{n,i}, y) + \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from condition (A2) that

$$\|y - u_{n,i}\| \frac{\|Ju_{n,i} - Jy_n\|}{r_{n,i}} \geq \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy_n\rangle \geq F_i(y, u_{n,i}), \quad \forall y \in C.$$

By taking the limit as $n \to \infty$ in the above inequality, from condition (A4) we obtain that

$$F_i(y, p) \leq 0, \quad \forall y \in C.$$

For $0 < t_i < 1$ and $y \in C$, define $y_{t_i} = t_i y + (1 - t_i)p$. It follows that $y_{t,i} \in C$, which yields that $F_i(y_{t,i}, p) \leq 0$. It follows from conditions (A1) and (A4) that

$$0 = F_i(y_{t,i}, y_{t,i}) \leq t_i F_i(y_{t,i}, y) + (1 - t_i)F_i(y_{t,i}, p) \leq t_i F_i(y_{t,i}, y).$$

That is,

$$F_i(y_{t,i}, y) \geq 0.$$

Letting $t_i \downarrow 0$, we find from condition (A3) that $F_i(p, y) \geq 0$, $\forall y \in C$. This implies that $p \in EP(F_i)$. This completes the proof that $p \in \Omega$.

Finally, we prove that $p = \Pi_\Omega x_0$. From $x_n = \Pi_{C_n} x_0$, we see that

$$\langle x_n - z, Jx_0 - Jx_n \rangle \geq 0, \quad \forall z \in C_n.$$

In view of $\Omega \subset C_n$, we find that

$$\langle x_n - w, Jx_0 - Jx_n \rangle \geq 0, \quad \forall w \in \Omega.$$

Letting $n \to \infty$ in the above inequality, we see that

$$\langle p - w, Jx_0 - Jp \rangle \geq 0, \quad \forall w \in \Omega.$$

In view of Lemma 1.7, we can obtain that $p = \Pi_\Omega x_0$. This completes the proof. □

If T is asymptotically quasi-ϕ-nonexpansive, then we find from Theorem 2.1 the following result.

Corollary 2.2 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and let C be a nonempty closed and convex subset of E. Let Δ be an index set. Let F_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *for every $i \in \Delta$. Let $T : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping. Assume that T is closed asymptotically regular on C and $\Omega := F(T) \cap \bigcap_{i \in \Delta} EF(F_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E, \quad \textit{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \bigcap_{i \in \Delta} C_{1,i}, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T^n x_n), \\ u_{n,i} \in C \textit{ such that } F_i(u_{n,i}, y) + \frac{1}{r_{n,i}} \langle y - u_{n,i}, J u_{n,i} - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1,i} = \{z \in C_n : \phi(z, u_{n,i}) \leq \phi(z, x_n) + \mu_n M_n\}, \\ C_{n+1} = \bigcap_{i \in \Delta} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $M_n = \sup\{\phi(z, x_n) : z \in \Omega\}$, $\{\alpha_n\}$ is a real number sequence in $(0,1)$ *such that* $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$, *$\{r_{n,i}\}$ is a real number sequence in $[a_i, \infty)$, where $\{a_i\}$ is a positive real number sequence. Then the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection from E onto Ω.*

Remark 2.3 Since the index set Δ is arbitrary, Corollary 2.2 is an improvement of the corresponding results in Kim [25].

Remark 2.4 Corollary 2.2 also improves the corresponding results in Qin *et al.* [5] in the following aspects:

(a) from a uniformly smooth and uniformly convex space to a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property;
(b) from a single bifunction to a family of bifunctions;
(c) from a quasi-ϕ-nonexpansive mapping to an asymptotically quasi-ϕ-nonexpansive mapping.

In the framework of Hilbert spaces, the theorem is reduced to the following.

Corollary 2.5 *Let E be a Hilbert space, and let C be a nonempty closed and convex subset of E. Let Δ be an index set. Let F_i be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *for every $i \in \Delta$. Let $T : C \to C$ be a generalized asymptotically quasi-nonexpansive mapping. Assume that T is closed asymptotically regular on C and $\Omega := F(T) \cap \bigcap_{i\in\Delta} EF(F_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E, \quad \textit{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \bigcap_{i\in\Delta} C_{1,i}, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n) T^n x_n, \\ u_{n,i} \in C \textit{ such that } F_i(u_{n,i}, y) + \frac{1}{r_{n,i}} \langle y - u_{n,i}, u_{n,i} - y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1,i} = \{z \in C_n : \|z - u_{n,i}\|^2 \leq \|z - x_n\|^2 + \mu_n M_n + \xi_n\}, \\ C_{n+1} = \bigcap_{i\in\Delta} C_{n+1,i}, \\ x_{n+1} = \operatorname{Proj}_{C_{n+1}} x_0, \end{cases}$$

where $M_n = \sup\{\|z - x_n\|^2 : z \in \Omega\}$, $\{\alpha_n\}$ is a real number sequence in $(0,1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, *$\{r_{n,i}\}$ is a real number sequence in $[a_i,\infty)$, where $\{a_i\}$ is a positive real number sequence. Then the sequence $\{x_n\}$ converges strongly to* $\operatorname{Proj}_\Omega x_0$, *where* $\operatorname{Proj}_\Omega$ *is the metric projection from E onto Ω.*

Proof In the framework of Hilbert spaces, we find that $\phi(x,y) = \|x-y\|^2$, J is reduced to the identity mapping and the generalized projection Π_C is reduced to the metric projection Proj_C. This completes the proof. □

For a single bifunction, we also have the following.

Corollary 2.6 *Let E be a Hilbert space, and let C be a nonempty closed and convex subset of E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $T : C \to C$ be a generalized asymptotically quasi-nonexpansive mapping. Assume that T is closed asymptotically regular on C and $\Omega := F(T) \cap EF(F)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence* *generated in the following manner*:

$$\begin{cases} x_0 \in E, \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n) T^n x_n, \\ u_n \in C \textit{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|z - u_n\|^2 \leq \|z - x_n\|^2 + \mu_n M_n + \xi_n\}, \\ x_{n+1} = \operatorname{Proj}_{C_{n+1}} x_0, \end{cases}$$

where $M_n = \sup\{\|z - x_n\|^2 : z \in \Omega\}$, $\{\alpha_n\}$ *is a real number sequence in* $(0,1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\{r_{n,i}\}$ *is a real number sequence in* $[a, \infty)$, *where* a *is a positive real number. Then the sequence* $\{x_n\}$ *converges strongly to* $\operatorname{Proj}_\Omega x_0$, *where* $\operatorname{Proj}_\Omega$ *is the metric projection from* E *onto* Ω.

Proof In the framework of Hilbert spaces, we find that $\phi(x,y) = \|x - y\|^2$, J is reduced to the identity mapping, and the generalized projection Π_C is reduced to the metric projection Proj_C. In view of Corollary 2.5, we may immediately conclude the desired results. □

Competing interests
The author declares that he has no competing interests.

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