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Research Article

On a Nonlinear Wave Equation Associated with Dirichlet Conditions: Solvability and Asymptotic Expansion of Solutions in Many Small Parameters

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A Dirichlet problem for a nonlinear wave equation is investigated. Under suitable assumptions, we prove the solvability and the uniqueness of a weak solution of the above problem. On the other hand, a high-order asymptotic expansion of a weak solution in many small parameters is studied. Our approach is based on the Faedo-Galerkin method, the compact imbedding theorems, and the Taylor expansion of a function.

1. Introduction

In this paper, we consider the following Dirichlet problem:

$$u_{tt} - \frac{\partial}{\partial x}\left(\mu(x,t,u)u_x\right) = f(x,t,u,u_x,u_t), \quad 0 < x < 1,\ 0 < t < T, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x), \qquad u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where $\tilde{u}_0, \tilde{u}_1, \mu$, and f are given functions satisfying conditions specified later.

In the special cases, when the function $\mu(x,t,u)$ is independent of $u, \mu(x,t,u) \equiv 1$, or $\mu(x,t,u) = \mu(x,t)$, and the nonlinear term f has the simple forms, the problem (1.1), with various initial-boundary conditions, has been studied by many authors, for example, Ortiz and Dinh [1], Dinh and Long [2, 3], Long and Diem [4], Long et al. [5], Long and Truong [6, 7], Long et al. [8], Ngoc et al. [9], and the references therein.

Ficken and Fleishman [10] and Rabinowitz [11] studied the periodic-Dirichlet problem for hyperbolic equations containing a small parameter ε, in particular, the differential equation

$$u_{tt} - u_{xx} = 2\alpha u_t + \varepsilon f(t,x,u,u_t,u_x). \tag{1.4}$$

In [12], Kiguradze has established the existence and uniqueness of a classical solution $u \in C^2([0,a] \times \mathbb{R}^n)$ of the periodic-Dirichlet problem for the following nonlinear wave equation:

$$u_{tt} - u_{xx} = g(t,x,u) + g_1(u)u_t, \tag{1.5}$$

under the assumption that g and g_1 are continuously differentiable functions (these conditions are sharp and cannot be weakened). Moreover, it is shown that the same results are valid for the equation

$$u_{tt} - u_{xx} = g(t,x,u) + g_1(u)u_t + \varepsilon q(t,x,u,u_t,u_x), \tag{1.6}$$

with sufficiently small ε and continuously differentiable q.

In [13], a unified approach to the previous cases was presented discussing the existence unique and asymptotic stability of classical solutions for a class of nonlinear continuous dynamical systems.

In [8], Long et al. have studied the linear recursive schemes and asymptotic expansion for the nonlinear wave equation

$$u_{tt} - u_{xx} = f(x,t,u,u_x,u_t) + \varepsilon f_1(x,t,u,u_x,u_t), \tag{1.7}$$

with the mixed nonhomogeneous conditions

$$u_x(0,t) - h_0 u(0,t) = g_0(t), \qquad u(1,t) = g_1(t). \tag{1.8}$$

In the case of $g_0, g_1 \in C^3(\mathbb{R}_+)$, $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, $f_1 \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, and some other conditions, an asymptotic expansion of the weak solution u_ε of order $N+1$ in ε is considered.

This paper consists of four sections. In Section 2, we present some preliminaries. Using the Faedo-Galerkin method and the compact imbedding theorems, in Section 3, we prove the solvability and the uniqueness of a weak solution of the problem (1.1)–(1.3). In Section 4, based on the ideals and the techniques used in the above-mentioned papers, we study a high-order asymptotic expansion of a weak solution for the problem (1.1)–(1.3), where (1.1) has the form of a linear wave equation with nonlinear perturbations containing many small parameters. In order to avoid making the treatment too complicated without losing of generality, at first, an asymptotic expansion of a weak solution $u = u_{\varepsilon_1,\varepsilon_2}(x,t)$ of order $N+1$ in two small parameters $\varepsilon_1, \varepsilon_2$ for the following equation:

$$u_{tt} - \frac{\partial}{\partial x}\left(\left[\mu_0(x,t) + \varepsilon_1 \mu_1(x,t,u)\right]u_x\right) = f_0(x,t) + \varepsilon_2 f_1(x,t,u,u_x,u_t), \tag{1.9}$$

associated with (1.2), (1.3), with $\mu_0 \in C^2([0,1]\times\mathbb{R}_+)$, $\mu_1 \in C^{N+1}([0,1]\times\mathbb{R}_+\times\mathbb{R})$, $\mu_0(x,t) \geq \mu_* > 0$, $\mu_1(x,t,z) \geq 0$, for all $(x,t,z) \in [0,1]\times\mathbb{R}_+\times\mathbb{R}$, $f_0 \in C^1([0,1]\times\mathbb{R}_+)$, and $f_1 \in C^N([0,1]\times\mathbb{R}_+\times\mathbb{R}^3)$ is established. Next, we note that the same results are valid for the equation in p small parameters $\varepsilon_1,\ldots,\varepsilon_p$ as follows

$$u_{tt} - \frac{\partial}{\partial x}\left[\left(\mu_0(x,t) + \sum_{i=1}^{p}\varepsilon_i \mu_i(x,t,u)\right)u_x\right] = f_0(x,t) + \sum_{i=1}^{p}\varepsilon_i f_i(x,t,u,u_x,u_t), \tag{1.10}$$

associated with (1.2), (1.3). The result obtained here is a relative generalization of [5–7, 14], where asymptotic expansion of a weak solution in two or three small parameters is given.

2. Preliminaries

Put $\Omega = (0,1)$. Let us omit the definitions of usual function spaces that will be used in what follows such as $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$, $H_0^m = H_0^m(\Omega)$. The norm in L^2 is denoted by $\|\cdot\|$. We denote by $\langle\cdot,\cdot\rangle$ the scalar product in L^2 or a pair of dual products of continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X. We denote $L^p(0,T;X)$, $1 \leq p \leq \infty$, the Banach space of real functions $u : (0,T) \to X$ measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X, & \text{if } p = \infty. \end{cases} \tag{2.1}$$

Let $u(t), u'(t) = u_t(t) = \dot{u}(t), u''(t) = u_{tt}(t) = \ddot{u}(t), u_x(t) = \nabla u(t), u_{xx}(t) = \Delta u(t)$ denote $u(x,t), \partial u/\partial t(x,t), \partial^2 u/\partial t^2(x,t), \partial u/\partial x(x,t), \partial^2 u/\partial x^2(x,t)$, respectively. With $f \in C^k([0,1]\times\mathbb{R}_+\times\mathbb{R}^3)$, $f = f(x,t,u,v,w)$, we put $D_1 f = \partial f/\partial x, D_2 f = \partial f/\partial t, D_3 f = \partial f/\partial u, D_4 f = \partial f/\partial v, D_5 f = \partial f/\partial w$ and $D^\alpha f = D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}D_4^{\alpha_4}D_5^{\alpha_5} f$; $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \in \mathbb{Z}_+^5$, $|\alpha| = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 = k$, $D^{(0,0,\ldots,0)} f = f$.

Similarly, with $\mu \in C^k([0,1]\times\mathbb{R}_+\times\mathbb{R})$, $\mu = \mu(x,t,z)$, we put $D_1\mu = \partial\mu/\partial x, D_2\mu = \partial\mu/\partial t, D_3\mu = \partial\mu/\partial z$ and $D^\beta \mu = D_1^{\beta_1}D_2^{\beta_2}D_3^{\beta_3}$, $\beta = (\beta_1,\beta_2,\beta_3) \in \mathbb{Z}_+^3$, $|\beta| = \beta_1+\beta_2+\beta_3 = k$.

On H^1, we will use the following norms:

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \tag{2.2}$$

Then, we have the following lemma.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\|v\|_{H^1} \quad \forall v \in H^1. \tag{2.3}$$

The proof of Lemma 2.1 is easy, hence we omit the details.

Remark 2.2. On $H_0^1, v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v_x\|$ are two equivalent norms. Furthermore, we have the following inequalities:

$$\|v\|_{C^0(\overline{\Omega})} \leq \|v_x\| \quad \forall v \in H_0^1. \tag{2.4}$$

Remark 2.3. (i) Let us note more that a unique weak solution u of the problem (1.1)–(1.3) will be obtained in Section 3 (Theorem 3.2) in the following manner.

Find $u \in \widetilde{W} = \{u \in L^\infty(0,T;H_0^1 \cap H^2) : u' \in L^\infty(0,T;H_0^1), u'' \in L^\infty(0,T;L^2)\}$ such that u verifies the following variational equation:

$$\langle u''(t), w\rangle + \langle \mu(\cdot,t,u(t))u_x(t), w_x\rangle = \langle f(\cdot,t,u(t),u_x(t),u'(t)), w\rangle, \quad \forall w \in H_0^1, \tag{2.5}$$

and the initial conditions

$$u(0) = \widetilde{u}_0, \qquad u'(0) = \widetilde{u}_1. \tag{2.6}$$

(ii) With the regularity obtained by $u \in \widetilde{W}$, it also follows from Theorem 3.2 that the problem (1.1)–(1.3) has a unique strong solution u that satisfies

$$u \in C^0\left(0,T;H^1\right) \cap C^1\left(0,T;L^2\right) \cap L^\infty\left(0,T;H^2\right), \quad u_t \in L^\infty\left(0,T;H^1\right), \quad u_{tt} \in L^\infty\left(0,T;L^2\right). \tag{2.7}$$

On the other hand, by $u \in \widetilde{W}$, we can see that $u, u_x, u_t, u_{xx}, u_{xt}, u_{tt} \in L^\infty(0,T;L^2) \subset L^2(Q_T)$.

Also, if $(u_0, u_1) \in (H_0^1 \cap H^2) \times H_0^1$, then the weak solution u of the problem (1.1)–(1.3) belongs to $H^2(Q_T)$. So, the solution is almost classical which is rather natural, since the initial data (u_0, u_1) do not belong necessarily to $C^2(\overline{\Omega}) \times C^1(\overline{\Omega})$.

3. The Existence and the Uniqueness of a Weak Solution

We make the following assumptions:

(H_1) $\widetilde{u}_0 \in H_0^1 \cap H^2, \widetilde{u}_1 \in H_0^1$,

(H_2) $\mu \in C^2([0,1] \times \mathbb{R}_+ \times \mathbb{R}), \mu(x,t,z) \geq \mu_* > 0$, for all $(x,t,z) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}$,

(H_3) $f \in C^1(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^3)$.

With μ and f satisfying the assumptions (H_2) and (H_3), respectively, for each $T^* > 0$ and $M > 0$ are given, we put the following constants:

$$\widetilde{K}_M(\mu) = \|\mu\|_{C^2(\widetilde{D}^*_M)}, \tag{3.1}$$

$$K_M(f) = \|f\|_{C^1(D^*_M)}, \tag{3.2}$$

where $\widetilde{D}^*_M = \{(x,t,z) : 0 \le x \le 1, 0 \le t \le T^*, |z| \le M\}$ and $D^*_M = \{(x,t,u,v,w) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3 : 0 \le x \le 1, 0 \le t \le T^*, |u|, |v|, |w| \le M\}$.

For each $T \in (0, T^*]$ and $M > 0$, we get

$$\begin{aligned} W(M,T) = \Big\{ & v \in L^\infty\left(0,T; H_0^1 \cap H^2\right) : v_t \in L^\infty\left(0,T; H_0^1\right),\ v_{tt} \in L^2(Q_T), \\ & \text{with } \|v\|_{L^\infty(0,T;H_0^1\cap H^2)}, \|v_t\|_{L^\infty(0,T;H_0^1)}, \|v_{tt}\|_{L^2(Q_T)} \le M \Big\}, \end{aligned} \tag{3.3}$$

$$W_1(M,T) = \left\{ v \in W(M,T) : v_{tt} \in L^\infty\left(0,T;L^2\right) \right\}, \tag{3.4}$$

where $Q_T = \Omega \times (0,T)$.

We choose the first term $u_0 \equiv \tilde{u}_0 \in W_1(M,T)$. Suppose that

$$u_{m-1} \in W_1(M,T), \quad m \ge 1. \tag{3.5}$$

The problem (1.1)–(1.3) is associated with the following variational problem.

Find $u_m \in W_1(M,T)$ such that

$$\langle u_m''(t), v\rangle + \langle \mu_m(t)\nabla u_m(t), \nabla v\rangle = \langle F_m(t), v\rangle, \quad \forall v \in H_0^1, \tag{3.6}$$

$$u_m(0) = \tilde{u}_0, \qquad u_m'(0) = \tilde{u}_1, \tag{3.7}$$

where

$$\mu_m(x,t) = \mu(x,t,u_{m-1}(t)), \qquad F_m(x,t) = f\left(x,t,u_{m-1}(x,t), \nabla u_{m-1}(x,t), u_{m-1}'(x,t)\right). \tag{3.8}$$

Then, we have the following theorem.

Theorem 3.1. *Let (H_1)–(H_3) hold. Then, there exist two constants $M > 0, T > 0$ and the linear recurrent sequence $\{u_m\} \subset W_1(M,T)$ defined by* (3.6)–(3.8).

Proof. The proof consists of three steps.

Step 1. The Faedo-Galerkin approximation (introduced by Lions [15]).

Consider a special basis $\{w_j\}$ on $H_0^1 : w_j(x) = \sqrt{2}\sin(j\pi x), j \in \mathbb{N}$, formed by the eigenfunctions of the Laplacian $-\Delta = -\partial^2/\partial x^2$. Put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.9}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of linear differential equations

$$\left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle + \left\langle \mu_m(t) \nabla u_m^{(k)}(t), \nabla w_j \right\rangle = \langle F_m(t), w_j \rangle, \quad 1 \le j \le k, \tag{3.10}$$

$$u_m^{(k)}(0) = \widetilde{u}_{0k}, \qquad \dot{u}_m^{(k)}(0) = \widetilde{u}_{1k}, \tag{3.11}$$

where

$$\begin{gathered} \widetilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \longrightarrow \widetilde{u}_0 \quad \text{strongly in } H_0^1 \cap H^2, \\ \widetilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \longrightarrow \widetilde{u}_1 \quad \text{strongly in } H_0^1. \end{gathered} \tag{3.12}$$

Note that by (3.5), it is not difficult to prove that the system (3.10), (3.11) has a unique solution $u_m^{(k)}(t)$ on interval $[0, T]$, so let us omit the details.

Step 2. A priori estimates. At first, put

$$\begin{aligned} s_m^{(k)}(t) &= p_m^{(k)}(t) + q_m^{(k)}(t) + \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds, \\ p_m^{(k)}(t) &= \left\| \dot{u}_m^{(k)}(t) \right\|^2 + \left\| \sqrt{\mu_m(t)} \nabla u_m^{(k)}(t) \right\|^2, \\ q_m^{(k)}(t) &= \left\| \nabla \dot{u}_m^{(k)}(t) \right\|^2 + \left\| \sqrt{\mu_m(t)} \Delta u_m^{(k)}(t) \right\|^2. \end{aligned} \tag{3.13}$$

Then, it follows from (3.9)–(3.11), (3.13) that

$$\begin{aligned} s_m^{(k)}(t) = s_m^{(k)}(0) &+ 2\langle \nabla \mu_m(0) \nabla \widetilde{u}_{0k}, \Delta \widetilde{u}_{0k} \rangle + 2\langle F_m(0), \Delta \widetilde{u}_{0k} \rangle \\ &+ \int_0^t ds \int_0^1 \mu_m'(x,s) \left(\left| \nabla u_m^{(k)}(x,s) \right|^2 + \left| \Delta u_m^{(k)}(x,s) \right|^2 \right) dx + 2\int_0^t \left\langle F_m(s), \dot{u}_m^{(k)}(s) \right\rangle ds \\ &+ 2\int_0^t \left\langle \frac{\partial}{\partial s} \left(\nabla \mu_m(s) \nabla u_m^{(k)}(s) \right), \Delta u_m^{(k)}(s) \right\rangle ds - 2\left\langle \nabla \mu_m(t) \nabla u_m^{(k)}(t), \Delta u_m^{(k)}(t) \right\rangle \end{aligned}$$

$$- 2\left\langle F_m(t), \Delta u_m^{(k)}(t) \right\rangle + 2\int_0^t \left\langle \frac{\partial F_m}{\partial t}(s), \Delta u_m^{(k)}(s) \right\rangle ds + \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds$$

$$= q_m^{(k)}(0) + 2\langle \nabla \mu_m(0) \nabla \widetilde{u}_{0k}, \Delta \widetilde{u}_{0k} \rangle + 2\langle F_m(0), \Delta \widetilde{u}_{0k} \rangle + \sum_{j=1}^{7} I_j. \tag{3.14}$$

Next, we will estimate the terms $I_j, j = 1, 2, \ldots, 7$ on the right-hand side of (3.14) as follows.

First Term I_1

We have

$$\mu_m'(t) = D_2\mu(x, t, u_{m-1}(t)) + D_3\mu(x, t, u_{m-1}(t)) u_{m-1}'(t). \tag{3.15}$$

From (3.1), (3.5), and (3.8), we have

$$\left| \mu_m'(x, t) \right| \le (1 + M) \widetilde{K}_M(\mu). \tag{3.16}$$

Hence,

$$I_1 = \int_0^t ds \int_0^1 \mu_m'(x, s) \left(\left| \nabla u_m^{(k)}(x, s) \right|^2 + \left| \Delta u_m^{(k)}(x, s) \right|^2 \right) dx \le \frac{1 + M}{\mu_*} \widetilde{K}_M(\mu) \int_0^t s_m^{(k)}(s) ds. \tag{3.17}$$

Second Term

By using (H_3), we obtain from (3.2), (3.5), and $(3.13)_2$ that

$$I_2 = 2\int_0^t \left\langle F_m(s), \dot{u}_m^{(k)}(s) \right\rangle ds \le T K_M^2(f) + \int_0^t p_m^{(k)}(s) ds. \tag{3.18}$$

Third Term

The Cauchy-Schwartz inequality yields

$$|I_3| = 2\left| \int_0^t \left\langle \frac{\partial}{\partial s} \left(\nabla \mu_m(s) \nabla u_m^{(k)}(s) \right), \Delta u_m^{(k)}(s) \right\rangle ds \right| \le \frac{2}{\sqrt{\mu_*}} \int_0^t r_m^{(k)}(s) \sqrt{q_m^{(k)}(s)} ds, \tag{3.19}$$

where $r_m^{(k)}(s) = \|\partial / \partial s (\nabla \mu_m(s) \nabla u_m^{(k)}(s))\|$.

We note

$$r_m^{(k)}(s) = \left\| \nabla \mu_m(s) \nabla \dot{u}_m^{(k)}(s) + \frac{\partial}{\partial s}(\nabla \mu_m(s)) \nabla u_m^{(k)}(s) \right\| \\ \leq \left(\|\nabla \mu_m(s)\|_{C^0(\overline{\Omega})} + \frac{1}{\sqrt{\mu_*}} \left\| \frac{\partial}{\partial s} \nabla \mu_m(s) \right\| \right) \sqrt{s_m^{(k)}(s)}. \tag{3.20}$$

On the other hand, by $\nabla \mu_m(x,s) = D_1\mu(x,s,u_{m-1}(x,s)) + D_3\mu(x,s,u_{m-1}(x,s)) \nabla u_{m-1}(x,s)$, it is implies that

$$\|\nabla \mu_m(s)\|_{C^0(\overline{\Omega})} \leq \widetilde{K}_M(\mu)\left(1 + \|\nabla u_{m-1}(s)\|_{C^0(\overline{\Omega})}\right) \leq 2(1+M)\widetilde{K}_M(\mu). \tag{3.21}$$

Similarly, the following equality

$$\frac{\partial}{\partial s} \nabla \mu_m(x,s) = D_1D_1\mu(x,s,u_{m-1}(x,s)) + D_3D_1\mu(x,s,u_{m-1}(x,s))u'_{m-1}(x,s) \\ + \left[D_1D_3\mu(x,s,u_{m-1}(x,s)) + D_3D_3\mu(x,s,u_{m-1}(x,s))u'_{m-1}(x,s)\right] \nabla u_{m-1}(x,s) \\ + D_3\mu(x,s,u_{m-1}(x,s)) \nabla u'_{m-1}(x,s) \tag{3.22}$$

gives

$$\left\| \frac{\partial}{\partial s} \nabla \mu_m(s) \right\| \leq \left(1 + 3M + M^2\right) \widetilde{K}_M(\mu). \tag{3.23}$$

It follows from (3.20)–(3.23) that

$$r_m^{(k)}(s) \leq \left[2(1+M) + \frac{1+3M+M^2}{\sqrt{\mu_*}}\right] \widetilde{K}_M(\mu) \sqrt{s_m^{(k)}(s)}. \tag{3.24}$$

Hence, we obtain from (3.19) and (3.24) that

$$|I_3| \leq \frac{2}{\sqrt{\mu_*}} \left[2(1+M) + \frac{1+3M+M^2}{\sqrt{\mu_*}}\right] \widetilde{K}_M(\mu) \int_0^t s_m^{(k)}(s)\,ds. \tag{3.25}$$

Fourth Term I_4

By the Cauchy-Schwartz inequality, we have

$$|I_4| = \left| -2\left\langle \nabla \mu_m(t) \nabla u_m^{(k)}(t), \Delta u_m^{(k)}(t) \right\rangle \right| \leq \frac{1}{\beta} \left\| \nabla \mu_m(t) \nabla u_m^{(k)}(t) \right\|^2 + \beta \left\| \Delta u_m^{(k)}(t) \right\|^2, \tag{3.26}$$

for all $\beta > 0$. On the other hand

$$\begin{aligned}\left\|\nabla\mu_m(t)\nabla u_m^{(k)}(t)\right\| &= \left\|\nabla\mu_m(0)\nabla\widetilde{u}_{0k} + \int_0^t \frac{\partial}{\partial s}\left(\nabla\mu_m(s)\nabla u_m^{(k)}(s)\right)ds\right\| \\ &\leq \|\nabla\mu_m(0)\|_{C^0(\overline{\Omega})}\|\nabla\widetilde{u}_{0k}\| + \int_0^t r_m^{(k)}(s)ds.\end{aligned} \tag{3.27}$$

Hence, we obtain from (3.26), (3.27) that

$$\begin{aligned}|I_4| \leq{}& \frac{\beta}{\mu_*}q_m^{(k)}(t) + \frac{2}{\beta}\|\nabla\mu_m(0)\|^2_{C^0(\overline{\Omega})}\|\nabla\widetilde{u}_{0k}\|^2 \\ &+ \frac{2}{\beta}T\left[2(1+M) + \frac{1+3M+M^2}{\sqrt{\mu_*}}\right]^2 \widetilde{K}_M^2(\mu)\int_0^t s_m^{(k)}(s)ds,\end{aligned} \tag{3.28}$$

for all $\beta > 0$.

Fifth Term I_5

By (3.5), (3.8), and (3.13), we obtain

$$\begin{aligned}|I_5| &= \left|-2\left\langle F_m(t), \Delta u_m^{(k)}(t)\right\rangle\right| \leq \frac{1}{\beta}\|F_m(t)\|^2 + \beta\left\|\Delta u_m^{(k)}(t)\right\|^2 \\ &\leq \frac{2}{\beta}\|F_m(0)\|^2 + \frac{2}{\beta}T\int_0^T \left\|\frac{\partial F_m}{\partial s}(s)\right\|^2 ds + \frac{\beta}{\mu_*}s_m^{(k)}(t), \quad \forall\beta > 0.\end{aligned} \tag{3.29}$$

Note that

$$\frac{\partial F_m}{\partial t}(t) = D_2 f[u_{m-1}] + D_3 f[u_{m-1}]u'_{m-1}(t) + D_4 f[u_{m-1}]\nabla u'_{m-1}(t) + D_5 f[u_{m-1}]u''_{m-1}(t), \tag{3.30}$$

where we use the notation $D_i f[u_{m-1}] = D_i f(x,t,u_{m-1}(x,t),\nabla u_{m-1}(x,t),u'_{m-1}(x,t))$, $i = 2,\ldots,5$. By (3.2), (3.5), and (3.30), we obtain

$$\left\|\frac{\partial F_m}{\partial t}(t)\right\| \leq K_M(f)\left(1 + 2M + \left\|u''_{m-1}(t)\right\|\right). \tag{3.31}$$

Hence, we deduce from (3.29) and (3.31) that

$$|I_5| \leq \frac{2}{\beta}\|F_m(0)\|^2 + \frac{4}{\beta}TK_M^2(f)\left[(1+2M)^2T + M^2\right] + \frac{\beta}{\mu_*}s_m^{(k)}(t), \quad \forall\beta > 0. \tag{3.32}$$

Sixth Term I_6

By (3.2), (3.5), $(3.13)_3$, and (3.31), we get

$$|I_6| = 2\left|\int_0^t \left\langle \frac{\partial F_m}{\partial t}(s), \Delta u_m^{(k)}(s)\right\rangle ds\right| \le \int_0^t \left\|\frac{\partial F_m}{\partial t}(s)\right\| ds + \int_0^t \left\|\frac{\partial F_m}{\partial t}(s)\right\| \left\|\Delta u_m^{(k)}(s)\right\|^2 ds$$

$$\le K_M(f)\left[(1+2M)T + \sqrt{T}\left(\int_0^T \left\|u''_{m-1}(s)\right\|^2 ds\right)^{1/2}\right]$$

$$+ \frac{1}{\mu_*}K_M(f)\int_0^t \left(1+2M+\left\|u''_{m-1}(s)\right\|\right) q_m^{(k)}(s)ds$$

$$\le K_M(f)\left[(1+2M)T + \sqrt{T}M\right] + \frac{1}{\mu_*}K_M(f)\int_0^t \left(1+2M+\left\|u''_{m-1}(s)\right\|\right) q_m^{(k)}(s)ds. \quad (3.33)$$

Seventh Term I_7

Equation (3.10) is rewritten as follows:

$$\left\langle \ddot{u}_m^{(k)}(t), w_j\right\rangle - \left\langle \frac{\partial}{\partial x}\left(\mu_m(t)\nabla u_m^{(k)}(t)\right), w_j\right\rangle = \langle F_m(t), w_j\rangle, \quad 1 \le j \le k. \quad (3.34)$$

Hence, by replacing w_j with $\ddot{u}_m^{(k)}(t)$ and integrating

$$I_7 = \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|^2 ds \le 2\int_0^t \left\|\frac{\partial}{\partial x}\left(\mu_m(s)\nabla u_m^{(k)}(s)\right)\right\|^2 ds + 2\int_0^t \|F_m(s)\|^2 ds$$
$$\le 2\int_0^t \left\|\frac{\partial}{\partial x}\left(\mu_m(s)\nabla u_m^{(k)}(s)\right)\right\|^2 ds + 2TK_M^2(f), \quad (3.35)$$

we need, estimate $\|\partial/\partial x(\mu_m(s)\nabla v_m^{(k)}(s))\|$.

Combining (3.1), (3.5), and (3.13) yields

$$\left\|\frac{\partial}{\partial x}\left(\mu_m(s)\nabla u_m^{(k)}(s)\right)\right\| = \left\|\nabla\mu_m(s)\nabla u_m^{(k)}(s) + \mu_m(s)\Delta u_m^{(k)}(s)\right\|$$
$$\le \left\|\nabla\mu_m(s)\right\|_{C^0(\overline{\Omega})}\left\|\nabla u_m^{(k)}(s)\right\| + \left\|\mu_m(s)\right\|_{C^0(\overline{\Omega})}\left\|\Delta u_m^{(k)}(s)\right\|$$
$$\le \frac{2}{\sqrt{\mu_*}}(1+M)\widetilde{K}_M(\mu)\sqrt{p_m^{(k)}(s)} + \frac{1}{\sqrt{\mu_*}}\widetilde{K}_M(\mu)\sqrt{q_m^{(k)}(s)}$$
$$\le \frac{3}{\sqrt{\mu_*}}(1+M)\widetilde{K}_M(\mu)\sqrt{s_m^{(k)}(s)}. \quad (3.36)$$

Therefore, from (3.35) and (3.36), we obtain

$$I_7 \leq 2TK_M^2(f) + \frac{18}{\mu_*}(1+M)^2 \widetilde{K}_M^2(\mu) \int_0^t s_m^{(k)}(s)ds. \tag{3.37}$$

Choosing $\beta > 0$, with $2\beta/\mu_* \leq 1/2$, it follows from (3.13), (3.14), (3.17), (3.18), (3.25), (3.28), (3.32), (3.33), and (3.37) that

$$s_m^{(k)}(t) \leq \widetilde{C}_{0k} + \widetilde{C}_1(M,T) + \int_0^t \left(\widetilde{C}_2(M,T) + \frac{2}{\mu_*} K_M(f) \left\| u''_{m-1}(s) \right\| \right) s_m^{(k)}(s)ds, \tag{3.38}$$

where

$$\begin{aligned}
\widetilde{C}_{0k} &= \widetilde{C}_{0k}(\beta, f, \mu, \widetilde{u}_0, \widetilde{u}_1, \widetilde{u}_{0k}, \widetilde{u}_{1k}) \\
&= 2s_m^{(k)}(0) + 4\langle \nabla\mu_m(0)\nabla\widetilde{u}_{0k}, \Delta\widetilde{u}_{0k} \rangle + 4\langle F_m(0), \Delta\widetilde{u}_{0k} \rangle \\
&\quad + \frac{4}{\beta}\|\nabla\mu_m(0)\|^2_{C^0(\overline{\Omega})}\|\nabla\widetilde{u}_{0k}\|^2 + \frac{4}{\beta}\|F_m(0)\|^2, \\
\widetilde{C}_1(M,T) &= \widetilde{C}_1(\beta, f, M, T) \\
&= 2\left(3 + \frac{4}{\beta}\left[(1+2M)^2 T + M^2\right]\right) T K_M^2(f) \\
&\quad + 2\left[M + (1+2M)\sqrt{T}\right]\sqrt{T} K_M(f), \\
\widetilde{C}_2(M,T) &= \widetilde{C}_2(\beta, f, \mu, M, T) \\
&= 2 + \frac{2}{\mu_0}(1+2M)K_M(f) \\
&\quad + \frac{2}{\mu_*}\left[(1+4\sqrt{\mu_*})(1+M) + 2\left(1+3M+M^2\right)\right]\widetilde{K}_M(\mu) \\
&\quad + \frac{4}{\mu_*}\left[\frac{1}{\beta}T\left(2(1+M)\sqrt{\mu_*} + 1 + 3M + M^2\right)^2 + 9(1+M)^2\right]\widetilde{K}_M^2(\mu).
\end{aligned} \tag{3.39}$$

By (H_1), we deduce from (3.12), $(3.39)_1$ that there exists $M > 0$ independent of m and k, such that

$$\widetilde{C}_{0k} \leq \frac{1}{2}M^2. \tag{3.40}$$

Notice that by (H_3), we deduce from $(3.39)_{2,3}$ that

$$\lim_{T \to 0_+} \widetilde{C}_1(M,T) = \lim_{T \to 0_+} T\widetilde{C}_2(M,T) = 0. \tag{3.41}$$

So, from (3.39) and (3.41), we can choose $T > 0$ such that

$$\left(\frac{1}{2}M^2 + \widetilde{C}_1(M,T)\right)\exp\left(T\widetilde{C}_2(M,T) + \frac{2}{\mu_0}K_M(f)\sqrt{T}M\right) \le M^2, \tag{3.42}$$

$$k_T = \left(1 + \frac{1}{\sqrt{\mu_*}}\right)\sqrt{T}\sqrt{4K_M^2(f) + (4+M)^2M^2\widetilde{K}_M^2(\mu)e^{T[1+((1+M)/2\mu_*)\widetilde{K}_M(\mu)]}} < 1. \tag{3.43}$$

Finally, it follows from (3.38), (3.40), and (3.42) that

$$\begin{aligned} s_m^{(k)}(t) \le{}& M^2\exp\left(-T\widetilde{C}_2(M,T) - \frac{2}{\mu_0}K_M(f)\sqrt{T}M\right) \\ &+ \int_0^t \left(\widetilde{C}_2(M,T) + \frac{2}{\mu_0}K_M(f)\left\|u''_{m-1}(s)\right\|\right)s_m^{(k)}(s)ds. \end{aligned} \tag{3.44}$$

By using Gronwall's lemma, we deduce from (3.44) that

$$\begin{aligned} s_m^{(k)}(t) \le{}& M^2\exp\left(-T\widetilde{C}_2(M,T) - \frac{2}{\mu_0}K_M(f)\sqrt{T}M\right) \\ &\times\exp\left[\int_0^T\left(\widetilde{C}_2(M,T) + \frac{2}{\mu_0}K_M(f)\left\|u''_{m-1}(s)\right\|\right)ds\right] \\ \le{}& M^2\exp\left(-T\widetilde{C}_2(M,T) - \frac{2}{\mu_0}K_M(f)\sqrt{T}M\right) \\ &\times\exp\left[T\widetilde{C}_2(M,T) + \frac{2}{\mu_0}K_M(f)\sqrt{T}\left\|u''_{m-1}\right\|_{L^2(Q_T)}\right] \le M^2. \end{aligned} \tag{3.45}$$

Therefore, we have

$$u_m^{(k)} \in W(M,T), \quad \forall m,k \in \mathbb{N}. \tag{3.46}$$

Step 3. Limiting process.

From (3.46), we can extract from $\{u_m^{(k)}\}$ a subsequence still denoted by $\{u_m^{(k)}\}$ such that

$$\begin{aligned} u_m^{(k)} &\longrightarrow u_m \quad \text{in } L^\infty\left(0,T;H_0^1\cap H^2\right) \text{ weak}^*, \\ \dot{u}_m^{(k)} &\longrightarrow u'_m \quad \text{in } L^\infty\left(0,T;H_0^1\right) \text{ weak}^*, \\ \ddot{u}_m^{(k)} &\longrightarrow u''_m \quad \text{in } L^2(Q_T) \text{ weak}, \end{aligned} \tag{3.47}$$

as $k \to \infty$, and

$$u_m \in W(M,T). \tag{3.48}$$

Based on (3.47), passing to limit in (3.10), (3.11) as $k \to \infty$, we have u_m satisfying (3.6)–(3.8). On the other hand, it follows from (3.5), (3.6), and (3.47) that

$$u_m'' = \nabla \mu_m \nabla u_m + \mu_m \Delta u_m + f(x, t, u_{m-1}, \nabla u_{m-1}, u_{m-1}') \in L^\infty\left(0, T; L^2\right). \tag{3.49}$$

Hence, $u_m \in W_1(M, T)$, and the proof of Theorem 3.1 is complete. □

Theorem 3.2. *Let* (H_1)*–*(H_3) *hold. Then, there exist* $M > 0$ *and* $T > 0$ *satisfying* (3.40), (3.42), *and* (3.43) *such that the problem* (1.1)–(1.3) *has a unique weak solution* $u \in W_1(M, T)$.

Furthermore, the linear recurrent sequence $\{u_m\}$ *defined by* (3.6)–(3.8) *converges to the solution* u *strongly in the space*

$$W_1(T) = \left\{ w \in L^\infty\left(0, T; H_0^1\right) : w' \in L^\infty\left(0, T; L^2\right) \right\}, \tag{3.50}$$

with the following estimation:

$$\|u_m - u\|_{L^\infty(0,T;H_0^1)} + \left\|u_m' - u'\right\|_{L^\infty(0,T;L^2)} \le C k_T^m, \quad \forall m \in \mathbb{N}, \tag{3.51}$$

where $k_T < 1$ *as in* (3.43) *and* C *is a constant depending only on* $T, \tilde{u}_0, \tilde{u}_1$ *and* k_T.

Proof. (i) *The existence*. First, we note that $W_1(T)$ is a Banach space with respect to the norm (see Lions [15])

$$\|w\|_{W_1(T)} = \|w\|_{L^\infty(0,T;H_0^1)} + \left\|w'\right\|_{L^\infty(0,T;L^2)}. \tag{3.52}$$

Next, we prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $v_m = u_{m+1} - u_m$. Then, v_m satisfies the variational problem

$$\begin{aligned} \langle v_m''(t), w\rangle + \langle \mu_{m+1}(t) \nabla v_m(t), \nabla w\rangle &= \left\langle \frac{\partial}{\partial x}\left[(\mu_{m+1}(t) - \mu_m(t)) \nabla u_m(t)\right], w \right\rangle \\ &\quad + \langle F_{m+1}(t) - F_m(t), w\rangle, \quad \forall w \in H_0^1, \\ v_m(0) = v_m'(0) &= 0. \end{aligned} \tag{3.53}$$

Taking $w = v_m'$ in $(3.53)_1$, after integrating in t, we get

$$\begin{aligned} Z_m(t) &= \int_0^t ds \int_0^1 \mu_{m+1}'(x, s) |\nabla v_m(s)|^2 dx + 2\int_0^t \langle F_{m+1}(s) - F_m(s), v_m'(s)\rangle ds \\ &\quad + 2\int_0^t \left\langle \frac{\partial}{\partial x}\left[(\mu_{m+1}(s) - \mu_m(s)) \nabla u_m(s)\right], v_m'(s) \right\rangle ds = \sum_{i=1}^3 J_i, \end{aligned} \tag{3.54}$$

in which

$$Z_m(t) = \left\|v_m'(t)\right\|^2 + \left\|\sqrt{\mu_{m+1}(t)}\nabla v_m(t)\right\|^2, \tag{3.55}$$

and all integrals on the right-hand side of (3.54) are estimated as follows.

First Integral

By (3.16), we obtain

$$|J_1| \leq \left|\int_0^t ds \int_0^1 \mu_{m+1}'(x,s)|\nabla v_m(s)|^2 dx\right| \leq \frac{1+M}{\mu_*}\widetilde{K}_M(\mu)\int_0^t Z_m(s)ds. \tag{3.56}$$

Second Integral

By (H_3),

$$\|F_{m+1}(t) - F_m(t)\| \leq 2K_M(f)\left[\|\nabla v_{m-1}(t)\| + \left\|v_{m-1}'(t)\right\|\right] \leq 2K_M(f)\|v_{m-1}\|_{W_1(T)}, \tag{3.57}$$

so

$$\begin{aligned} |J_2| &\leq 2\left|\int_0^t \left\langle F_{m+1}(s) - F_m(s), v_m'(s)\right\rangle ds\right| \leq 4K_M(f)\|v_{m-1}\|_{W_1(T)}\int_0^t \left\|v_m'(s)\right\| ds \\ &\leq 4TK_M^2(f)\|v_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s)ds. \end{aligned} \tag{3.58}$$

Third Integral

Using (H_2) again, we get

$$\begin{aligned} |J_3| &= 2\left|\int_0^t \left\langle \frac{\partial}{\partial x}\left[(\mu_{m+1}(s) - \mu_m(s))\nabla u_m(s)\right], v_m'(s)\right\rangle ds\right| \\ &\leq \int_0^t \left\|\frac{\partial}{\partial x}\left[(\mu_{m+1}(s) - \mu_m(s))\nabla u_m(s)\right]\right\|^2 ds + \int_0^t Z_m(s)ds. \end{aligned} \tag{3.59}$$

Note that

$$\begin{aligned} &\frac{\partial}{\partial x}\left[(\mu_{m+1}(s) - \mu_m(s))\nabla u_m(s)\right] \\ &\quad = (\mu_{m+1}(s) - \mu_m(s))\Delta u_m(s) \\ &\qquad + (D_1\mu[u_m] - D_1\mu[u_{m-1}])\nabla u_m(s) + (D_3\mu[u_m] - D_3\mu[u_{m-1}])|\nabla u_m(s)|^2 \\ &\qquad + D_3\mu[u_{m-1}]\nabla v_{m-1}(s)\nabla u_m(s). \end{aligned} \tag{3.60}$$

Hence,

$$\begin{aligned}\left\|\frac{\partial}{\partial x}\left[\left(\mu_{m+1}(s)-\mu_m(s)\right)\nabla u_m(s)\right]\right\| \leq{}& \left\|\mu_{m+1}(s)-\mu_m(s)\right\|_{C^0(\overline{\Omega})}\|\Delta u_m(s)\| \\ &+\left\|\left(D_1\mu[u_m]-D_1\mu[u_{m-1}]\right)\right\|_{C^0(\overline{\Omega})}\|\nabla u_m(s)\| \\ &+\left\|\left(D_1\mu[u_m]-D_1\mu[u_{m-1}]\right)\right\|_{C^0(\overline{\Omega})}\|\nabla u_m(t)\|^2_{C^0(\overline{\Omega})} \\ &+\left\|D_3\mu[u_{m-1}]\right\|_{C^0(\overline{\Omega})}\|\nabla u_m(s)\|_{C^0(\overline{\Omega})}\|\nabla v_{m-1}(s)\|.\end{aligned} \tag{3.61}$$

We also note that

$$\begin{gathered}\left\|\mu_{m+1}(s)-\mu_m(s)\right\|_{C^0(\overline{\Omega})}\leq\widetilde{K}_M(\mu)\|w_{m-1}\|_{W_1(T)}, \\ \left\|D_i\mu[u_m]-D_i\mu[u_{m-1}]\right\|_{C^0(\overline{\Omega})}\leq\widetilde{K}_M(\mu)\|w_{m-1}\|_{W_1(\mathrm{T})},\quad i=1,3, \\ \|\nabla u_m(s)\|_{C^0(\overline{\Omega})}\leq\sqrt{2}\|\nabla u_m(s)\|_{H^1}\leq\sqrt{2}\sqrt{\|\nabla u_m(s)\|^2+\|\Delta u_m(s)\|^2}\leq 2M, \\ \left\|D_3\mu[u_m]\right\|_{C^0(\overline{\Omega})}\leq\widetilde{K}_M(\mu),\end{gathered} \tag{3.62}$$

where we use the notation $D_i\mu[u_{m-1}] = D_i\mu(x,t,u_m(x,t))$, $i = 1,2,3$. Therefore, it implies from (3.61) and (3.62) that

$$\left\|\frac{\partial}{\partial x}\left[\left(\mu_{m+1}(s)-\mu_m(s)\right)\nabla u_m(s)\right]\right\|\leq(4+M)M\widetilde{K}_M(\mu)\|v_{m-1}\|_{W_1(T)}. \tag{3.63}$$

Hence,

$$|J_3|\leq(4+M)^2M^2T\widetilde{K}^2_M(\mu)\|v_{m-1}\|^2_{W_1(T)}+\int_0^t Z_m(s)ds. \tag{3.64}$$

Combining (3.54)–(3.56), (3.58), and (3.64) yields

$$Z_m(t)\leq T\left[4K^2_M(f)+(4+M)^2M^2\widetilde{K}^2_M(\mu)\right]\|v_{m-1}\|^2_{W_1(T)}+\left(2+\frac{1+M}{\mu_*}\widetilde{K}_M(\mu)\right)\int_0^t Z_m(s)ds. \tag{3.65}$$

Using Gronwall's lemma, (3.65) gives

$$\|v_m\|_{W_1(T)}\leq k_T\|v_{m-1}\|_{W_1(T)}\quad\forall m\in\mathbb{N}, \tag{3.66}$$

where $k_T < 1$ as in (3.43).

Hence, we obtain from (3.66) that

$$\left\|u_{m+p}-u_m\right\|_{W_1(T)}\leq\frac{k_T^m}{1-k_T}\|u_1-u_0\|_{W_1(T)}\quad\forall m,p\in\mathbb{N}, \tag{3.67}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then, there exists $u \in W_1(T)$ such that

$$u_m \longrightarrow u \quad \text{strongly in } W_1(T). \tag{3.68}$$

On the other hand, from (3.48), we deduce the existence of a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{gathered} u_{m_j} \longrightarrow u \quad \text{in } L^\infty\left(0,T;H_0^1 \cap H^2\right) \text{ weak}^*, \\ u'_{m_j} \longrightarrow u' \quad \text{in } L^\infty\left(0,T;H_0^1\right) \text{ weak}^*, \\ u''_{m_j} \longrightarrow u'' \quad \text{in } L^2(Q_T) \text{ weak}, \end{gathered} \tag{3.69}$$

$$u \in W(M,T). \tag{3.70}$$

Note that

$$\begin{gathered} \left|\mu_m(x,t) - \mu(x,t,u(x,t))\right| \le \widetilde{K}_M(\mu)\|u_{m-1} - u\|_{W_1(T)}, \\ \left\|F_m(t) - f\left(\cdot,t,u(t),u_x(t),u'(t)\right)\right\| \le 2K_M(f)\|u_{m-1} - u\|_{W_1(T)}. \end{gathered} \tag{3.71}$$

Hence, from (3.68) and (3.71), we obtain

$$\begin{gathered} \mu_m \longrightarrow \mu(\cdot,\cdot,u) \quad \text{strongly in } L^\infty(Q_T), \\ F_m \longrightarrow f\left(\cdot,t,u(t),u_x(t),u'(t)\right) \quad \text{strongly in } L^\infty\left(0,T;L^2\right). \end{gathered} \tag{3.72}$$

Finally, passing to limit in (3.6)–(3.8) as $m = m_j \to \infty$, it implies from (3.68), (3.69), and (3.72) that there exists $u \in W(M,T)$ satisfying the equation

$$\begin{gathered} \left\langle u''(t),w\right\rangle + \left\langle \mu(\cdot,t,u(t))u_x(t),\nabla w\right\rangle = \left\langle f\left(\cdot,t,u(t),u_x(t),u'(t)\right),w\right\rangle, \quad \forall w \in H_0^1, \\ u(0) = \widetilde{u}_0, \qquad u'(0) = \widetilde{u}_1. \end{gathered} \tag{3.73}$$

On the other hand, by (H_2), we obtain from (3.70), $(3.72)_2$, and $(3.73)_1$ that

$$u'' = D_1\mu[u]u_x + D_3\mu[u]u_x^2 + \mu[u]u_{xx} + f\left(x,t,u,u_x,u'\right) \in L^\infty\left(0,T;L^2\right), \tag{3.74}$$

thus $u \in W_1(M,T)$, and Step 1 follows.

(ii) *The uniqueness of the solution*.

Let $u_1, u_2 \in W_1(M,T)$ be two weak solutions of the problem (1.1)–(1.3). Then, $u = u_1 - u_2$ satisfies the variational problem

$$\begin{aligned}
\langle u''(t), w\rangle + \langle \mu_1(t) u_x(t), w_x\rangle &= \left\langle \frac{\partial}{\partial x}\left([\mu_1(t) - \mu_2(t)] u_{2x}(t)\right), w\right\rangle \\
&\quad + \langle F_2(t) - F_1(t), w\rangle, \quad \forall w \in H_0^1, \\
u(0) &= u'(0) = 0, \\
\mu_i(t) &= \mu(x,t,u_i(t)) \equiv \mu[u_i], F_i(t) \\
&= f\left(x,t,u_i(t),u_{ix}(t),u_i'(t)\right), \quad i = 1,2.
\end{aligned} \tag{3.75}$$

We take $w = u'$ in $(3.75)_1$ and integrate in t to get

$$\begin{aligned}
\rho(t) &= \int_0^t ds \int_0^1 \mu_1'(x,s) u_x^2(x,s)dx + 2\int_0^t \langle F_1(s) - F_2(s), u'(s)\rangle ds \\
&\quad + 2\int_0^t \left\langle \frac{\partial}{\partial x}\left([\mu_1(s) - \mu_2(s)] u_{2x}(s)\right), u'\right\rangle ds \equiv \sum_{i=1}^3 \rho_i(t),
\end{aligned} \tag{3.76}$$

where

$$\rho(t) = \left\|u'(t)\right\|^2 + \left\|\sqrt{\mu_1(t)} u_x(t)\right\|^2. \tag{3.77}$$

We now estimate the terms on the right-hand side of (3.76) as follows:

$$\rho_1(t) = \int_0^t ds \int_0^1 \mu_1'(x,s) u_x^2(x,s)dx \le \frac{1}{\mu_*}(1+M)\widetilde{K}_M(\mu)\int_0^t \rho(s)ds \equiv \rho_M^{(1)} \int_0^t \rho(s)ds, \tag{3.78}$$

$$\begin{aligned}
\rho_2(t) &= 2\int_0^t \langle F_1(s) - F_2(s), u'(s)\rangle ds \le 4K_M(f)\int_0^t \left(\|u_x(s)\| + \left\|u'(s)\right\|\right)\left\|u'(s)\right\| ds \\
&\le 4\left(1 + \frac{1}{\sqrt{\mu_*}}\right) K_M(f) \int_0^t \rho(s)ds \equiv \rho_M^{(2)} \int_0^t \rho(s)ds,
\end{aligned} \tag{3.79}$$

$$\rho_3(t) = 2\int_0^t \left\langle \frac{\partial}{\partial x}\left([\mu_1(s) - \mu_2(s)] u_{2x}(s)\right), u'\right\rangle ds \le 2\int_0^t \left\| \frac{\partial}{\partial x}\left([\mu_1(s) - \mu_2(s)] u_{2x}(s)\right)\right\| \left\|u'(s)\right\| ds. \tag{3.80}$$

On the other hand

$$\begin{aligned}
\frac{\partial}{\partial x}\left([\mu_1(s) - \mu_2(s)] u_{2x}(s)\right) &= [\mu_1(s) - \mu_2(s)] u_{2xx}(s) + \left(D_1\mu[u_1] - D_1\mu[u_2]\right) u_{2x}(s) \\
&\quad + \left(D_3\mu[u_1] - D_3\mu[u_2]\right) u_{1x} u_{2x} + \mathrm{D}_3\mu[u_2] u_x u_{2x}.
\end{aligned} \tag{3.81}$$

Hence,

$$\begin{aligned}\left\|\frac{\partial}{\partial x}\left([\mu_1(s)-\mu_2(s)]u_{2x}(s)\right)\right\| &\le \|\mu_1(s)-\mu_2(s)\|_{C^0(\overline{\Omega})}\|u_{2xx}(s)\| \\ &\quad + \|D_1\mu[u_1]-D_1\mu[u_2]\|_{C^0(\overline{\Omega})}\|u_{2x}(s)\| \\ &\quad + \|D_3\mu[u_1]-D_3\mu[u_2]\|_{C^0(\overline{\Omega})}\|u_{1x}(s)\|_{C^0(\overline{\Omega})}\|u_{2x}(s)\|_{C^0(\overline{\Omega})} \\ &\quad + \|D_3\mu[u_2]\|_{C^0(\overline{\Omega})}\|u_x(s)\|\|u_{2x}(s)\|_{C^0(\overline{\Omega})} \\ &\le (3+M)M\widetilde{K}_M(\mu)\|u_x(s)\|.\end{aligned} \tag{3.82}$$

It follows from (3.80), (3.82) that

$$\rho_3(t) \le \frac{1}{\sqrt{\mu_*}}(3+M)M\widetilde{K}_M(\mu)\int_0^t \rho(s)ds \equiv \rho_M^{(3)}\int_0^t \rho(s)ds. \tag{3.83}$$

Combining (3.76)–(3.79) and (3.83) yields

$$\rho(t) \le \left(\rho_M^{(1)}+\rho_M^{(2)}+\rho_M^{(3)}\right)\int_0^t \rho(s)ds. \tag{3.84}$$

Using Gronwall's lemma, it follows from (3.84) that $\rho \equiv 0$ that is, $u_1 \equiv u_2$.
Theorem 3.2 is proved completely. □

Remark 3.3. (i) In the case of $\mu \equiv 1, f \in C^1(\overline{\Omega}\times\mathbb{R}_+\times\mathbb{R}^3)$ and the boundary condition in [4] standing for (1.2), we obtained some similar results in [4].

(ii) In the case of $\mu \equiv 1, f \in C^1(\overline{\Omega}\times\mathbb{R}_+\times\mathbb{R}^3), f(1,t,u,v,w) = 0$, for all$t \ge 0$, for all$(u,v,w) \in \mathbb{R}^3$, and the boundary condition in [8] standing for (1.2), some results as above were given in [8].

Remark 3.4. By Galerkin method, as in Remark 2.3, the local existence of a strong solution $u \in H^2(Q_T)$ of the problem (1.1)–(1.3) is proved.

In the case of $\mu = \mu(x,t)$ and $f = f(x,t)$, obviously, the problem (1.1)–(1.3) is linear. Then, by the same method and applying Banach's theorem [16, Chapter 5, Theorem 17.1], it is not difficult to prove that the problem (1.1)–(1.3) is global solvability. To strengthen some hypotheses, it is possible to prove existence of a classical solution $u \in C^2(Q_T)\cap C^1(\overline{Q}_T)$.

4. Asymptotic Expansion of a Weak Solution in Many Small Parameters

In this section, we will study a high-order asymptotic expansion of a weak solution for the problem (1.1)–(1.3), in which (1.1) has the form of a linear wave equation with nonlinear perturbations containing many small parameters.

The Problem with Two Small Parameters

At first, we consider the case of the nonlinear perturbations containing two small parameters.

Let (H_1) hold. We make the following assumptions:

(H_4) $\mu_0 \in C^2([0,1] \times \mathbb{R}_+)$, $\mu_1 \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, $\mu_0 \geq \mu_* > 0$, $\mu_1 \geq 0$,

(H_5) $f_0 \in C^1([0,1] \times \mathbb{R}_+)$, $f_1 \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$.

We consider the following perturbed problem, where $\varepsilon_1, \varepsilon_2$ are two small parameters such that $0 \leq \varepsilon_i \leq \varepsilon_{i*} < 1$, $i = 1, 2$:

$$u_{tt} - \frac{\partial}{\partial x}\left(\mu_{\varepsilon_1}(x,t,u)u_x\right) = F_{\varepsilon_2}(x,t,u,u_x,u_t), \quad 0 < x < 1,\ 0 < t < T,$$

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = \widetilde{u}_0(x), \qquad u_t(x,0) = \widetilde{u}_1(x), \tag{$P_{\vec{\varepsilon}}$}$$

$$\mu_{\varepsilon_1}(x,t,u) = \mu_0(x,t) + \varepsilon_1 \mu_1(x,t,u),$$

$$F_{\varepsilon_2}(x,t,u,u_x,u_t) = f_0(x,t) + \varepsilon_2 f_1(x,t,u,u_x,u_t).$$

By Theorem 3.2, the problem $(P_{\vec{\varepsilon}})$ has a unique weak solution u depending on $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2) : u_{\vec{\varepsilon}} = u(\varepsilon_1, \varepsilon_2)$. When $\vec{\varepsilon} = (0,0)$, $(P_{\vec{\varepsilon}})$ is denoted by (P_0). We will study the asymptotic expansion of $u_{\vec{\varepsilon}}$ with respect to $\varepsilon_1, \varepsilon_2$.

We use the following notations. For a multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, and $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$, we put

$$|\alpha| = \alpha_1 + \alpha_2, \qquad \alpha! = \alpha_1!\alpha_2!,$$

$$\|\vec{\varepsilon}\| = \sqrt{\varepsilon_1^2 + \varepsilon_2^2}, \qquad \vec{\varepsilon}^{\,\alpha} = \varepsilon_1^{\alpha_1}\varepsilon_2^{\alpha_2}, \tag{4.1}$$

$$\alpha, \beta \in \mathbb{Z}_+^2, \quad \alpha \leq \beta \Longleftrightarrow \alpha_i \leq \beta_i \quad \forall i = 1,2.$$

We first note the following lemma.

Lemma 4.1. *Let $m, N \in \mathbb{N}$ and $u_\alpha \in \mathbb{R}, \alpha \in \mathbb{Z}_+^2, 1 \leq |\alpha| \leq N$. Then,*

$$\left(\sum_{1\leq|\alpha|\leq N} u_\alpha \vec{\varepsilon}^{\,\alpha}\right)^m = \sum_{m\leq|\alpha|\leq mN} T_\alpha^{(m)}[u]\vec{\varepsilon}^{\,\alpha}, \tag{4.2}$$

where the coefficients $T_\alpha^{(m)}[u]$, $m \le |\alpha| \le mN$ *depending on* $u = (u_\alpha), \alpha \in \mathbb{Z}_+^2, 1 \le |\alpha| \le N$*are defined by the recurrent formulas*

$$T_\alpha^{(1)}[u] = u_\alpha, \quad 1 \le |\alpha| \le N,$$

$$T_\alpha^{(m)}[u] = \sum_{\beta \in A_\alpha^{(m)}} u_{\alpha-\beta} T_\beta^{(m-1)}[u], \quad m \le |\alpha| \le mN, \ m \ge 2, \tag{4.3}$$

$$A_\alpha^{(m)} = \left\{ \beta \in \mathbb{Z}_+^2 : \beta \le \alpha, \ 1 \le \left|\alpha - \beta\right| \le N, \ m-1 \le \left|\beta\right| \le (m-1)N \right\}.$$

The proof of Lemma 4.1 can be found in [6].

We also use the notations $f_1[u] = f_1(x,t,u,u_x,u_t), \mu_1[u] = \mu_1(x,t,u)$.

Let u_0 be a unique weak solution of the problem (P_0) corresponding to $\vec{\varepsilon} = (0,0)$ that is,

$$\begin{gathered} u_0'' - \frac{\partial}{\partial x}(\mu_0(x,t)u_{0x}) = f_0(x,t), \quad 0 < x < 1, \ 0 < t < T, \\ u_0(0,t) = u_0(1,t) = 0, \\ u_0(x,0) = \tilde{u}_0(x), \quad u_0'(x,0) = \tilde{u}_1(x), \\ u_0 \in W_1(M,T). \end{gathered} \tag{P_0}$$

Let us consider the sequence of weak solutions u_γ, $\gamma \in \mathbb{Z}_+^2, 1 \le |\gamma| \le N$, defined by the following problems:

$$\begin{gathered} u_\gamma'' - \frac{\partial}{\partial x}(\mu_0(x,t)u_{\gamma x}) = F_\gamma, \quad 0 < x < 1, 0 < t < T, \\ u_\gamma(0,t) = u_\gamma(1,t) = 0, \\ u_\gamma(x,0) = u_\gamma'(x,0) = 0, \\ u_\gamma \in W_1(M,T), \end{gathered} \tag{$\tilde{P}_\gamma$}$$

where $F_\gamma, \gamma \in \mathbb{Z}_+^2, 1 \le |\gamma| \le N$ are defined by the recurrent formulas as follows:

$$F_\gamma = \pi_\gamma^{(2)}[f_1] + \sum_{2 \le |\nu| \le |\gamma|, \nu \le \gamma} \frac{\partial}{\partial x}\left(\rho_\nu^{(1)}[\mu_1] \nabla u_{\gamma-\nu}\right), \quad 1 \le \left|\gamma\right| \le N, \tag{4.4}$$

with $\rho_\delta[\mu_1] = \rho_\delta[\mu_1; \{u_\gamma\}_{\gamma\le\delta}]$, $\rho_\delta^{(1)}[\mu_1] = \rho_\delta^{(1)}[\mu_1; \{u_\gamma\}_{\gamma\le\delta}]$, $\pi_\delta[f_1] = \pi_\delta[f_1; \{u_\gamma\}_{\gamma\le\delta}]$, $\pi_\delta^{(2)}[f_1] = \pi_\delta^{(2)}[f_1; \{u_\gamma\}_{\gamma\le\delta}]$, $|\delta| \le N-1$ defined by

$$\rho_\delta[\mu_1] = \begin{cases} \mu_1[u_0], & |\delta| = 0, \\ \displaystyle\sum_{m=1}^{|\delta|} \frac{1}{m!} D_3^m \mu_1[u_0] T_\delta^{(m)}[u], & 1 \le |\delta| \le N-1, \end{cases} \tag{4.5}$$

$$\begin{gathered} \rho_\delta^{(1)}[\mu_1] = \rho_{\delta_1-1,\delta_2}[\mu_1], \quad \delta = (\delta_1,\delta_2) \in \mathbb{Z}_+^2, \\ \rho_\delta^{(1)}[\mu_1] = \rho_{0,\delta_2}^{(1)}[\mu_1] = \rho_{-1,\delta_2}[\mu_1] = 0, \quad \text{if } \delta_1 = 0, \end{gathered} \tag{4.6}$$

$$\pi_\delta[f_1] = \begin{cases} f_1[u_0], & |\delta| = 0, \\ \displaystyle\sum_{\substack{1\le|m|\le|\delta| \\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N) \\ \alpha+\beta+\gamma=\delta}} \frac{1}{m!} D^m f_1[u_0] T_\alpha^{(m_1)} & \\ \quad \times [u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}[u'], & 1 \le |\delta| \le N-1, \end{cases} \tag{4.7}$$

where $m = (m_1, m_2, m_3) \in \mathbb{Z}_+^3$, $|m| = m_1 + m_2 + m_3$, $m! = m_1!m_2!m_3!$, $D^m f_j = D_3^{m_1} D_4^{m_2} D_5^{m_3} f_j$, $A(m,N) = \{(\alpha,\beta,\gamma) \in (\mathbb{Z}_+^2)^3 : m_1 \le |\alpha| \le m_1 N, m_2 \le |\beta| \le m_2 N, m_3 \le |\gamma| \le m_3 N\}$,

$$\begin{gathered} \pi_\delta^{(2)}[f_1] = \pi_{\delta_1,\delta_2-1}[f_1], \quad \delta = (\delta_1,\delta_2) \in \mathbb{Z}_+^2, \\ \pi_\delta^{(2)}[f_1] = \pi_{\delta_1,0}^{(2)}[f_1] = \pi_{\delta_1,-1}[f_1] = 0, \quad \text{if } \delta_2 = 0. \end{gathered} \tag{4.8}$$

Then, we have the following lemma.

Lemma 4.2. *Let $\rho_\nu[\mu_1]$, $\pi_\nu[f_1]$, $|\nu| \le N-1$ be the functions defined by* (4.5) *and* (4.7)*. Put $h = \sum_{|\gamma|\le N} u_\gamma \vec{\varepsilon}^{\,\gamma}$, then one has*

$$\mu_1[h] = \sum_{|\nu|\le N-1} \rho_\nu[\mu_1] \vec{\varepsilon}^{\,\nu} + \|\vec{\varepsilon}\|^N \widetilde{R}_{N-1}^{(1)}[\mu_1, \vec{\varepsilon}], \tag{4.9}$$

$$f_1[h] = \sum_{|\nu|\le N-1} \pi_\nu[f_1] \vec{\varepsilon}^{\,\nu} + \|\vec{\varepsilon}\|^N R_{N-1}^{(1)}[f_1, \vec{\varepsilon}], \tag{4.10}$$

where $\|\widetilde{R}_{N-1}^{(1)}[\mu_1, \vec{\varepsilon}]\|_{L^\infty(0,T;L^2)} + \|R_{N-1}^{(1)}[f_1, \vec{\varepsilon}]\|_{L^\infty(0,T;L^2)} \le C$, with C is a constant depending only on $N, T, f_1, \mu_1, u_\gamma$, $|\gamma| \le N$.

Proof. (i) In the case of $N = 1$, the proof of (4.9) is easy, hence we omit the details. We only prove with $N \ge 2$. We write $h = u_0 + \sum_{1\le|\gamma|\le N} u_\gamma \vec{\varepsilon}^{\,\gamma} \equiv u_0 + h_1$.

Using Taylor's expansion of the function $\mu_1[h] = \mu_1[u_0 + h_1]$ around the point u_0 up to order N, we obtain from (4.2) that

$$\begin{aligned}
\mu_1[u_0 + h_1] &= \mu_1[u_0] + \sum_{m=1}^{N-1} \frac{1}{m!} D_3^m \mu_1[u_0] h_1^m + \frac{1}{(N-1)!} \int_0^1 (1-\theta)^{N-1} D_3^N \mu_1[u_0 + \theta h_1] h_1^N d\theta \\
&= \mu_1[u_0] + \sum_{m=1}^{N-1} \frac{1}{m!} D_3^m \mu_1[u_0] \sum_{m \le |\nu| \le mN} T_\nu^{(m)}[u] \vec{\varepsilon}^{\,\nu} + \widetilde{R}_{N-1}^{(1)}[\mu_1, h_1] \\
&= \mu_1[u_0] + \sum_{m=1}^{N-1} \frac{1}{m!} D_3^m \mu_1[u_0] \sum_{m \le |\nu| \le N-1} T_\nu^{(m)}[u] \vec{\varepsilon}^{\,\nu} \\
&\quad + \sum_{m=1}^{N-1} \frac{1}{m!} D_3^m \mu_1[u_0] \sum_{N \le |\nu| \le mN} T_\nu^{(m)}[u] \vec{\varepsilon}^{\,\nu} + \widetilde{R}_{N-1}^{(1)}[\mu_1, h_1],
\end{aligned} \tag{4.11}$$

where

$$\widetilde{R}_{N-1}^{(1)}[\mu_1, h_1] = \frac{1}{(N-1)!} \int_0^1 (1-\theta)^{N-1} D_3^N \mu_1[u_0 + \theta h_1] h_1^N d\theta. \tag{4.12}$$

We note that

$$\sum_{m=1}^{N-1} \frac{1}{m!} D_3^m \mu_1[u_0] \sum_{m \le |\nu| \le N-1} T_\nu^{(m)}[u] \vec{\varepsilon}^{\,\nu} = \sum_{1 \le |\nu| \le N-1} \left(\sum_{m=1}^{|\nu|} \frac{1}{m!} D_3^m \mu_1[u_0] T_\nu^{(m)}[u] \right) \vec{\varepsilon}^{\,\nu}. \tag{4.13}$$

On the other hand, if we put

$$\widetilde{R}_{N-1}^{(1)}[\mu_1, \vec{\varepsilon}] = \|\vec{\varepsilon}\|^{-N} \left(\sum_{m=1}^{N-1} \frac{1}{m!} D_3^m \mu_1[u_0] \sum_{N \le |\nu| \le mN} T_\nu^{(m)}[u] \vec{\varepsilon}^{\,\nu} + \widetilde{R}_{N-1}^{(1)}[\mu_1, h_1] \right), \tag{4.14}$$

then by the boundedness of the functions $u_\gamma, \nabla u_\gamma, u'_\gamma$, $|\gamma| \le N$ in the function space $L^\infty(0,T;H^1)$, we obtain from (4.3), (4.12), and (4.14) that $\|\widetilde{R}_{N-1}^{(1)}[\mu_1, \vec{\varepsilon}]\|_{L^\infty(0,T;L^2)} \le C$, with and C is a constant depending only on N, T, μ_1, u_γ, $|\gamma| \le N$. Therefore, we obtain from (4.5), (4.11), (4.13), and (4.14) that

$$\begin{aligned}
\mu_1[u_0 + h_1] &= \mu_1[u_0] + \sum_{1 \le |\nu| \le N-1} \left(\sum_{m=1}^{|\nu|} \frac{1}{m!} D_3^m \mu_1[u_0] T_\nu^{(m)}[u] \right) \vec{\varepsilon}^{\,\nu} + \|\vec{\varepsilon}\|^N \widetilde{R}_{N-1}^{(1)}[\mu_1, \vec{\varepsilon}] \\
&= \sum_{|\nu| \le N-1} \rho_\nu[\mu_1] \vec{\varepsilon}^{\,\nu} + \|\vec{\varepsilon}\|^N \widetilde{R}_{N-1}^{(1)}[\mu_1, \vec{\varepsilon}].
\end{aligned} \tag{4.15}$$

Hence, (4.9) in Lemma 4.2 is proved.

(ii) We also only prove (4.10) with $N \geq 2$. Using Taylor's expansion of the function $f_1[u_0 + h_1]$ around the point u_0 up to order $N+1$, we obtain from (4.2) that

$$\begin{aligned}
f_1[u_0+h_1] &= f_1[u_0] + D_3 f_1[u_0]h_1 + D_4 f_1[u_0]\nabla h_1 + D_5 f_1[u_0]h_1' \\
&\quad + \sum_{\substack{2\le|m|\le N-1\\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \frac{1}{m!} D^m f_1[u_0] h_1^{m_1} (\nabla h_1)^{m_2} \left(h_1'\right)^{m_3} + R_{N-1}^{(1)}[f_1,h_1] \\
&= f_1[u_0] + D_3 f_1[u_0]h_1 + D_4 f_1[u_0]\nabla h_1 + D_5 f_1[u_0]h_1' \\
&\quad + \sum_{\substack{2\le|m|\le N-1\\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{|m|\le|\nu|\le|m|N} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N)\\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f_1[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}\left[u'\right] \vec{\varepsilon}^{\nu} \\
&\quad + R_{N-1}^{(1)}[f_1,h_1] \\
&= f_1[u_0] + D_3 f_1[u_0]h_1 + D_4 f_1[u_0]\nabla h_1 + D_5 f_1[u_0]h_1' \\
&\quad + \sum_{\substack{2\le|m|\le N-1\\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{|m|\le|\nu|\le N-1} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N)\\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f_1[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}\left[u'\right] \vec{\varepsilon}^{\nu} \\
&\quad + \sum_{\substack{2\le|m|\le N-1\\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{N\le|\nu|\le|m|N} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N)\\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f_1[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}\left[u'\right] \vec{\varepsilon}^{\nu} \\
&\quad + R_{N-1}^{(1)}[f_1,h_1],
\end{aligned} \tag{4.16}$$

where

$$R_{N-1}^{(1)}[f_1,h_1] = \sum_{\substack{|m|=N\\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \frac{N}{m!} \int_0^1 (1-\theta)^{N-1} D^m f_1[u_0+\theta h_1] h_1^{m_1} (\nabla h_1)^{m_2} \left(h_1'\right)^{m_3} d\theta. \tag{4.17}$$

We also note that

$$\begin{aligned}
&f_1[u_0] + D_3 f_1[u_0]h_1 + D_4 f_1[u_0]\nabla h_1 + D_5 f_1[u_0]h_1' \\
&\quad + \sum_{\substack{2\le|m|\le N-1\\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{|m|\le|\nu|\le N-1} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N)\\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f_1[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}\left[u'\right] \vec{\varepsilon}^{\nu} \\
&= f_1[u_0] + \sum_{\substack{1\le|m|\le N-1\\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{|m|\le|\nu|\le N-1} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N)\\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f_1[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}\left[u'\right] \vec{\varepsilon}^{\nu}
\end{aligned}$$

$$= f_1[u_0] + \sum_{1\le|\nu|\le N-1} \sum_{\substack{1\le|m|\le|\nu| \\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N) \\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f_1[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}[u'] \vec{\varepsilon}^{\,\nu}$$

$$= \sum_{|\nu|\le N-1} \pi_\nu[f_1] \vec{\varepsilon}^{\,\nu}. \tag{4.18}$$

Similarly,

$$\sum_{\substack{2\le|m|\le N-1 \\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{N\le|\nu|\le|m|N} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N) \\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f_1[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}[u'] \vec{\varepsilon}^{\,\nu}$$
$$+ R_{N-1}^{(1)}[f_1, h_1] = \|\vec{\varepsilon}\|^N R_{N-1}^{(1)}[f_1, \vec{\varepsilon}], \tag{4.19}$$

where $\|R_{N-1}^{(1)}[f_1, \vec{\varepsilon}]\|_{L^\infty(0,T;L^2)} \le C$, with C is a constant depending only on $N, T, f_1, u_\gamma,\ |\gamma| \le N$.

Then, (4.10) holds. Lemma 4.2 is proved. □

Remark 4.3. Lemma 4.2 is a generalization of the formula given in [17, page 262, formula (4.38)], and it is useful to obtain Lemma 4.4 below. These lemmas are the key to the asymptotic expansion of a weak solution $u = u(\varepsilon_1, \varepsilon_2)$ of order $N + 1$ in two small parameters $\varepsilon_1, \varepsilon_2$.

By $u_{\vec{\varepsilon}} = u(\varepsilon_1, \varepsilon_2) \in W_1(M, T)$ as a unique weak solution of $(P_{\vec{\varepsilon}})$, $v = u_{\vec{\varepsilon}} - \sum_{|\gamma|\le N} u_\gamma \vec{\varepsilon}^{\,\gamma} \equiv u_{\vec{\varepsilon}} - h$ satisfies the problem

$$\begin{aligned} v'' - \frac{\partial}{\partial x}\left(\mu_{\varepsilon_1}[v+h] v_x\right) &= \varepsilon_2\left(f_1[v+h] - f_1[h]\right) + \varepsilon_1 \frac{\partial}{\partial x}\left[\left(\mu_1[v+h] - \mu_1[h]\right) h_x\right] \\ &\quad + E_{\vec{\varepsilon}}(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ v(0,t) &= v(1,t) = 0, \\ v(x,0) &= v'(x,0) = 0, \\ \mu_{\varepsilon_1}[v] &= \mu_0 + \varepsilon_1 \mu_1[v] = \mu_0(x,t) + \varepsilon_1 \mu_1(x,t,v), \\ f_1[v] &= f_1(x,t,v,v_x,v'), \quad \mu_1[v] = \mu_1(x,t,v), \end{aligned} \tag{4.20}$$

where

$$E_{\vec{\varepsilon}}(x,t) = \varepsilon_2 f_1[h] + \varepsilon_1 \frac{\partial}{\partial x}\left[\left(\mu_1[h] - \mu_1[u_0]\right) h_x\right] - \sum_{1\le|\gamma|\le N} F_\gamma \vec{\varepsilon}^{\,\gamma}. \tag{4.21}$$

Lemma 4.4. *Let* $(H_1), (H_4)$ *and* (H_5) *hold. Then*

$$\|E_{\vec{\varepsilon}}\|_{L^\infty(0,T;L^2)} \le E_* \|\vec{\varepsilon}\|^{N+1}, \tag{4.22}$$

where E_* *is a constant depending only on* $N, T, f_0, f_1, \mu_0, \mu_1, u_\gamma,\ |\gamma| \le N$.

Proof. We only need prove with $N \geq 2$.

Using (4.9) for the function $\mu_1[h]$, we obtain

$$\mu_1[h] = \mu_1[u_0] + \sum_{1\leq|\nu|\leq N-1} \rho_\nu[\mu_1]\overrightarrow{\varepsilon}^{\nu} + \|\overrightarrow{\varepsilon}\|^N \widetilde{R}^{(1)}_{N-1}[\mu_1, \overrightarrow{\varepsilon}]. \tag{4.23}$$

By (4.6), (4.8), we write

$$\begin{aligned}
\varepsilon_1(\mu_1[h] - \mu_1[u_0]) &= \sum_{1\leq|\nu|\leq N-1} \rho_\nu[\mu_1]\varepsilon_1\overrightarrow{\varepsilon}^{\nu} + \varepsilon_1\|\overrightarrow{\varepsilon}\|^N \widetilde{R}^{(1)}_{N-1}[\mu_1, \overrightarrow{\varepsilon}] \\
&= \sum_{2\leq|\nu|\leq N, \nu_1\geq 1} \rho_{\nu_1-1,\nu_2}[\mu_1]\overrightarrow{\varepsilon}^{\nu} + \varepsilon_1\|\overrightarrow{\varepsilon}\|^N \widetilde{R}^{(1)}_{N-1}[\mu_1, \overrightarrow{\varepsilon}] \\
&= \sum_{2\leq|\nu|\leq N} \rho^{(1)}_\nu[\mu_1]\overrightarrow{\varepsilon}^{\nu} + \varepsilon_1\|\overrightarrow{\varepsilon}\|^N \widetilde{R}^{(1)}_{N-1}[\mu_1, \overrightarrow{\varepsilon}].
\end{aligned} \tag{4.24}$$

On the other hand, from (4.24), we compute

$$\begin{aligned}
\varepsilon_1(\mu_1[h] - \mu_1[u_0])h_x &= \left(\sum_{2\leq|\nu|\leq N} \rho^{(1)}_\nu[\mu_1]\ \overrightarrow{\varepsilon}^{\nu} + \varepsilon_1\|\overrightarrow{\varepsilon}\|^N \widetilde{R}^{(1)}_{N-1}[\mu_1, \overrightarrow{\varepsilon}]\right)h_x \\
&= \left(\sum_{2\leq|\nu|\leq N} \rho^{(1)}_\nu[\mu_1]\overrightarrow{\varepsilon}^{\nu}\right)\sum_{|\alpha|\leq N} \nabla u_\alpha \overrightarrow{\varepsilon}^{\alpha} + \varepsilon_1\|\overrightarrow{\varepsilon}\|^N \widetilde{R}^{(1)}_{N-1}[\mu_1, \overrightarrow{\varepsilon}]h_x \\
&= \sum_{2\leq|\nu|\leq N, |\alpha|\leq N} \rho^{(1)}_\nu[\mu_1]\nabla u_\alpha \overrightarrow{\varepsilon}^{\nu+\alpha} + \|\overrightarrow{\varepsilon}\|^{N+1} \widetilde{R}^{(1)}_{N}[\mu_1, \overrightarrow{\varepsilon}] \\
&= \sum_{2\leq|\nu|\leq N, |\alpha|\leq N} \rho^{(1)}_\nu[\mu_1]\nabla u_\alpha \overrightarrow{\varepsilon}^{\nu+\alpha} + \|\overrightarrow{\varepsilon}\|^{N+1} \widetilde{R}^{(1)}_{N}[\mu_1, \overrightarrow{\varepsilon}] \\
&= \sum_{2\leq|\gamma|\leq 2N}\ \sum_{2\leq|\nu|\leq N, |\gamma-\nu|\leq N} \rho^{(1)}_\nu[\mu_1]\nabla u_{\gamma-\nu} \overrightarrow{\varepsilon}^{\gamma} + \|\overrightarrow{\varepsilon}\|^{N+1} \widetilde{R}^{(1)}_{N}[\mu_1, \overrightarrow{\varepsilon}] \\
&= \sum_{2\leq|\gamma|\leq N}\ \sum_{2\leq|\nu|\leq N, |\gamma-\nu|\leq N} \rho^{(1)}_\nu[\mu_1]\nabla u_{\gamma-\nu} \overrightarrow{\varepsilon}^{\gamma} \\
&\quad + \sum_{N+1\leq|\gamma|\leq 2N}\ \sum_{2\leq|\nu|\leq N, |\gamma-\nu|\leq N} \rho^{(1)}_\nu[\mu_1]\nabla u_{\gamma-\nu} \overrightarrow{\varepsilon}^{\gamma} + \|\overrightarrow{\varepsilon}\|^{N+1} \widetilde{R}^{(1)}_{N}[\mu_1, \overrightarrow{\varepsilon}] \\
&= \sum_{2\leq|\gamma|\leq N}\ \sum_{2\leq|\nu|\leq N, |\gamma-\nu|\leq N} \rho^{(1)}_\nu[\mu_1]\nabla u_{\gamma-\nu} \overrightarrow{\varepsilon}^{\gamma} + \|\overrightarrow{\varepsilon}\|^{N+1} \widetilde{R}^{(2)}_{N}[\mu_1, \overrightarrow{\varepsilon}] \\
&= \sum_{2\leq|\gamma|\leq N}\ \sum_{2\leq|\nu|\leq N, \nu\leq\gamma} \rho^{(1)}_\nu[\mu_1]\nabla u_{\gamma-\nu} \overrightarrow{\varepsilon}^{\gamma} + \|\overrightarrow{\varepsilon}\|^{N+1} \widetilde{R}^{(2)}_{N}[\mu_1, \overrightarrow{\varepsilon}],
\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}\widetilde{R}_N^{(1)}[\mu_1,\vec{\varepsilon}] &= \frac{\varepsilon_1}{\|\vec{\varepsilon}\|}\widetilde{R}_{N-1}^{(1)}[\mu_1,\vec{\varepsilon}]h_x,\\ \|\vec{\varepsilon}\|^{N+1}\widetilde{R}_N^{(2)}[\mu_1,\vec{\varepsilon}] &= \sum_{N+1\le|\gamma|\le 2N}\ \sum_{2\le|\nu|\le N,|\gamma-\nu|\le N}\rho_\nu^{(1)}[\mu_1]\nabla u_{\gamma-\nu}\vec{\varepsilon}^{\gamma} + \|\vec{\varepsilon}\|^{N+1}\widetilde{R}_N^{(1)}[\mu_1,\vec{\varepsilon}].\end{aligned} \tag{4.26}$$

Hence,

$$\begin{aligned}\varepsilon_1\frac{\partial}{\partial x}[(\mu_1[h]-\mu_1[u_0])h_x] &= \frac{\partial}{\partial x}\left[\sum_{2\le|\gamma|\le N}\ \sum_{2\le|\nu|\le N,\nu\le\gamma}\rho_\nu^{(1)}[\mu_1]\nabla u_{\gamma-\nu}\vec{\varepsilon}^{\gamma} + \|\vec{\varepsilon}\|^{N+1}\widetilde{R}_N^{(2)}[\mu_1,\vec{\varepsilon}]\right]\\ &= \sum_{2\le|\gamma|\le N}\ \sum_{2\le|\nu|\le N,\nu\le\gamma}\frac{\partial}{\partial x}\left[\rho_\nu^{(1)}[\mu_1]\nabla u_{\gamma-\nu}\right]\vec{\varepsilon}^{\gamma} + \|\vec{\varepsilon}\|^{N+1}\frac{\partial}{\partial x}\widetilde{R}_N^{(2)}[\mu_1,\vec{\varepsilon}].\end{aligned} \tag{4.27}$$

Similarly, we write

$$\begin{aligned}\varepsilon_2 f_1[h] &= \varepsilon_2\left(\sum_{|\nu|\le N-1}\pi_\nu[f_1]\vec{\varepsilon}^{\nu} + \|\vec{\varepsilon}\|^{N}R_{N-1}^{(1)}[f_1,\vec{\varepsilon}]\right)\\ &= \sum_{1\le|\nu|\le N}\pi_\nu^{(2)}[f_1]\vec{\varepsilon}^{\nu} + \|\vec{\varepsilon}\|^{N+1}\overline{R}_N^{(1)}[f_1,\vec{\varepsilon}],\end{aligned} \tag{4.28}$$

where $\overline{R}_N^{(1)}[f_1,\vec{\varepsilon}] = \varepsilon_2/\|\vec{\varepsilon}\|R_{N-1}^{(1)}[f_1,\vec{\varepsilon}]$ is bounded in the function space $L^\infty(0,T;L^2)$ by a constant depending only on $N,T,f_1,u_\gamma,\ |\gamma|\le N$.

Combining (4.4), (4.21), (4.27), and (4.28) yields

$$\begin{aligned}E_{\vec{\varepsilon}}(x,t) &= \varepsilon_2 f_1[h] + \varepsilon_1\frac{\partial}{\partial x}[(\mu_1[h]-\mu_1[u_0])h_x] - \sum_{1\le|\gamma|\le N}F_\gamma\vec{\varepsilon}^{\gamma}\\ &= \sum_{1\le|\gamma|\le N}\left\{\left[\pi_\nu^{(2)}[f_1] + \sum_{2\le|\nu|\le N,\nu\le\gamma}\frac{\partial}{\partial x}\left[\rho_\nu^{(1)}[\mu_1]\nabla u_{\gamma-\nu}\right]\right] - F_\gamma\right\}\vec{\varepsilon}^{\gamma}\\ &\quad + \|\vec{\varepsilon}\|^{N+1}\left(\overline{R}_N^{(1)}[f_1,\vec{\varepsilon}] + \frac{\partial}{\partial x}\widetilde{R}_N^{(2)}[\mu_1,\vec{\varepsilon}]\right)\\ &= \|\vec{\varepsilon}\|^{N+1}\left(\overline{R}_N^{(1)}[f_1,\vec{\varepsilon}] + \frac{\partial}{\partial x}\widetilde{R}_N^{(2)}[\mu_1,\vec{\varepsilon}]\right).\end{aligned} \tag{4.29}$$

By the boundedness of the functions $u_\gamma, \nabla u_\gamma, u'_\gamma,\ |\gamma| \le N$ in the function space $L^\infty(0,T;H^1)$, we obtain from (4.26) and (4.29) that

$$\|E_{\vec{\varepsilon}}\|_{L^\infty(0,T;L^2)} \le E_*\|\vec{\varepsilon}\|^{N+1}, \tag{4.30}$$

where E_* is a constant depending only on $N,T,f_0,f_1,\mu_0,\mu_1,u_\gamma,\ |\gamma|\le N$.

The proof of Lemma 4.4 is complete. □

Now, we consider the sequence of functions $\{v_m\}$ defined by

$$
\begin{gathered}
v_0 \equiv 0, \\
v''_{m-1} - \frac{\partial}{\partial x}\left(\mu_{\varepsilon_1}[v_{m-1}+h]v_{mx}\right) = \varepsilon_2\left(f_1[v_{m-1}+h] - f_i[h]\right) \\
+\varepsilon_1 \frac{\partial}{\partial x}\left[\left(\mu_1[v_{m-1}+h] - \mu_1[h]\right)h_x\right] + E_{\vec{\varepsilon}}(x,t), \quad 0<x<1,\ 0<t<T, \\
v_m(0,t) = v_m(1,t) = 0, \\
v_m(x,0) = v'_m(x,0) = 0, \quad m \geq 1.
\end{gathered} \tag{4.31}
$$

With $m = 1$, we have the problem

$$
\begin{gathered}
v''_1 - \frac{\partial}{\partial x}\left(\mu_{\varepsilon_1}[h]v_{1x}\right) = E_{\vec{\varepsilon}}(x,t), \quad 0<x<1,\ 0<t<T, \\
v_1(0,t) = v_1(1,t) = 0, \\
v_1(x,0) = v'_1(x,0) = 0.
\end{gathered} \tag{4.32}
$$

Multiplying two sides of $(4.32)_1$ by v'_1, we compute without difficulty from (4.22) that

$$
\begin{aligned}
\|v'_1(t)\|^2 + \left\|\sqrt{\mu_{1,\varepsilon_1}(t)}v_{1x}(t)\right\|^2 &= 2\int_0^t \langle E_{\vec{\varepsilon}}(s), v'_1(s)\rangle ds + \int_0^t ds \int_0^1 \mu'_{1,\varepsilon_1}(x,s)v^2_{1x}(x,s)dx \\
&\leq T E_*^2 \|\vec{\varepsilon}\|^{2N+2} + \int_0^t \|v'_1(s)\|^2 ds + \int_0^t ds \int_0^1 \left|\mu'_{1,\varepsilon_1}(x,s)\right| v^2_{1x}(x,s)dx,
\end{aligned} \tag{4.33}
$$

where $\mu_{1,\varepsilon_1}(x,t) = \mu_{\varepsilon_1}[h(x,t)] = \mu_0(x,t) + \varepsilon_1\mu_1(x,t,h(x,t))$. By

$$
\mu'_{1,\varepsilon_1}(x,t) = \mu'_0(x,t) + \varepsilon_1\left[D_2\mu_1(x,t,h(x,t)) + D_3\mu_1(x,t,h(x,t))h'(x,t)\right], \tag{4.34}
$$

we get

$$
\left|\mu'_{1,\varepsilon_1}(x,t)\right| \leq \widetilde{K}(\mu_0) + (1+M_*)\widetilde{K}_{M_*}(\mu_1) \equiv \zeta_0, \tag{4.35}
$$

with $M_* = (N+1)M$, $\widetilde{K}(\mu_0) = \|\mu_0\|_{C^1(\overline{Q}_{T^*})}$.

It follows from (4.33), (4.35) that

$$
\|v'_1(t)\|^2 + \mu_*\|v_{1x}(t)\|^2 \leq T E_*^2 \|\vec{\varepsilon}\|^{2N+2} + \int_0^t \|v'_1(s)\|^2 ds + \zeta_0 \int_0^t \|v_{1x}(s)\|^2 ds. \tag{4.36}
$$

Using Gronwall's lemma, (4.36) gives

$$\|v_1'\|_{L^\infty(0,T;L^2)} + \|v_{1x}\|_{L^\infty(0,T;L^2)} \le \left(1+\frac{1}{\sqrt{\mu_*}}\right)\sqrt{T}E_*\|\vec{\varepsilon}\|^{N+1}\exp\left[\frac{(\mu_*+\zeta_0)T}{2\mu_*}\right]. \tag{4.37}$$

We will prove that there exists a constant C_T, independent of m and $\vec{\varepsilon}$, such that

$$\|v_m'\|_{L^\infty(0,T;L^2)} + \|v_{mx}\|_{L^\infty(0,T;L^2)} \le C_T\|\vec{\varepsilon}\|^{N+1}, \quad \text{with } \|\vec{\varepsilon}\| \le \varepsilon^* < 1,\ \forall m. \tag{4.38}$$

Multiplying two sides of $(4.31)_1$ with v_m' and after integrating in t, we obtain without difficulty from (4.22) that

$$\begin{aligned}
\|v_m'(t)\|^2 + \mu_*\|v_{mx}(t)\|^2 \le{}& TE_*^2\|\vec{\varepsilon}\|^{2N+2} + \int_0^t \|v_m'(s)\|^2 ds + \int_0^t ds\int_0^1 \left|\mu_{m,\varepsilon_1}'(x,s)\right| v_{mx}^2(x,s)dx \\
&+ 2\varepsilon_2\int_0^t \|f_1[v_{m-1}+h]-f_1[h]\|\,\|v_m'(s)\|ds \\
&+ 2\varepsilon_1\int_0^t \left\|\frac{\partial}{\partial x}\left[(\mu_1[v_{m-1}+h]-\mu_1[h])h_x\right]\right\|\,\|v_m'(s)\|ds \\
={}& TE_*^2\|\vec{\varepsilon}\|^{2N+2} + \int_0^t \|v_m'(s)\|^2 ds + \widehat{J}_1(t)+\widehat{J}_2(t)+\widehat{J}_3(t),
\end{aligned} \tag{4.39}$$

where $\mu_{m,\varepsilon_1}(x,t) = \mu_{\varepsilon_1}[v_{m-1}+h] = \mu_0(x,t)+\varepsilon_1\mu_1(x,t,v_{m-1}(x,t)+h(x,t))$. We will estimate the integrals on the right-hand side of (4.39) as follows.

First Integral $\widehat{J}_1(t)$

We have

$$\mu_{m,\varepsilon_1}'(x,t) = \mu_0'(x,t) + \varepsilon_1\left[D_2\mu_1(x,t,v_{m-1}+h) + D_3\mu_1(x,t,v_{m-1}+h)\left(v_{m-1}'+h'\right)\right], \tag{4.40}$$

hence

$$\left|\mu_{m,\varepsilon_1}'(x,t)\right| \le \widetilde{K}(\mu_0) + (1+M_{1*})\widetilde{K}_{M_{1*}}(\mu_1) \equiv \chi_1, \quad \text{with } M_{1*} = (N+2)M. \tag{4.41}$$

It follows from (4.41) that

$$\widehat{J}_1(t) = \int_0^t ds\int_0^1 \left|\mu_{m,\varepsilon_1}'(x,s)\right| v_{mx}^2(x,s)dx \le \chi_1\int_0^t \|v_{mx}(s)\|^2 ds. \tag{4.42}$$

Second Integral $\widehat{J}_2(t)$

We note that

$$\left\| f_1[v_{m-1}+h] - f_1[h] \right\| \leq 2K_{M_{1*}}(f_1) \|v_{m-1}\|_{W_1(T)}. \tag{4.43}$$

Therefore,

$$\widehat{J}_2(t) = 2\varepsilon_2 \int_0^t \left\| f_1[v_{m-1}+h] - f_1[h] \right\| \left\| v_m'(s) \right\| ds \leq T\chi_2^2 \|\overrightarrow{\varepsilon}\|^2 \|v_{m-1}\|_{W_1(T)}^2 + \int_0^t \left\| v_m'(s) \right\|^2 ds, \tag{4.44}$$

where $\chi_2 = \chi_2(M_{1*}, f_1) = 2K_{M_{1*}}(f_1)$.

Third Integral $\widehat{J}_3(t)$

First, we need to estimate $\|\partial/\partial x[(\mu_1[v_{m-1}+h] - \mu_1[h])h_x]\|$.

From the equation

$$\begin{aligned} &\frac{\partial}{\partial x}\left[(\mu_1[v_{m-1}+h] - \mu_1[h])h_x\right] \\ &\quad = (\mu_1[v_{m-1}+h] - \mu_1[h])h_{xx} + \frac{\partial}{\partial x}(\mu_1[v_{m-1}+h] - \mu_1[h])h_x \\ &\quad = (\mu_1[v_{m-1}+h] - \mu_1[h])h_{xx} + (D_1\mu_1[v_{m-1}+h] - D_1\mu_1[h])h_x \\ &\qquad + (D_3\mu_1[v_{m-1}+h] - D_3\mu_1[h])(\nabla v_{m-1} + \nabla h)h_x + D_3\mu_1[h]\nabla v_{m-1}h_x, \end{aligned} \tag{4.45}$$

it implies that

$$\begin{aligned} &\left\| \frac{\partial}{\partial x}\left[(\mu_1[v_{m-1}+h] - \mu_1[h])h_x\right] \right\| \\ &\quad \leq \left\| \mu_1[v_{m-1}+h] - \mu_1[h] \right\|_{C^0(\overline{\Omega})} \|h_{xx}\| \\ &\qquad + \left\| D_1\mu_1[v_{m-1}+h] - D_1\mu_1[h] \right\|_{C^0(\overline{\Omega})} \|h_x\| \\ &\qquad + \left\| D_3\mu_1[v_{m-1}+h] - D_3\mu_1[h] \right\|_{C^0(\overline{\Omega})} \|\nabla v_{m-1} + \nabla h\|_{C^0(\overline{\Omega})} \|h_x\| \\ &\qquad + \left\| D_3\mu_1[h] \right\|_{C^0(\overline{\Omega})} \|v_{m-1}\|_{W_1(T)} \|h_x\|_{C^0(\overline{\Omega})}. \end{aligned} \tag{4.46}$$

On the other hand, we have

$$\begin{gathered} \left\| \mu_1[v_{m-1}+h] - \mu_1[h] \right\|_{C^0(\overline{\Omega})} \leq \widetilde{K}_{M_{1*}}(\mu_1) \|v_{m-1}\|_{W_1(T)}, \\ \left\| D_j\mu_1[v_{m-1}+h] - D_j\mu_1[h] \right\|_{C^0(\overline{\Omega})} \leq \widetilde{K}_{M_{1*}}(\mu_1) \|v_{m-1}\|_{W_1(T)}, \quad j = 1, 3, \\ \left\| D_3\mu_1[h] \right\|_{C^0(\overline{\Omega})} \leq \widetilde{K}_{M_{1*}}(\mu_1). \end{gathered} \tag{4.47}$$

We deduce from (4.46) and (4.47) that

$$\left\| \frac{\partial}{\partial x} \left[\left(\mu_1[v_{m-1}+h] - \mu_1[h] \right) h_x \right] \right\| \le (3+2M_{1*}) M_{1*} \widetilde{K}_{M_{1*}}(\mu_1) \|v_{m-1}\|_{W_1(T)}. \tag{4.48}$$

Next, by (4.48), it follows that

$$\begin{aligned} \widehat{J}_3(t) &= 2\varepsilon_1 \int_0^t \left\| \frac{\partial}{\partial x} \left[\left(\mu_1[v_{m-1}+h] - \mu_1[h] \right) h_x \right] \right\| \left\| v_m'(s) \right\| ds \\ &\le T\chi_3^2 \|\vec{\varepsilon}\|^2 \|v_{m-1}\|_{W_1(T)}^2 + \int_0^t \left\| v_m'(s) \right\|^2 ds, \end{aligned} \tag{4.49}$$

where $\chi_3 = \chi_3(M_{1*}, \mu_1) = (3+2M_{1*}) M_{1*} \widetilde{K}_{M_{1*}}(\mu_1)$.

Combining (4.39), (4.42), (4.44), and (4.49) gives

$$\begin{aligned} \|v_m'(t)\|^2 + \mu_* \|v_{mx}(t)\|^2 &\le T E_*^2 \|\vec{\varepsilon}\|^{2N+2} + T\left(\chi_2^2 + \chi_3^2\right) \|\vec{\varepsilon}\|^2 \|v_{m-1}\|_{W_1(T)}^2 \\ &\quad + 3\int_0^t \left\| v_m'(s) \right\|^2 ds + \chi_1 \int_0^t \|v_{mx}(s)\|^2 ds \\ &\le T E_*^2 \|\vec{\varepsilon}\|^{2N+2} + T\left(\chi_2^2 + \chi_3^2\right) \|\vec{\varepsilon}\|^2 \|v_{m-1}\|_{W_1(T)}^2 \\ &\quad + \left(3 + \frac{\chi_1}{\mu_*}\right) \int_0^t \left(\left\| v_m'(s) \right\|^2 + \mu_* \|v_{mx}(s)\|^2 \right) ds. \end{aligned} \tag{4.50}$$

Using Gronwall's lemma, we deduce from (4.50) that

$$\|v_m\|_{W_1(T)} \le \sigma_T \|v_{m-1}\|_{W_1(T)} + \delta, \quad \forall m \ge 1, \tag{4.51}$$

where

$$\sigma_T = \sqrt{\chi_2^2 + \chi_3^2}\,\eta_T, \quad \delta = \eta_T E_* \|\vec{\varepsilon}\|^{N+1}, \quad \eta_T = \left(1 + \frac{1}{\sqrt{\mu_*}}\right) \sqrt{T} \exp\left[\frac{T}{2}\left(3 + \frac{\chi_1}{\mu_*}\right)\right]. \tag{4.52}$$

We can assume that

$$\sigma_T < 1, \tag{4.53}$$

with sufficiently small $T > 0$.

Lemma 4.5. *Let the sequence* $\{\zeta_m\}$ *satisfy*

$$\zeta_m \le \sigma \zeta_{m-1} + \delta \quad \forall m \ge 1, \ \zeta_0 = 0, \tag{4.54}$$

where $0 \le \sigma < 1, \delta \ge 0$ *are the given constants. Then,*

$$\zeta_m \leq \frac{\delta}{(1-\sigma)} \quad \forall m \geq 1. \tag{4.55}$$

This lemma is useful, as it will be said below, and it is easy to prove.

Applying Lemma 4.5 with $\zeta_m = \|v_m\|_{W_1(T)}, \sigma = \sigma_T = \sqrt{\chi_2^2 + \chi_3^2}\eta_T < 1, \delta = \eta_T E_* \|\overrightarrow{\varepsilon}\|^{N+1}$, it follows from (4.55) that

$$\left\|v_m'\right\|_{L^\infty(0,T;L^2)} + \|v_{mx}\|_{L^\infty(0,T;L^2)} = \|v_m\|_{W_1(T)} \leq \frac{\delta}{(1-\sigma_T)} \equiv C_T \left\|\overrightarrow{\varepsilon}\right\|^{N+1}. \tag{4.56}$$

On the other hand, the linear recurrent sequence $\{v_m\}$ defined by (4.31) converges strongly in the space $W_1(T)$ to the solution v of the problem (4.20). Hence, letting $m \to +\infty$ in (4.56) yields

$$\left\|v'\right\|_{L^\infty(0,T;L^2)} + \|v_x\|_{L^\infty(0,T;L^2)} \leq C_T \left\|\overrightarrow{\varepsilon}\right\|^{N+1}, \tag{4.57}$$

it means that

$$\left\| u' - \sum_{|\gamma| \leq N} u_\gamma' \overrightarrow{\varepsilon}^\gamma \right\|_{L^\infty(0,T;L^2)} + \left\| u_x - \sum_{|\gamma| \leq N} u_{\gamma x} \overrightarrow{\varepsilon}^\gamma \right\|_{L^\infty(0,T;L^2)} \leq C_T \left\|\overrightarrow{\varepsilon}\right\|^{N+1}. \tag{4.58}$$

Consequently, we obtain the following theorem.

Theorem 4.6. *Let* $(H_1), (H_4)$ *and* (H_5) *hold. Then there exist constants* $M > 0$ *and* $T > 0$ *such that, for every* $\overrightarrow{\varepsilon}$, *with* $\|\overrightarrow{\varepsilon}\| \leq \varepsilon_* < 1$, *the problem* $(P_{\overrightarrow{\varepsilon}})$ *has a unique weak solution* $u = u_{\overrightarrow{\varepsilon}} \in W_1(M,T)$ *satisfying an asymptotic expansion up to order* $N+1$ *as in* (4.58), *where the functions* $u_\gamma, |\gamma| \leq N$ *are the weak solutions of the problems* (P_0), $(\widetilde{P}_\gamma)$, $1 \leq |\gamma| \leq N$, *respectively.*

The Problem with Many Small Parameters

Next, we note that the results as above still hold for the problem in p small parameters $\varepsilon_1, \ldots, \varepsilon_p$ as follows:

$$u_{tt} - \frac{\partial}{\partial x}\left[\left(\mu_0(x,t) + \sum_{i=1}^{p} \varepsilon_i \mu_i(x,t,u)\right) u_x\right]$$

$$= f_0(x,t) + \sum_{i=1}^{p} \varepsilon_i f_i(x,t,u,u_x,u_t), \quad 0 < x < 1,\ 0 < t < T, \tag{$\widehat{P}_{\overrightarrow{\varepsilon}}$}$$

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = \widetilde{u}_0(x), \qquad u_t(x,0) = \widetilde{u}_1(x).$$

For more detail, we also make the following assumptions:

$(\widehat{H}_4)$ $\mu \in C^2([0,1] \times \mathbb{R}_+)$, $\mu_i \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, $\mu_0 \geq \mu_* > 0$, $\mu_i \geq 0$, $i = 1,2,\ldots,p$,

$(\widehat{H}_5)$ $f_0 \in C^1([0,1] \times \mathbb{R}_+)$, $f_i \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, $i = 1,2,\ldots,p$.

For a multi-index $\alpha = (\alpha_1,\ldots,\alpha_p) \in \mathbb{Z}_+^p$, and $\vec{\varepsilon} = (\varepsilon_1,\ldots,\varepsilon_p) \in \mathbb{R}^p$, we also put

$$|\alpha| = \alpha_1 + \cdots + \alpha_p, \quad \alpha! = \alpha_1! \cdots \alpha_p!,$$
$$\|\vec{\varepsilon}\| = \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_p^2}, \quad \vec{\varepsilon}^{\alpha} = \varepsilon_1^{\alpha_1} \cdots \varepsilon_p^{\alpha_p}, \tag{4.59}$$
$$\alpha, \beta \in \mathbb{Z}_+^p, \quad \alpha \leq \beta \Longleftrightarrow \alpha_i \leq \beta_i \quad \forall i = 1,\ldots,p.$$

Let u_0 be a unique weak solution of the problem (P_0), which is $(\widehat{P}_{\vec{\varepsilon}})$ corresponding to $\vec{\varepsilon} = (0,\ldots,0)$. Let the sequence of weak solutions $u_\gamma, \gamma \in \mathbb{Z}_+^p, 1 \leq |\gamma| \leq N$ be defined by the problems $(\widetilde{P}_\gamma)$, in which $F_\gamma, \gamma \in \mathbb{Z}_+^p, 1 \leq |\gamma| \leq N$, are defined by suitable recurrent formulas. Then, the following similar theorem holds.

Theorem 4.7. *Let (H_1),$(\widehat{H}_4)$ and $(\widehat{H}_5)$ hold. Then there exist constants $M > 0$ and $T > 0$ such that, for every $\vec{\varepsilon}$, with $\|\vec{\varepsilon}\| \leq \varepsilon_* < 1$, the problem $(\widehat{P}_{\vec{\varepsilon}})$ has a unique weak solution $u = u_{\vec{\varepsilon}} \in W_1(M,T)$ satisfying an asymptotic estimation up to order $N + 1$ as follows:*

$$\left\| u' - \sum_{|\gamma| \leq N} u_\gamma' \vec{\varepsilon}^{\gamma} \right\|_{L^\infty(0,T;L^2)} + \left\| u_x - \sum_{|\gamma| \leq N} u_{\gamma x} \vec{\varepsilon}^{\gamma} \right\|_{L^\infty(0,T;L^2)} \leq C_T \left\| \vec{\varepsilon} \right\|^{N+1}. \tag{4.60}$$

The proof of Theorem 4.7 is similar the one as above let us omit it.

Remark 4.8. Typical examples about asymptotic expansion of solutions in a small parameter can be found in the research of many authors such as [1, 3, 4, 8, 9, 17–19]. However, to our knowledge, in the case of asymptotic expansion in many small parameters, there is only partial results, for example, [5–7, 14], concerning asymptotic expansion of solutions in two or three small parameters.

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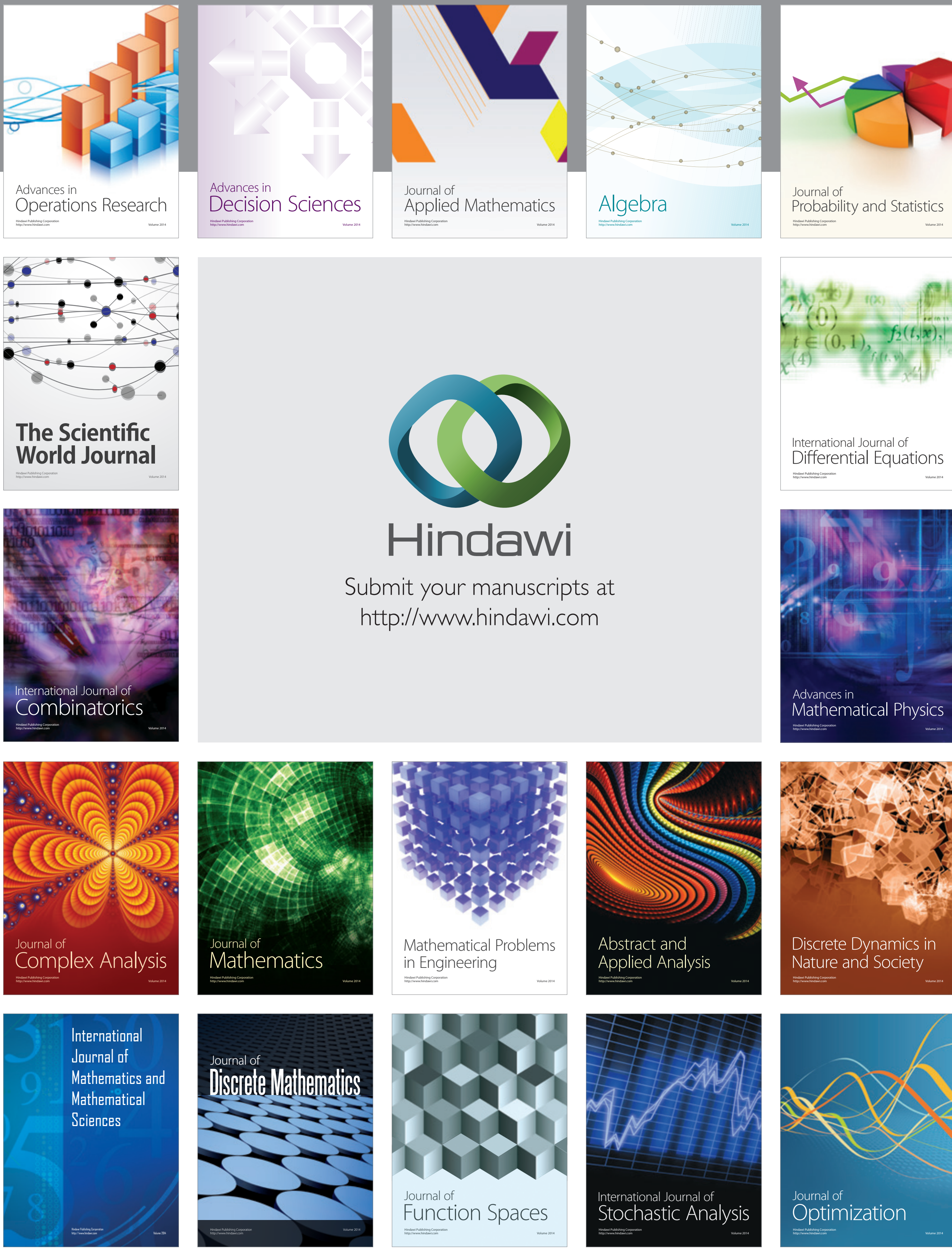
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