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Boundedness of solutions of forced isochronous oscillators with singularity at resonance

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Full list of author information is available at the end of the article**Abstract**

In this paper we study the second-order periodic system: $x'' + V'(x) + p(t)g(x) = F_x(x, t)$ where $V(x)$ has a singularity. Under some assumptions on the V, F, g , and p , by Ortega's small twist theorem, we obtain the existence of quasi-periodic solutions and boundedness of all the solutions.

Keywords: isochronous oscillators; repulsive singularity; small twist theorem; quasi-periodic solutions; boundedness of solution

1 Introduction and main result

The question of boundedness of all solutions of

$$x'' + V'(x, t) = 0, \quad (1.1)$$

initiated by Littlewood [1], has been the subject of numerous studies. Morris [2] has shown that when $V'(x, t) = x^3 - p(t)$, where $p(t)$ is continuous, all solutions are bounded. Levi [3] considered (1.1), where the potential V satisfies a superquadratic growth and has a singularity. The author reached a similar conclusion as in [1]. In [4], Liu has proved boundedness of all the solutions of the following equation with an asymmetric nonlinearity

$$x'' + \alpha x^+ - \beta x^- = p(t),$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, $\alpha, \beta > 0$. For this case, all the solutions of the unperturbed equation

$$x'' + \alpha x^+ - \beta x^- = 0, \quad (1.2)$$

have a common period $T(1/\sqrt{a} + 1/\sqrt{b})$, which means that (1.2) is isochronous.

In [5], Bonheure *et al.* study the following equation, which is a perturbation of the isochronous oscillator:

$$x'' + V'(x) + g(x) - p(t) = 0, \quad (1.3)$$

where $V(x)$, defined on $(a, +\infty)$, has a repulsive singularity and satisfies

$$\lim_{n \rightarrow +\infty} \frac{2V(x)}{x^2} = \frac{m^2}{4}, \quad \lim_{n \rightarrow a^+} V(x) = +\infty,$$

m is a positive integer, $a \in (-\infty, 0)$. The perturbation $g(x) - p(t)$ satisfies the Lazer-Landsman condition

$$4g^+ > \max_{\theta} p_*(\theta),$$

where the function $g(x)$ is smooth and bounded, $\lim_{n \rightarrow +\infty} g(x) = g^+$, $p_*(\theta) = \int_0^{2\pi} p(t + \theta) |\sin(mt/2)| dt$ and $p(t) = p(t + 2\pi)$. The authors assume that all solutions of the unperturbed equation

$$x'' + V'(x) = 0 \tag{1.4}$$

are $2\pi/m$, which means that (1.4) is an isochronous oscillator with period $2\pi/m$. The authors proved the existence of 2π -periodic solutions of (1.3). We refer for more details on the isochronous system to [5] and the references therein.

Recently, Liu [6] studied the quasi-periodic solutions and boundedness of all solutions of the isochronous oscillators equation (1.3) where $V(x)$ has a singularity, the nonlinearity $g(x)$ is a bounded perturbation, and $p(t)$ is 2π -periodic. Moreover, the following assumptions hold:

- (1) $V(0) = V'(0) = 0$, $V''(x) > 0$ for $x \neq 0$ and $\lim_{x \rightarrow a^+} V(x) = +\infty$, $a \in (-\infty, 0)$, and $V(x)$ is defined on $(a, +\infty)$.
- (2) The function

$$W(x) := \frac{V(x)}{V'(x)}$$

is smooth in $(-1, \infty)$ and the limit $\lim_{x \rightarrow -1} W(x)$ exists. Furthermore, the following estimates hold: for each $1 \leq k \leq 6$ there is a constant c_0 such that

$$|W(x)| \leq c_0(1+x), \quad |W^k(x)| \leq c_0, \quad \text{for } x \in [-1, \infty).$$

- (3) The positive function V is smooth and, for $0 \leq k \leq 6$,

$$|(1+x)^k V^k(x)| \leq c'_0 V(x),$$

where c'_0 is a positive constant.

- (4) For $x > 0$, let $\Phi(x) = V(x) - \frac{1}{8}x^2$; the function Φ satisfies

$$\lim_{x \rightarrow +\infty} x^k \Phi^k(x) = 0,$$

for every positive integer k . By the results in [5], the auxiliary autonomous system $x'' + V'(x) = 0$ is an isochronous system with period $T = 2\pi$.

(5) The function g is bounded on the interval $[-1, +\infty)$ and $g(x) > 0$ for $x > 0$. Moreover, the following inequalities hold:

$$\lim_{x \rightarrow +\infty} (1+x)^k \frac{d^k}{dx^k} g(x) = 0.$$

(6) If limit $\lim_{x \rightarrow +\infty} g(x) = g^+$ exists and $p^*(\theta) = \int_0^{2\pi} p(t+\theta) |\sin(t/2)| dt$, the following condition of Lazer-Landesman type holds:

$$4g^+ > \max_{\theta} p^*(\theta).$$

Liu first reduced the system to a normal form and then applied a variant of Moser’s twist theorem of invariant curves to prove the existence of quasi-periodic solution and the boundedness of all solutions. This result relies on the fact that the nonlinearity can guarantee the twist condition of the KAM theorem. The assumptions (5) and (6) satisfy the Lazer-Landesman condition which plays a key role in the boundedness problem. In fact, it has been shown by Alonso and Ortega [7] that, when the Lazer-Landesman condition cannot be satisfied, the solutions with large initial conditions are unbounded either in the past or in the future.

We observe that in [5, 6], the perturbation $g(x) - p(t)$ is smooth and bounded, so a natural question is to find sufficient conditions on $g(x)$ and $p(t)$ such that all solutions of (1.3) are bounded when the perturbation is unbounded. The purpose of this paper is to deal with this problem. We can refer to more papers on the Littlewood Problem on unbounded perturbation such as [8, 9].

Motivated by the papers [5, 6], we consider the following equation:

$$x'' + V'(x) + p(t)g(x) = F_x(x, t), \tag{1.5}$$

where $0 < \alpha < 1$. We suppose that (1)-(4) hold; moreover, we have

(5') $F(x, t) \in C^{7,6}(S^1 \times R)$ and

$$\lim_{x \rightarrow +\infty} \frac{F(x, t)}{|x|^{\alpha} x} = 0, \quad F(x, t + 2\pi) = F(x, t).$$

(6') $p(t) = p(t + 2\pi)$ and

$$\lim_{x \rightarrow +\infty} \frac{g(x)}{|x|^{\alpha}} = 1, \quad G(x) = \int_0^x g(s) ds,$$

where $0 < \alpha < 1$.

(7') We mention a condition similar to the Lazer-Landesman condition: $\int_0^{2\pi} \sin \frac{\theta}{2} \times |\sin \frac{\theta}{2}|^{\alpha} p(t_0 + \theta) d\theta > 0$.

Our main result is the following theorem.

Theorem 1 *Suppose the assumptions (1)-(4) and (5')-(7') hold, then (1.5) has infinitely many quasi-periodic solutions and all the solutions of (1.5) are bounded.*

The main idea of our proof is acquired from [6]. The proof of Theorem 1 is based on the small twist theorem due to Ortega [10]. It mainly consists of two steps. The first one is to transform (1.5) into a perturbation a integrable Hamilton system. The second one is to show that the Poincaré map of the equivalent system satisfies Ortega’s twist theorem, then the desired result can be obtained.

2 The proof of the theorem

2.1 Action-angle variables

Observe that (1.5) is equivalent to the following Hamiltonian system:

$$x' = \frac{\partial H}{\partial x}, \quad y' = -\frac{\partial H}{\partial x} \tag{2.1}$$

with the Hamiltonian function

$$H(x, y, t) = \frac{1}{2}y^2 + V(x) + p(t)G(x) - F(x, t).$$

In order to introduce action and angle variables, we first consider the auxiliary autonomous equation:

$$x' = y, \quad y' = -V'(x), \tag{2.2}$$

which is an integrable Hamiltonian system with Hamiltonian function

$$H_1(x, y, t) = \frac{1}{2}y^2 + V(x).$$

The closed curves $H_1(x, y, t) = h > 0$ are just the integral curves of (2.2).

Denote by $T(h)$ the time period of the integral curve Γ_h of (2.2) defined by $H_1(x, y, t) = h$ and by I the area enclosed by the closed curve Γ_h for every $h > 0$. Let $-1 < -\alpha_h < \beta_h$ be such that $V(-\alpha_h) = V(\beta_h) = h$. Then by assumption (1), we have

$$\lim_{h \rightarrow +\infty} (\alpha_h) = 1, \quad \lim_{h \rightarrow +\infty} (\beta_h) = +\infty.$$

It is easy to see that

$$I(h) = \int_{-\alpha_h}^{\beta_h} \sqrt{2(h - V(s))} ds, \quad \forall h > 0.$$

Let us denote

$$T_-(h) = \int_{-\alpha_h}^0 \frac{1}{\sqrt{2(h - V(s))}} ds, \quad T_+(h) = \int_0^{\beta_h} \frac{1}{\sqrt{2(h - V(s))}} ds.$$

By assumption (4), we know that the auxiliary autonomous equation is isochronous and we have the periodic $T(h) = T_-(h) + T_+(h) = 2\pi$, which yields $I(h) = 2\pi h$. Moreover, similar to the estimate in [5, 6], we have

$$|h^k T_+^{(k)}(h)| \leq C \cdot \frac{1}{\sqrt{h}}, \quad k > 0, \tag{2.3}$$

and

$$|h^k T_-^{(k)}(h)| \leq C \cdot \frac{1}{\sqrt{h}}, \quad k \geq 0. \tag{2.4}$$

We now carry out the standard reduction to the action-angle variables. For this purpose, we define the generating function $S(x, I) = \int_C \sqrt{2(h - V(s))} ds$, where C is the part of the closed curve Γ_h connecting the point on the y -axis and the point (x, y) .

We define the well-known map $(\theta, I) \rightarrow (x, y)$ by

$$y = \frac{\partial S}{\partial x}(x, I), \quad \theta = \frac{\partial S}{\partial I}(x, I),$$

which is symplectic, since

$$\begin{aligned} dx \wedge dy &= dx \wedge (S_{xx} dx + S_{xI} dI) = S_{xI} dx \wedge dI, \\ d\theta \wedge dI &= (S_{Ix} dx + S_{II} dI) \wedge dI = S_{Ix} dx \wedge dI. \end{aligned}$$

From the above discussion, we easily get

$$\begin{aligned} \theta &= \begin{cases} \int_{-\alpha_h}^x \frac{1}{\sqrt{2(h(x,y)-V(s))}} ds, & \text{if } y \geq 0, \\ 2\pi - \int_{-\alpha_h}^x \frac{1}{\sqrt{2(h(x,y)-V(s))}} ds, & \text{if } y < 0, \end{cases} \\ I(x, y) &= \int_{-\alpha_h}^{\beta_h} \sqrt{2(h(x, y) - V(s))} ds. \end{aligned}$$

In the new variables (θ, I) , the system (2.1) becomes

$$\theta' = \frac{\partial H}{\partial I}, \quad I' = -\frac{\partial H}{\partial \theta}, \tag{2.5}$$

where

$$H(\theta, I, t) = I + 2\pi p(t)G(x(\theta, I)) - 2\pi F(x(\theta, I), t). \tag{2.6}$$

2.2 New action and angle variables

Now we are concerned with the Hamiltonian system (2.5) with the Hamiltonian function $H(\theta, I, t)$ given by (2.6). Note that

$$I d\theta - H dt = -(H dt - I d\theta).$$

This means that if one can solve I from (2.6) as a function of H (θ and t as parameters), then

$$\frac{dH}{d\theta} = -\frac{\partial I}{\partial t}(t, H, \theta), \quad \frac{dt}{d\theta} = -\frac{\partial I}{\partial H}(t, H, \theta) \tag{2.7}$$

is also a Hamiltonian system with Hamiltonian function I and now the action, angle, and time variables are H , t , and θ .

Let

$$R(\theta, I, t) = 2\pi p(t)G(x(\theta, I)) - 2\pi F(x(\theta, I), t). \tag{2.8}$$

In order to estimate R , we need the estimate on the functions $x(I, \theta)$. For this purpose, we introduce the following lemma proved in [3, 6].

Lemma 1 *There is a constant C such that*

$$I^k \frac{\partial^k x}{\partial I^k} \leq C(1+x) \leq C\sqrt{I} \quad \text{for } 0 \leq k \leq 6.$$

By (5'), (2.8), and Lemma 1, we can directly get the following lemma.

Lemma 2 *The following estimates hold:*

$$\left| \frac{\partial^{k+l} R(\theta, I, t)}{\partial I^k \partial t^l} \right| < I^{\frac{\alpha+1}{2}}$$

for $k+l \leq 6$.

Using (2.6), we have

$$\frac{\partial H}{\partial I} = 1 + \frac{\partial R}{\partial I} \rightarrow 1, \quad \text{as } I \rightarrow +\infty.$$

Hence, by the implicit function theorem, there exists a function $R_1 = R_1(t, H, \theta)$ such that

$$I = H - R_1(t, H, \theta).$$

The function R_1 is defined implicitly by

$$R_1(t, H, \theta) = R(\theta, H - R_1, t). \tag{2.9}$$

Now we give the estimates of $R_1(t, H, \theta)$. By a direct computation, we have the following.

Lemma 3 $|H^k \frac{\partial^{k+l} R_1(t, H, \theta)}{\partial H^k \partial t^l}| < H^{\frac{\alpha+1}{2}}$ for $k+l \leq 6$.

Proof Case $m = 0$. By (2.9), Lemma 2 and noticing that $\frac{H}{I} \rightarrow 1$ as $I \rightarrow +\infty$, we have

$$|R_1(t, H, \theta)| = |R(t, H - R_1, \theta)| \leq C \cdot |H - R_1|^{\frac{\alpha+1}{2}} \leq C \cdot |H|^{\frac{\alpha+1}{2}}.$$

Case $m = 1$. Taking derivatives on both sides of (2.9) with respect to H , we have

$$\frac{\partial R_1}{\partial H} = \frac{1}{\pi} \frac{1}{1 + \frac{1}{\frac{\partial R}{\partial I}}}.$$

By Lemma 2, we have

$$\left| \frac{\partial R}{\partial I} \right| \leq C \cdot I^{-\frac{1+\alpha}{2}} \leq C \cdot H^{-\frac{1+\alpha}{2}}.$$

Since

$$\left| \frac{1}{\frac{\partial R}{\partial H}} \right| \gg 1 \quad \text{as } H \rightarrow +\infty,$$

we have

$$\left| H \frac{\partial R_1}{\partial H} \right| \leq C \cdot H^{\frac{1+\alpha}{2}}.$$

We suppose that

$$\left| H^m \frac{\partial^m R_1(H, t, \theta)}{\partial H^m} \right| \leq H^{\frac{\alpha+1}{2}} \tag{2.10}$$

holds where $m = k - 1$. We will prove that (2.10) also holds when $m = k, 1 \leq k \leq 6$.

By direct calculation, we have

$$\frac{\partial^k R_1}{\partial H^k} = \frac{\sum c_{nj_1 \dots j_n} \frac{\partial^n R}{\partial I^n} \frac{\partial^{j_1}}{\partial H^{j_1}} (H - R_1) \dots \frac{\partial^{j_n}}{\partial H^{j_n}} (H - R_1)}{1 + \frac{\partial R}{\partial I}}, \tag{2.11}$$

where $1 \leq n \leq k, j_1 + \dots + j_n = k, 1 \leq j_1, \dots, j_n < k$.

By (2.10), we have

$$\left| \frac{\partial^\sigma (H - R_1)}{\partial H^\sigma} \right| \leq C \cdot \left| \frac{\partial (H - R_1)}{\partial H} \right|^\sigma, \tag{2.12}$$

where $1 < \sigma \leq k - 1$.

By (2.11), (2.12), and Lemma 2, we have

$$\left| \frac{\partial^k R_1}{\partial H^k} \right| \leq C \cdot \left| \frac{\partial^k R}{\partial I^k} \left(\frac{\partial (H - R_1)}{\partial H} \right)^k \right| \leq C \cdot \left| \frac{\partial^k R}{\partial I^k} \right| \leq C \cdot H^{\frac{\alpha+1}{2}-k},$$

then we proved that (2.10) holds when $m = k$. Thus we proved Lemma 3. □

Set

$$\begin{aligned} R_2(t, H, \theta) &= R(\theta, H, t) - R_1(t, H, \theta) \\ &= - \int_0^1 \frac{\partial R}{\partial I}(\theta, H - sR_1, t) R_1 ds. \end{aligned}$$

By similarity to the proofs of Lemmas 2 and 3, we have the following.

Lemma 4 *The following estimates hold:*

$$\left| \frac{\partial^{k+l} R_2(t, H, \theta)}{\partial H^k \partial t^l} \right| < H^\alpha$$

for $k + l \leq 6$.

Remark 1 By the proofs and estimates of R, R_1, R_2 , we can easily see that R_1 is the main (twist) term of R and R_2 is the small term of R .

Now the new Hamiltonian function $I = I(t, H, \theta)$ is written in the form

$$\begin{aligned} I &= H - R_1(\theta, H, t) + R_2(t, H, \theta) \\ &= H - 2\pi p(t)G(x(\theta, H)) + 2\pi F(x(\theta, H), t) + R_2. \end{aligned}$$

The system (2.7) is of the form

$$\begin{cases} \frac{dt}{d\theta} = \frac{\partial I}{\partial H} = 1 - 2\pi \frac{\partial x}{\partial H}(\theta, H)(g(x(\theta, H))p(t) - F_x(x(\theta, H), t)) + \frac{\partial R_2}{\partial H}(t, H, \theta), \\ \frac{dH}{d\theta} = -\frac{\partial I}{\partial t} = 2\pi p'(t)G(x(\theta, H)) - 2\pi F_t(x(\theta, H), t) - \frac{\partial R_2}{\partial t}(t, H, \theta). \end{cases} \quad (2.13)$$

Introduce a new action variable $\rho \in [1, 2]$ and a parameter $\epsilon > 0$ by $H = \epsilon^{-2}\rho$. Then $H \gg 1 \Leftrightarrow 0 < \epsilon \ll 1$. Under this transformation, the system (2.13) is changed into the form

$$\begin{cases} \frac{dt}{d\theta} = 1 - 2\pi \frac{\partial x}{\partial H}(\theta, \epsilon^{-2}\rho)(g(\theta, \epsilon^{-2}\rho)p(t) - F_x(x(\theta, \epsilon^{-2}\rho), t)) + \frac{\partial R_2}{\partial H}(t, \epsilon^{-2}\rho, \theta), \\ \frac{d\rho}{d\theta} = 2\pi \epsilon^2 p'(t)G(x(\theta, \epsilon^{-2}\rho)) - 2\pi \epsilon^2 F_t(x(\theta, \rho), t) - \epsilon^2 \frac{\partial R_2}{\partial t}(t, \epsilon^{-2}\rho, \theta), \end{cases} \quad (2.14)$$

which is also a Hamiltonian system, with the new Hamiltonian function

$$\Gamma(t, \rho, \theta; \epsilon) = \rho - 2\pi \epsilon^{-2} p(t)G(x(\theta, \epsilon^{-2}\rho)) + 2\pi \epsilon^{-2} F(x(\theta, \epsilon^{-2}\rho), t) + \epsilon^{-2} \frac{\partial R_2}{\partial t}(t, \epsilon^{-2}\rho, \theta).$$

Obviously, if $\epsilon \ll 1$, the solution $(t(\theta, t_0, \rho_0), \rho(\theta, t_0, \rho_0))$ of (2.14) with the initial date $(t_0, \rho_0) \in R \times [1, 2]$ is defined in the interval $\theta \in [0, 2\pi]$ and $\rho(\theta, t_0, \rho_0) \in [\frac{1}{2}, 3]$. So the Poincaré map of (2.14) is well defined in the domain $R \times [1, 2]$.

Lemma 5 ([6, Lemma 5.1]) *The Poincaré map of (2.14) has the intersection property.*

The proof is similar to the corresponding one in [6].

For convenience we introduce the notation $O_k(1)$ and $o_k(1)$. We say a function $f(t, \rho, \theta, \epsilon) \in O_k(1)$ if f is smooth in (t, ρ) and for $k_1 + k_2 \leq k$,

$$\left| \frac{\partial^{k_1+k_2}}{\partial t^{k_1} \partial \rho^{k_2}} f(t, \rho, \theta, \epsilon) \right| \leq C,$$

for some constant $C > 0$ which is independent of the arguments $t, \rho, \theta, \epsilon$.

Similarly, we say $f(t, \rho, \theta, \epsilon) \in o_k(1)$ if f is smooth in (t, ρ) and for $k_1 + k_2 \leq k$,

$$\lim_{\epsilon \rightarrow 0} \left| \frac{\partial^{k_1+k_2}}{\partial t^{k_1} \partial \rho^{k_2}} f(t, \rho, \theta, \epsilon) \right| = 0,$$

uniformly in (t, ρ, θ) .

2.3 Poincaré map and twist theorems

We will use Ortega's small twist theorem to prove that the Poincaré map P has an invariant closed curve, if ϵ is sufficiently small. The earlier result was due to Moser [11–13]. Later, Orgeta [10] improved the previous results. Let us first recall the theorem in [10].

Lemma 6 (Ortega’s Theorem) *Let $A = \mathbb{S}^1 \times [a, b]$ be a finite cylinder with universal cover $\mathbb{A} = \mathbb{R} \times [a, b]$. The coordinate in \mathbb{A} is denoted by (τ, ν) . Consider the map*

$$\bar{f} : A \rightarrow \mathbb{S} \times \mathbb{R}.$$

We assume that the map has the intersection property. Suppose that $f : A \rightarrow \mathbb{R} \times \mathbb{R}$, $(\tau_0, \nu_0) \rightarrow (\tau_1, \nu_1)$ is a lift of \bar{f} and it has the form

$$\begin{cases} \tau_1 = \tau_0 + 2N\pi + \delta l_1(\tau_0, \nu_0) + \delta \tilde{g}_1(\tau_0, \nu_0), \\ \nu_1 = \nu_0 + \delta l_2(\tau_0, \nu_0) + \delta \tilde{g}_2(\tau_0, \nu_0), \end{cases} \quad (2.15)$$

where N is an integer, $\delta \in (0, 1)$ is a parameter. The functions l_1, l_2, \tilde{g}_1 , and \tilde{g}_2 satisfy

$$\begin{aligned} l_1 \in C^6(A), \quad l_1(\tau_0, \nu_0) > 0, \quad \frac{\partial l_1}{\partial \nu_0}(\tau_0, \nu_0) > 0, \quad \forall (\tau_0, \nu_0) \in A, \\ l_2(\cdot, \cdot), \tilde{g}_1(\cdot, \cdot, \epsilon), \tilde{g}_2(\cdot, \cdot, \epsilon) \in C^5(A). \end{aligned} \quad (2.16)$$

In addition, we assume that there is a function $I : A \rightarrow \mathbb{R}$ satisfying

$$I \in C^6(A), \quad \frac{\partial I}{\partial \nu_0}(\tau_0, \nu_0) > 0, \quad \forall (\tau_0, \nu_0) \in A \quad (2.17)$$

and

$$l_1(\tau_0, \nu_0) \cdot \frac{\partial I}{\partial \tau_0}(\tau_0, \nu_0) + l_2(\tau_0, \nu_0) \cdot \frac{\partial I}{\partial \nu_0}(\tau_0, \nu_0) = 0, \quad \forall (\tau_0, \nu_0) \in A. \quad (2.18)$$

Moreover, suppose that there are two numbers \tilde{a} and \tilde{b} such that $a < \tilde{a} < \tilde{b} < b$ and

$$I_M(a) < I_m(\tilde{a}) \leq I_M(\tilde{a}) < I_m(\tilde{b}) \leq I_M(\tilde{b}) < I_m(b) \quad (2.19)$$

where

$$I_M(r) = \max_{\rho \in \mathbb{S}^1} I(\rho_0, \tau_0), \quad I_m(r) = \min_{\rho \in \mathbb{S}^1} I(\rho_0, \tau_0).$$

Then there exist $\epsilon > 0$ and $\Delta > 0$ such that, if $\delta < \Delta$ and

$$\|\tilde{g}_1(\cdot, \cdot, \epsilon)\|_{C^5(A)} + \|\tilde{g}_2(\cdot, \cdot, \epsilon)\|_{C^5(A)} < \epsilon,$$

the mapping \bar{f} has an invariant curve in Γ_A . The constant ϵ is independent of δ .

We make the ansatz that the solution of (2.14) with the initial condition $(t(0), \rho(0)) = (t_0, \rho_0)$ is of the form

$$t = t_0 + \theta + \epsilon^{1-\alpha} \Sigma_1(t_0, \rho_0, \theta; \epsilon), \quad \rho = \rho_0 + \epsilon^{1-\alpha} \Sigma_2(t_0, \rho_0, \theta; \epsilon).$$

Then the Poincaré map of (2.14) is

$$P : t_1 = t_0 + 2\pi + \epsilon^{1-\alpha} \Sigma_1(t_0, \rho_0, 2\pi; \epsilon), \quad \rho_1 = \rho_0 + \epsilon^{1-\alpha} \Sigma_2(t_0, \rho_0, 2\pi; \epsilon). \quad (2.20)$$

The functions Σ_1 and Σ_2 satisfy

$$\begin{cases} \Sigma_1 = 2\pi \epsilon^{\alpha-1} \int_0^\theta \frac{\partial x}{\partial H}(\theta, \epsilon^{-2}\rho)(g(x)p(t) - F_x(x(\theta, \epsilon^{-2}\rho), t)) d\theta \\ \quad + \epsilon^{\alpha-1} \int_0^\theta \frac{\partial R_2}{\partial H}(\theta, \epsilon^{-2}\rho) d\theta, \\ \Sigma_2 = 2\pi \epsilon^{\alpha+1} \int_0^\theta G(x(\theta, \epsilon^{-2}\rho))p'(t) - F_t(x(\theta, \epsilon^{-2}\rho), t) d\theta \\ \quad - \epsilon^{\alpha+1} \int_0^\theta \frac{\partial R_2}{\partial t}(\theta, \epsilon^{-2}\rho) d\theta, \end{cases} \quad (2.21)$$

where $t = t_0 + \theta + \epsilon^{1-\alpha}\Sigma_1$, $\rho = \rho_0 + \epsilon^{1-\alpha}\Sigma_2$. By Lemmas 1 and 3, we know that

$$|\Sigma_1| + |\Sigma_2| \leq C \quad \text{for } \theta \in [0, 2\pi]. \quad (2.22)$$

Moreover we can prove that

$$\Sigma_1, \Sigma_2 \in O_6(1). \quad (2.23)$$

Lemma 7 *The following estimates hold:*

$$\begin{aligned} G(x(\theta, \epsilon^{-2}\rho)) - G(x(\theta, \epsilon^{-2}\rho_0)) &\in \epsilon^{-2\alpha} O_6(1), \\ g(x(\theta, \epsilon^{-2}\rho)) \frac{\partial x}{\partial H} - g(x(\theta, \epsilon^{-2}\rho_0)) \frac{\partial x}{\partial H} &\in \epsilon^{2-2\alpha} O_6(1). \end{aligned}$$

Proof We only consider $x \geq 0$, since the case $x \leq 0$ can be proved similarly.

Let

$$\begin{aligned} \Delta(t - \dots_0, \rho_0, \theta) &= G(x(\theta, \epsilon^{-2}\rho)) - G(x(\theta, \epsilon^{-2}\rho_0)) \\ &= \int_0^1 g(x(\theta, \epsilon^{-2}\rho_0 + s\epsilon^{-1-\alpha}\Sigma_2)) \\ &\quad \cdot \frac{\partial x}{\partial H}(\theta, \epsilon^{-2}\rho_0 + s\epsilon^{-1-\alpha}\Sigma_2) \epsilon^{-1-\alpha}\Sigma_2 ds. \end{aligned}$$

By (6'), Lemma 2 and (2.21), we have

$$\begin{aligned} |\Delta(t - \dots_0, \rho_0, \theta)| &\leq C \cdot (\epsilon^{-2}\rho_0 + s\epsilon^{-1-\alpha}\Sigma_2)^{\frac{\alpha}{2}} (\epsilon^{-2}\rho_0 + s\epsilon^{-1-\alpha}\Sigma_2)^{-\frac{1}{2}} \epsilon^{-1-\alpha}\Sigma_2 \\ &\leq C \cdot (\epsilon^{-2}\rho_0 + s\epsilon^{-1-\alpha}\Sigma_2)^{\frac{\alpha-1}{2}} \epsilon^{-1-\alpha} \\ &\leq C \cdot \epsilon^{-2\alpha}. \end{aligned}$$

Taking the derivative with respect to ρ_0 in both sides of $\Delta(t - \dots_0, \rho_0, \theta)$, we have

$$\begin{aligned} \frac{\partial \Delta}{\partial \rho_0} &= \int_0^1 \left[g'(x) \frac{\partial x}{\partial H} \frac{1 + s\epsilon^{-1-\alpha} \frac{\partial \Sigma_2}{\partial \rho_0}}{\epsilon^2} \frac{\partial x}{\partial H} \epsilon^{-1-\alpha}\Sigma_2 + g(x) \frac{\partial^2 x}{\partial H^2} \frac{1 + s\epsilon^{-1-\alpha} \frac{\partial \Sigma_2}{\partial \rho_0}}{\epsilon^2} \epsilon^{-1-\alpha}\Sigma_2 \right. \\ &\quad \left. + g(x) \frac{\partial x}{\partial H} \frac{\partial \Sigma_2}{\partial \rho_0} \epsilon^{-1-\alpha} \right] ds. \end{aligned}$$

Similar to the estimates of $|\Delta|$ and noticing $|\Delta| \leq C \cdot \epsilon^{-2\alpha}$, we have

$$\left| \frac{\partial \Delta}{\partial \rho_0} \right| \leq C \cdot \epsilon^{-2\alpha}.$$

Now we will estimate $\frac{\partial^n \Delta}{\partial \rho^n}$, $n \geq 2$. By direct calculation, we have

$$\frac{\partial^n \Delta}{\partial \rho^n} = \int_0^1 \sum \frac{\partial^{\Sigma_{jk_1}} g(x)}{\partial \rho^{\Sigma_{jk_1}}} \cdot \frac{\partial^{\Sigma_{jk_2}} x_H}{\partial \rho^{\Sigma_{jk_2}}} \cdot \frac{\partial^{\Sigma_{jk_3}} \Phi}{\partial \rho^{\Sigma_{jk_3}}} ds, \tag{2.24}$$

where $\Phi = \epsilon^{-1-\alpha} \Sigma_2$, $\Sigma_{jkl} = 1$, $n \geq 2$, $k = 1, 2, \dots, n$, $l = 1, 2, 3$. For estimating $\frac{\partial^n \Delta}{\partial \rho^n}$, we need the estimates of x_H and $g(x)$.

We firstly give the estimates of x_H . By direct calculation, we get

$$\frac{\partial^n x_H}{\partial \rho^n} = \sum \frac{\partial^k g(x)}{\partial H^k} \cdot \frac{\partial^{i_1} \Psi}{\partial \rho^{i_1}} \cdots \frac{\partial^{i_k} \Psi}{\partial \rho^{i_k}}, \tag{2.25}$$

where $\Psi = \epsilon^{-2} + s\epsilon^{-1-\alpha} \frac{\partial(\Sigma_2)}{\partial \rho}$, $n \geq 1$, $k + i_1 + \dots + i_k = n$.

Since $\Psi = \epsilon^{-2} + s\epsilon^{-1-\alpha} \frac{\partial(\Sigma_2)}{\partial \rho}$, we have

$$\Psi \leq C \cdot \epsilon^{-2}, \quad \frac{\partial^\tau \Psi}{\partial \rho^\tau} \leq C \cdot \left(\frac{\partial \Psi}{\partial \rho} \right)^\tau, \quad \tau > 1. \tag{2.26}$$

By (2.25) and (2.26), we have

$$\frac{\partial^n x_H}{\partial \rho^n} \leq C \cdot \frac{\partial^n g(x)}{\partial H^n} (\Psi)^n \leq C \cdot \epsilon^{1+2n} \cdot (\epsilon^{-2})^n = C \cdot \epsilon. \tag{2.27}$$

We now give the estimates of $g(x)$. By (6'), (2.26), (2.27), and Lemma 1, we have

$$g(x) \leq C \cdot \epsilon^{-\alpha}, \quad \frac{\partial g(x)}{\partial \rho} = g'(x)x_H \Psi \leq C \cdot \epsilon^{1-\alpha} \cdot \epsilon \cdot \epsilon^{-\alpha} \leq C \cdot \epsilon^{-\alpha}.$$

We suppose that the following inequality holds:

$$\frac{\partial^{n-1} g(x)}{\partial \rho^{n-1}} \leq C \cdot \epsilon^{-\alpha}, \quad n \geq 1. \tag{2.28}$$

By direct calculation, we have

$$\frac{\partial^n g(x)}{\partial \rho^n} = \sum \frac{\partial^{j_1} g(x)}{\partial \rho^{j_1}} \cdot \frac{\partial^{j_2} x_H}{\partial \rho^{j_2}} \cdot \frac{\partial^{j_3} \Psi}{\partial \rho^{j_3}}, \tag{2.29}$$

where $j_i \geq 0$, $j_1 + j_2 + j_3 = n - 1$. By the estimates of Ψ (2.26), the estimates of x_H (2.27) and (2.28), (2.29), we have

$$\frac{\partial^n g(x)}{\partial \rho^n} \leq C \cdot \epsilon^{1-\alpha} \cdot \epsilon \cdot \epsilon^{-2} = C \cdot \epsilon^{-\alpha}. \tag{2.30}$$

Since we have the estimates of Ψ , x_H , and $g(x)$, now we can give the estimates of $\frac{\partial^n \Delta}{\partial \rho^n}$. By (2.26), (2.27), (2.30), and (2.24), we have the estimate of Δ :

$$\begin{aligned} \frac{\partial^n \Delta}{\partial \rho^n} &= \int_0^1 \sum \frac{\partial^{\Sigma_{jk_1}} g(x)}{\partial \rho^{\Sigma_{jk_1}}} \cdot \frac{\partial^{\Sigma_{jk_2}} x_H}{\partial \rho^{\Sigma_{jk_2}}} \cdot \frac{\partial^{\Sigma_{jk_3}} \Phi}{\partial \rho^{\Sigma_{jk_3}}} ds, \\ &\leq C \cdot \epsilon^{-\alpha} \cdot \epsilon \cdot \epsilon^{-1-\alpha} \\ &= \epsilon^{-2\alpha}, \end{aligned}$$

which means that

$$G(x(\theta, \epsilon^{-2}\rho)) - G(x(\theta, \epsilon^{-2}\rho_0)) \in \epsilon^{-2\alpha} O_6(1).$$

The estimates for $\frac{\partial x}{\partial H}(\theta, \epsilon^{-2}\rho)g(x(\theta, \epsilon^{-2}\rho)) - \frac{\partial x}{\partial H}(\theta, \epsilon^{-2}\rho_0)g(x(\theta, \epsilon^{-2}\rho_0))$ follow from a similar argument, and we omit it here. Thus Lemma 7 is proved. \square

Now we turn to giving an asymptotic expression of the Poincaré map of (2.13), that is, we study the behavior of the functions Σ_1 and Σ_2 at $\theta = 2\pi$ as $\epsilon \rightarrow 0$.

In order to estimate Σ_1 and Σ_2 , we introduce the following definition and lemma. Let $\Theta_+(I)$ be the subset of the interval $[0, 2\pi]$ such that $\theta \in \Theta_+(I)$, $x(\theta, I) > 0$ and $\Theta_-(I)$ satisfies $\theta \in \Theta_-(I)$, $x(\theta, I) < 0$. By (2.3) and (2.4), similar to the computation in [6], there is a function $\eta \in \epsilon O_6(1)$ such that $T_-(h) = T_-(\frac{I}{2\pi}) = 2\eta(t_0, \rho_0, \theta_0; \epsilon)$. By the definition of Θ_+ and Θ_- , we have

$$\text{measure } \Theta_- = 2\eta, \quad \text{measure } \Theta_+ = 2\pi - 2\eta. \tag{2.31}$$

Moreover we will use the following lemma.

Lemma 8 ([6, Lemma 4.1]) *For $\theta \in \Theta_+(I)$, the function x has the following expression:*

$$x(\theta, I) = 2\sqrt{\frac{I}{\pi}} \sin\left(\frac{\theta}{2} - \frac{T_-(h)}{4}\right) + \sqrt{\frac{I}{\pi}} X\left(\theta - \frac{T_-(h)}{4}, I\right),$$

and

$$\lim_{I \rightarrow +\infty} \sum_{k=1}^6 \left| I^{k-\frac{1}{2}} \left(\frac{\partial^k}{\partial I^k} \left(x(\theta, I) - 2\sqrt{\frac{I}{\pi}} \sin\left(\frac{\theta}{2} - \frac{T_-(h)}{4}\right) \right) \right) \right| = 0.$$

Now we can give the estimates of Σ_1 and Σ_2 .

Lemma 9

$$\Sigma_1(t_0, \rho_0, 2\pi; \epsilon) = -2^{\alpha+1} \pi^{\frac{1}{2}} \rho_0^{\frac{\alpha-1}{2}} \int_0^{2\pi} \sin \frac{\theta}{2} \left| \sin \frac{\theta}{2} \right|^\alpha p(t_0 + \theta) d\theta + o_6(1),$$

$$\Sigma_2(t_0, \rho_0, 2\pi; \epsilon) = -2^{2+\alpha} \pi^{\frac{1}{2}} \rho_0^{\frac{\alpha+1}{2}} \int_0^{2\pi} \sin \frac{\theta}{2} \left| \sin \frac{\theta}{2} \right|^\alpha p'(t_0 + \theta) d\theta + o_6(1).$$

Proof Firstly we consider Σ_1 . By Lemmas 4, 7, and (2.21), we have

$$\begin{aligned} & \Sigma_1(t_0, \rho_0, 2\pi; \epsilon) \\ &= -2\pi \epsilon^{\alpha-1} \int_0^{2\pi} \frac{\partial x}{\partial H}(\theta, \epsilon^{-2}\rho) (g(x(\theta, \epsilon^{-2}\rho))p(t) - F_x(x(\theta, \epsilon^{-2}\rho), t)) d\theta \\ & \quad + \epsilon^{\alpha-1} \int_0^{2\pi} \frac{\partial R_2}{\partial H}(\theta, \epsilon^{-2}\rho) d\theta \\ &= -2\pi \epsilon^{\alpha-1} \int_0^{2\pi} \frac{\partial x}{\partial H}(\theta, \epsilon^{-2}\rho_0) (g(x(\theta, \epsilon^{-2}\rho))p(t_0 + \theta) - F_x(x(\theta, \epsilon^{-2}\rho_0), t)) d\theta \end{aligned}$$

$$\begin{aligned}
 & + \epsilon^{1-\alpha} O_6(1) \\
 & = -2\pi \epsilon^{\alpha-1} \int_{\Theta_+} \frac{\partial x}{\partial H}(\theta, \epsilon^{-2} \rho_0) (g(x(\theta, \epsilon^{-2} \rho_0)) p(t_0 + \theta) - F_x(x(\theta, \epsilon^{-2} \rho_0), t)) d\theta \\
 & \quad - 2\pi \epsilon^{\alpha-1} \int_{\Theta_-} \frac{\partial x}{\partial H}(\theta, \epsilon^{-2} \rho_0) (g(x(\theta, \epsilon^{-2} \rho_0)) p(t_0 + \theta) - F_x(x(\theta, \epsilon^{-2} \rho_0), t)) d\theta \\
 & \quad + \epsilon^{1-\alpha} O_6(1).
 \end{aligned}$$

By Lemma 1, we know that when $x \in \Theta_-$

$$x((\theta, \epsilon^{-2} \rho_0)) \in O_6(1), \quad \frac{\partial x}{\partial H}(\theta, \epsilon^{-2} \rho_0) \in \epsilon^2 O_6(1). \tag{2.32}$$

By (2.32) and (6'), we have

$$\begin{aligned}
 \Sigma_1(t_0, \rho_0, 2\pi; \epsilon) & = -2\pi \epsilon^{\alpha-1} \int_{\Theta_+} \frac{\partial x}{\partial H}(\theta, \epsilon^{-2} \rho_0) (|x|^\alpha p(t_0 + \theta) - F_x(x(\theta, \epsilon^{-2} \rho_0), t)) d\theta \\
 & \quad + \epsilon^{1-\alpha} O_6(1).
 \end{aligned}$$

By Lemma 8, we have

$$\begin{aligned}
 \Sigma_1(t_0, \rho_0, 2\pi; \epsilon) & = -2\pi \epsilon^{\alpha-1} \int_{\Theta_+} \left[\frac{\epsilon}{\sqrt{\rho_0 \pi}} \sin\left(\frac{\theta}{2} - \frac{\eta}{4}\right) \right] \\
 & \quad \cdot \left[2 \left(\epsilon^{-1} \sqrt{\frac{\rho_0}{\pi}} \sin\left(\frac{\theta}{2} - \frac{\eta}{4}\right) \right)^\alpha p(t_0 + \theta) d\theta + \epsilon^{1-\alpha} O_6(1) \right] \\
 & = -2^{\alpha+1} \pi^{\frac{1-\alpha}{2}} \rho_0^{\frac{\alpha-1}{2}} \int_{\Theta_+} \sin \frac{\theta}{2} \left| \sin \frac{\theta}{2} \right|^\alpha p(t_0 + \theta) d\theta + o_6(1) \\
 & = -2^{\alpha+1} \pi^{\frac{1-\alpha}{2}} \rho_0^{\frac{\alpha-1}{2}} \int_0^{2\pi} \sin \frac{\theta}{2} \left| \sin \frac{\theta}{2} \right|^\alpha p(t_0 + \theta) d\theta + o_6(1).
 \end{aligned}$$

Now we consider Σ_2 :

$$\begin{aligned}
 \Sigma_2 & = 2\pi \epsilon^{\alpha+1} \int_0^{2\pi} G(x(\theta, \epsilon^{-2} \rho_0)) p'(t) d\theta - 2\pi \epsilon^{\alpha+1} \int_0^{2\pi} F_t(x(\theta, \epsilon^{-2} \rho_0), t) d\theta \\
 & \quad - \epsilon^{\alpha+1} \int_0^{2\pi} \frac{\partial R_2}{\partial t}(\theta, \epsilon^{-2} \rho_0) d\theta.
 \end{aligned}$$

By (6'), Lemmas 4 and 7, and noticing that

$$\lim_{x \rightarrow +\infty} \frac{F(x, t)}{|x|^{\alpha+1}} = 0,$$

we have

$$\begin{aligned}
 \Sigma_2 & = 2\pi \epsilon^{\alpha+1} \int_0^{2\pi} |x(\theta, \epsilon^{-2} \rho_0)|^\alpha x(\theta, \epsilon^{-2} \rho_0) p'(t_0 + \theta) + o_6(1) \\
 & = 2\pi \epsilon^{\alpha+1} \int_{\Theta_+} |x(\theta, \epsilon^{-2} \rho_0)|^\alpha x(\theta, \epsilon^{-2} \rho_0) p'(t_0 + \theta) \\
 & \quad + 2\pi \epsilon^{\alpha+1} \int_{\Theta_-} |x(\theta, \epsilon^{-2} \rho_0)|^\alpha x(\theta, \epsilon^{-2} \rho_0) p'(t_0 + \theta) + o_6(1).
 \end{aligned}$$

By Lemmas 1, 8, and (2.32)

$$\begin{aligned} \Sigma_2 &= 2\pi \epsilon^{\alpha+1} \int_{\Theta_+} |x(\theta, \epsilon^{-2}\rho_0)|^\alpha x(\theta, \epsilon^{-2}\rho_0) p'(t_0 + \theta) + o_6(1) \\ &= 2\pi \epsilon^{\alpha+1} \int_{\Theta_+} \left(2\epsilon^{-1} \sqrt{\frac{\rho_0}{\pi}} \sin\left(\frac{\theta}{2} - \frac{\eta}{4}\right) \right) \\ &\quad \cdot \left| 2\epsilon^{-1} \sqrt{\frac{\rho_0}{\pi}} \sin\left(\frac{\theta}{2} - \frac{\eta}{4}\right) \right|^\alpha \cdot p'(t) d\theta + o_6(1) \\ &= -2^{2+\alpha} \pi^{\frac{1-\alpha}{2}} \rho_0^{\frac{\alpha+1}{2}} \int_{\Theta_+} \sin \frac{\theta}{2} \left| \sin \frac{\theta}{2} \right|^\alpha p'(t_0 + \theta) d\theta + o_6(1) \\ &= -2^{2+\alpha} \pi^{\frac{1-\alpha}{2}} \rho_0^{\frac{\alpha+1}{2}} \int_0^{2\pi} \sin \frac{\theta}{2} \left| \sin \frac{\theta}{2} \right|^\alpha p'(t_0 + \theta) d\theta + o_6(1). \end{aligned}$$

Let

$$\begin{aligned} \Psi_1(t_0, \rho_0) &= -2^{\alpha+1} \pi^{\frac{1-\alpha}{2}} \rho_0^{\frac{\alpha-1}{2}} \int_0^{2\pi} \sin \frac{\theta}{2} \left| \sin \frac{\theta}{2} \right|^\alpha p(t_0 + \theta) d\theta, \\ \Psi_2(t_0, \rho_0) &= -2^{2+\alpha} \pi^{\frac{1-\alpha}{2}} \rho_0^{\frac{\alpha+1}{2}} \int_0^{2\pi} \sin \frac{\theta}{2} \left| \sin \frac{\theta}{2} \right|^\alpha p'(t_0 + \theta) d\theta. \end{aligned}$$

Then there are two functions ϕ_1 and ϕ_2 such that the Poincaré map of (2.14), given by (2.20), is of the form

$$P: \quad t_1 = t_0 + 2\pi + \epsilon^{1-\alpha} \Psi_1(t_0, \rho_0) + \epsilon^{1-\alpha} \phi_1, \quad \rho_1 = \rho_0 + \epsilon^{1-\alpha} \Psi_2(t_0, \rho_0) + \epsilon^{1-\alpha} \phi_2,$$

where $\phi_1, \phi_2 \in o_6(1)$.

Note that $p(t) > 0$,

$$\Psi_1 < 0, \quad \frac{\partial \Psi_1}{\partial \rho_0} > 0.$$

Let

$$L = \frac{\rho_0^{-\frac{\alpha+1}{2}}}{\int_0^{2\pi} \sin \frac{\theta}{2} \left| \sin \frac{\theta}{2} \right|^\alpha p(t_0 + \theta) d\theta}.$$

Then

$$\frac{\partial L}{\partial t_0} \Psi_1(t_0, \rho_0) + \frac{\partial L}{\partial \rho_0} \Psi_2(t_0, \rho_0) = 0.$$

The other assumptions of Ortega's theorem are easily verified. Hence, there is an invariant curve of P in the annulus $(t_0, \rho_0) \in S^1 \times [1, 2]$, which implies the boundedness of our original equation (1.5). Then Theorem 1 is proved. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The article is a joint work of two authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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