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Geometric interpretation of Blundon's inequality and Ciamberlini's inequality

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Abstract
In this paper, we present a geometric interpretation of Blundon's inequality and Ciamberlini's inequality. Our results provide a useful method for proving the inequalities concerning sides, circumradius, and inradius of a triangle. As applications, some improved inequalities are established to illustrate the effectiveness of the proposed method.
MSC: 26D15; 26D05

Keywords: Blundon's inequality; Ciamberlini's inequality; geometric interpretation; proving inequality

1 Introduction

Blundon's inequality states that, for any triangle with the circumradius R, the inradius r, and the semiperimeter s, it is true that (see [1])

$$\begin{aligned} &2R^2+10Rr-r^2-2(R-2r)\sqrt{R(R-2r)} \\ &\quad \le s^2 \le 2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}. \end{aligned} \tag{1}$$

The equality occurs in the left-side inequality if and only if the triangle is either equilateral or isosceles, having the vertex angle greater than $\pi/3$; the equality occurs in the right-side inequality if and only if the triangle is either equilateral or isosceles, having the vertex angle less than $\pi/3$.

Blundon's inequality expresses the necessary and sufficient conditions for the existence of a triangle with elements s, R, and r. In many references this inequality is called the fundamental triangle inequality.

Another fundamental inequality, related to non-obtuse triangle (or non-acute triangle), is known in the literature as Ciamberlini's inequality (see [2]). This inequality claims that, for any non-obtuse triangle, the inequality

$$s \ge 2R+r \tag{2}$$

holds true; inequality (2) is reverse for any non-acute triangle. The equality occurs in (2) if and only if the triangle is a right triangle.

Blundon's inequality and Ciamberlini's inequality have many applications in Euclidean geometry, particularly in the field of geometric inequalities. For more details we refer the reader to [3–8] and the references cited therein.

The main purpose of this paper is to present a geometric interpretation of Blundon's inequality and Ciamberlini's inequality. Also, we show some interesting applications of our results. This paper is organized as follows. Section 2 describes a geometric interpretation of Blundon's inequality and Ciamberlini's inequality. Section 3 gives some remarks on the geometric interpretation of Blundon's inequality and Ciamberlini's inequality, we display how to use the geometric interpretation of these inequalities to prove some geometric inequalities. Finally, Section 4 illustrates the applications of the results given in Section 2, some classical geometric inequalities such as Leuenberger's inequality, Walker's inequality, and Finsler-Hadwiger's inequality are improved. Moreover, an open problem proposed by Huang in [9] is also solved.

2 Geometric interpretation of Blundon's inequality and Ciamberlini's inequality

Theorem 1 *Let ΔABC be a triangle with circumcircle $\odot O$ and incircle $\odot I$, and let R, r, and s be the circumradius, inradius, and semiperimeter of the triangle, respectively. Then*

(i) *there exists an isosceles $\Delta A_1B_1C_1$ with vertex angle $A_1 = 2\arcsin(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2r}{R}})$ which inscribes the circumcircle $\odot O$, and which satisfies*

$$R_1 = R, \qquad r_1 = r, \qquad s_1 \le s, \tag{3}$$

where R_1, r_1, and s_1 are the circumradius, inradius, and the semiperimeter of $\Delta A_1B_1C_1$, respectively;

(ii) *there exists an isosceles $\Delta A_2B_2C_2$ with vertex angle $A_2 = 2\arcsin(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2r}{R}})$ which inscribes the circumcircle $\odot O$ and satisfies*

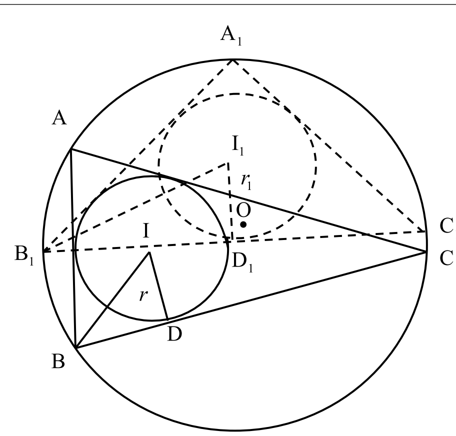
$$R_2 = R, \qquad r_2 = r, \qquad s_2 \ge s, \tag{4}$$

where R_2, r_2, and s_2 are the circumradius, inradius, and the semiperimeter of $\Delta A_2B_2C_2$, respectively.

Proof (i) As shown in the diagram (see Figure 1), we construct an isosceles $\Delta A_1B_1C_1$, inscribing the circumcircle $\odot O$, such that the vertex angle satisfies

$$A_1 = 2\arcsin\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2r}{R}}\right).$$

Figure 1 Proof of Theorem 1, part (i).



Since $\Delta A_1B_1C_1$ and ΔABC have common circumcircle $\odot O$, we conclude that

$$R_1 = R.$$

Next, we prove that the inradius of isosceles $\Delta A_1B_1C_1$ is equal to the inradius of ΔABC. By the law of sines we find that the congruent side lengths of $\Delta A_1B_1C_1$ is

$$B_1D_1 = \frac{1}{2}B_1C_1 = R\sin A_1.$$

Thus, we obtain

$$\begin{aligned} r_1 &= B_1D_1 \tan\frac{B_1}{2} = R\sin A_1 \tan\frac{B_1}{2} \\ &= R\sin A_1 \tan\frac{\pi - A_1}{4} \\ &= 2R\left(\sin\frac{A_1}{2}\cos\frac{A_1}{2}\right)\frac{1-\sin\frac{A_1}{2}}{\cos\frac{A_1}{2}} \\ &= 2R\sin\frac{A_1}{2}\left(1-\sin\frac{A_1}{2}\right). \end{aligned}$$

Substituting $A_1 = 2\arcsin(\frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{2r}{R}})$ into the above expression, it follows that

$$r_1 = 2R\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2r}{R}}\right)\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2r}{R}}\right) = r.$$

Finally, we shall verify that the semiperimeter of isosceles $\Delta A_1B_1C_1$ satisfies $s_1 \le s$. Since

$$\begin{aligned} s_1 &= A_1B_1 + B_1D_1 \\ &= 2R\cos\frac{A_1}{2} + 2R\cos\frac{A_1}{2}\sin\frac{A_1}{2} \\ &= 2R\left(1+\sin\frac{A_1}{2}\right)\cos\frac{A_1}{2} \end{aligned}$$

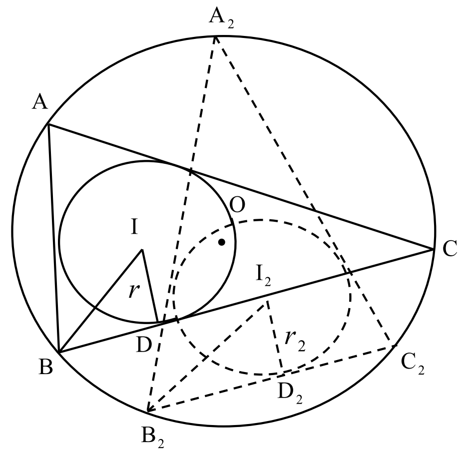
and

$$\sin\frac{A_1}{2} = \frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2r}{R}},$$

we get

$$\begin{aligned} s_1^2 &= 4R^2\left(\frac{3}{2}+\frac{1}{2}\sqrt{1-\frac{2r}{R}}\right)^2\left[1-\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2r}{R}}\right)^2\right] \\ &= \frac{1}{4}R^2\left(1-\sqrt{1-\frac{2r}{R}}\right)\left(3+\sqrt{1-\frac{2r}{R}}\right)^3 \\ &= 2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R(R-2r)}. \end{aligned}$$

Figure 2 Proof of Theorem 1, part (ii).



Applying the left-side Blundon's inequality (1) yields

$$s_1 \le s.$$

This proves the first part of Theorem 1.

(ii) By using the same method as in part (i) above, we construct an isosceles $\Delta A_2B_2C_2$, inscribing the circumcircle $\odot O$ (see Figure 2), such that the vertex angle satisfies

$$A_2 = 2\arcsin\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2r}{R}}\right).$$

Then we have

$$R_2 = R$$

and

$$r_2 = B_2D_2 \tan\frac{B_2}{2} = R\sin A_2 \tan\frac{\pi - A_2}{4} = 2R\sin\frac{A_2}{2}\left(1 - \sin\frac{A_2}{2}\right).$$

Substituting $A_2 = 2\arcsin(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2r}{R}})$ into the above expression, it follows that

$$r_2 = 2R\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2r}{R}}\right)\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{2r}{R}}\right) = r.$$

Next, we need to verify that the semiperimeter of isosceles $\Delta A_2B_2C_2$ satisfies $s_2 \ge s$. From

$$\begin{aligned} s_2 &= A_2B_2 + B_2D_2 \\ &= 2R\cos\frac{A_2}{2} + 2R\cos\frac{A_2}{2}\sin\frac{A_2}{2} \\ &= 2R\left(1 + \sin\frac{A_2}{2}\right)\cos\frac{A_2}{2} \end{aligned}$$

and

$$\sin\frac{A_2}{2}=\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2r}{R}},$$

we deduce that

$$\begin{aligned}s_2^2&=4R^2\left(\frac{3}{2}-\frac{1}{2}\sqrt{1-\frac{2r}{R}}\right)^2\left[1-\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{2r}{R}}\right)^2\right]\\&=\frac{1}{4}R^2\left(1+\sqrt{1-\frac{2r}{R}}\right)\left(3-\sqrt{1-\frac{2r}{R}}\right)^3\\&=2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}.\end{aligned}$$

Using the right-side Blundon's inequality (1) leads to

$$s_2\ge s.$$

The second part of Theorem 1 is proved. □

Theorem 2 *Let ΔABC be a non-obtuse triangle with circumcircle $\odot O$ and incircle $\odot I$, and let R, r, and s be the circumradius, inradius, and semiperimeter of the triangle, respectively.*

(i) *If $2r\le R<(\sqrt{2}+1)r$, then there exists an isosceles $\Delta A_1B_1C_1$ with vertex angle $A_1=2\arcsin(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2r}{R}})$ which inscribes the circumcircle $\odot O$ and satisfies*

$$R_1=R,\qquad r_1=r,\qquad s_1\le s,\tag{5}$$

where R_1, r_1, and s_1 are the circumradius, inradius, and the semiperimeter of $\Delta A_1B_1C_1$, respectively.

(ii) *If $R\ge(\sqrt{2}+1)r$, then there exists a right triangle $\Delta A_2B_2C_2$ with an acute angle $A_2=2\arctan(\frac{R-\sqrt{R^2-2Rr-r^2}}{2R+r})$ which inscribes the circumcircle $\odot O$ and satisfies*

$$R_2=R,\qquad r_2=r,\qquad s_2\le s,\tag{6}$$

where R_2, r_2, and s_2 are the circumradius, inradius, and the semiperimeter of $\Delta A_2B_2C_2$, respectively.

Proof The assertion in part (i) of Theorem 2 can be proved by using the same method as in the proof of Theorem 1, part (i), above.

We will now prove part (ii) of Theorem 2.

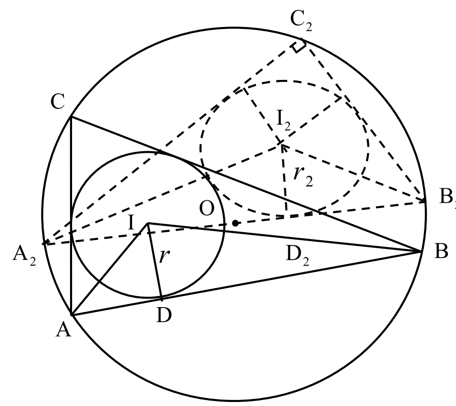
According to the assumption $R\ge(\sqrt{2}+1)r$, we can construct a right triangle $\Delta A_2B_2C_2$ inscribing the circumcircle $\odot O$ (see Figure 3), such that an acute angle satisfies

$$A_2=2\arctan\left(\frac{R-\sqrt{R^2-2Rr-r^2}}{2R+r}\right).$$

Since $\Delta A_2B_2C_2$ and ΔABC have common circumcircle $\odot O$, we conclude that

$$R_2=R.$$

Figure 3 Proof of Theorem 2, part (ii).



It is easily observed that

$$A_2B_2 = r_2\left(\cot\frac{A_2}{2} + \cot\frac{B_2}{2}\right) = r_2\left(\cot\frac{A_2}{2} + \cot\left(\frac{\pi}{4} - \frac{A_2}{2}\right)\right).$$

So, we have

$$\begin{aligned} r_2 &= \frac{A_2B_2}{\cot\frac{A_2}{2} + \cot(\frac{\pi}{4} - \frac{A_2}{2})} \\ &= \frac{2R}{\cot\frac{A_2}{2} + \cot(\frac{\pi}{4} - \frac{A_2}{2})} \\ &= \frac{2R(\tan\frac{A_2}{2} - \tan^2\frac{A_2}{2})}{1 + \tan^2\frac{A_2}{2}}. \end{aligned}$$

Now, from

$$\tan\frac{A_2}{2} = \frac{R - \sqrt{R^2 - 2Rr - r^2}}{2R + r},$$

it follows that

$$\begin{aligned} r_2 &= \frac{2R(\tan\frac{A_2}{2} - \tan^2\frac{A_2}{2})}{1 + \tan^2\frac{A_2}{2}} \\ &= \frac{2R(\frac{R-\sqrt{R^2-2Rr-r^2}}{2R+r} - (\frac{R-\sqrt{R^2-2Rr-r^2}}{2R+r})^2)}{1 + (\frac{R-\sqrt{R^2-2Rr-r^2}}{2R+r})^2} \\ &= r. \end{aligned}$$

Next, we verify that $s_2 \le s$.
In $\Delta A_2B_2C_2$, we have

$$\begin{aligned} s_2 &= \frac{A_2B_2 + B_2C_2 + C_2A_2}{2} \\ &= \frac{A_2B_2 + A_2B_2 + r_2 + r_2}{2} \end{aligned}$$

$$= 2R_2 + r_2$$
$$= 2R + r.$$

Using Ciamberlini's inequality (2) for non-obtuse triangles

$$s \geq 2R + r,$$

we obtain

$$s_2 = 2R + r \leq s.$$

The proof of Theorem 2 is completed. □

Theorem 3 *Let ΔABC be a non-acute triangle with circumcircle $\odot O$ and incircle $\odot I$, and let R, r, and s be the circumradius, inradius, and semiperimeter of the triangle, respectively. Then there exists a right triangle $\Delta A_1B_1C_1$ with an acute angle $A_1 = 2\arctan(\frac{R-\sqrt{R^2-2Rr-r^2}}{2R+r})$ which inscribes the circumcircle $\odot O$ and satisfies*

$$R_1 = R, \qquad r_1 = r, \qquad s_1 \geq s, \tag{7}$$

where R_1, r_1, and s_1 are the circumradius, inradius, and the semiperimeter of $\Delta A_1B_1C_1$, respectively.

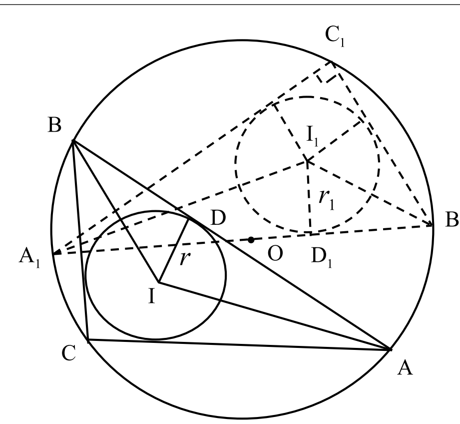
Proof Note that in any non-acute triangle we have the inequality (see [3])

$$R \geq (\sqrt{2} + 1)r.$$

This enables us to construct a right triangle $\Delta A_1B_1C_1$, inscribing the circumcircle $\odot O$ (see Figure 4), such that an acute angle satisfies

$$A_1 = 2\arctan\left(\frac{R - \sqrt{R^2 - 2Rr - r^2}}{2R + r}\right).$$

Figure 4 Proof of Theorem 3.



By using methods similar to those of Theorem 2, part (ii) together with an application of Ciamberlini's inequality for non-acute triangles, we can deduce that

$$R_1 = R, \qquad r_1 = r, \qquad s_1 \geq s,$$

which implies the desired results of Theorem 3. □

3 Remarks on geometric interpretation of Blundon's inequality, and Ciamberlini's inequality

The results of Theorems 1, 2 and 3 provide a useful method to prove the inequalities for triangles.

Remark 1 The result of Theorem 1 implies that:

(i) In order to prove the validity of the inequality

$$s \geq f(R, r) \tag{8}$$

for any triangle, it is sufficient to prove that inequality (8) is valid for the isosceles triangles with the vertex angle greater than or equal to $\pi/3$.

(ii) In order to prove the validity of the inequality

$$s \leq f(R, r) \tag{9}$$

for any triangle, it is sufficient to prove that inequality (9) is valid for the isosceles triangles with the vertex angle less than or equal to $\pi/3$.

Remark 2 The result of Theorem 2 implies that, in order to prove the validity of the inequality

$$s \geq f(R, r) \tag{10}$$

for any non-obtuse triangle, it is sufficient to prove that inequality (10) is valid for the isosceles triangles with the vertex angle greater than or equal to $\pi/3$ in the case when $2r \leq R < (\sqrt{2}+1)r$, and inequality (10) is valid for the right triangles in the case when $R \geq (\sqrt{2}+1)r$.

Remark 3 The result of Theorem 3 implies that, in order to prove the validity of the inequality

$$s \leq f(R, r) \tag{11}$$

for any non-acute triangle, it is sufficient to prove that inequality (11) is valid for the right triangles.

Remark 4 If the inequality under consideration is homogeneous with respect to R, r, and s, in order to convenient for computing, we may assume that the side lengths of the isosceles triangles in the form of

$$a = 2, \qquad b = \frac{1+x^2}{1-x^2}, \qquad c = \frac{1+x^2}{1-x^2}, \tag{12}$$

where $x \in (0, \sqrt{3}/3]$ for the case of vertex angle of isosceles triangles are greater than or equal to $\pi/3$; and $x \in [\sqrt{3}/3, 1)$ for the case of vertex angle of isosceles triangles is less than or equal to $\pi/3$.

It is easily observed that the function $\varphi(x) = \frac{1+x^2}{1-x^2}$, $\varphi : (0,1) \to (1,\infty)$ is strictly increasing. So, $b, c \in (1,\infty)$, $a = 2$, which are the side lengths constituting isosceles triangles.

Furthermore, the semiperimeter s, the inradius r, and circumradius R of the triangle can be calculated by the following formulas:

$$s = \frac{a+b+c}{2} = \frac{2}{1-x^2},$$

$$r = \sqrt{\frac{(b+c-a)(c+a-b)(a+b-c)}{4(a+b+c)}} = x,$$

$$R = \frac{abc}{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}} = \frac{(1+x^2)^2}{4x(1-x^2)}.$$

Remark 5 If the inequality under consideration is homogeneous with respect to R, r, and s, in order to convenient for computing, we may assume that the side lengths of the right triangles are in the form of

$$c = 1, \qquad b = \frac{1-x^2}{1+x^2}, \qquad a = \frac{2x}{1+x^2}, \tag{13}$$

where $0 < x < 1$.

It is easy to see that the function $\varphi(x) = \frac{1-x^2}{1+x^2}$, $\varphi : (0,1) \to (0,1)$ is strictly decreasing, thus $b \in (0,1)$. It follows from $c = 1$ and $a^2 + b^2 = c^2$ that a, b, c are the side lengths constituting right triangles.

Furthermore, the semiperimeter s, the inradius r, and circumradius R of the triangle can be calculated by the expressions below:

$$s = \frac{1+x}{1+x^2}, \qquad r = \frac{x(1-x)}{1+x^2}, \qquad R = \frac{1}{2}. \tag{14}$$

4 Some applications

In this section we illustrate the applications of the results given in Section 2. Based on these results, we establish some sharp geometric inequalities, which improves some classical geometric inequalities.

In [10], Blundon asked for the proof of the inequality

$$s \le 2R + (3\sqrt{3} - 4)r, \tag{15}$$

which holds in any triangle ABC. The solution given by the editors was in fact a comment made by Makowski [11], who refers the reader to [1], where Blundon originally published this inequality.

We establish a sharpened version of inequality (15), as follows.

Proposition 1 *In any* $\triangle ABC$ *we have the inequality*

$$s \le 2R + (3\sqrt{3} - 4)r - (3\sqrt{3} - 5)(R - 2r)\frac{r}{R}, \tag{16}$$

where the constant $3\sqrt{3}-5$ *is best possible, that is, it cannot be replaced by larger numbers.*

Proof By using Theorem 1, in order to prove that inequality (16) holds for any triangle, it is enough to prove that inequality (16) holds for the isosceles triangle. In view of inequality (16) being homogeneous with respect to R, r, and s, we may assume the side lengths of the triangle as

$$a = 2, \qquad b = \frac{1+x^2}{1-x^2}, \qquad c = \frac{1+x^2}{1-x^2},$$

where $0 < x < 1$. Further, the semiperimeter s, inradius r, and the circumradius R of the triangle can be formulated as follows:

$$s = \frac{2}{1-x^2}, \qquad r = x, \qquad R = \frac{(1+x^2)^2}{4x(1-x^2)}.$$

Note that inequality (16) is equivalent to

$$2R + (3\sqrt{3}-4)r - s - (3\sqrt{3}-5)\left(r - \frac{2r^2}{R}\right) \geq 0. \tag{17}$$

Substituting x for s, r, R in (17) gives

$$\begin{aligned}
&2R + (3\sqrt{3}-4)r - s - (3\sqrt{3}-5)\left(r - \frac{2r^2}{R}\right) \\
&\quad = \frac{(1+x^2)^2}{2x(1-x^2)} + (3\sqrt{3}-4)x - \frac{2}{1-x^2} - (3\sqrt{3}-5)\left(x - \frac{8x^3(1-x^2)}{(1+x^2)^2}\right) \\
&\quad = (48\sqrt{3}-81)\frac{(1-x)(x-\frac{1}{\sqrt{3}})^2}{2x(1+x)(x^2+1)^2}\left(x^4 + \left(\frac{2}{\sqrt{3}}+2\right)x^3\right. \\
&\qquad \left. + \left(\frac{4}{\sqrt{3}}+2\right)x^2 + \left(\frac{22}{39}\sqrt{3}+\frac{14}{13}\right)x + \frac{16}{39}\sqrt{3}+\frac{9}{13}\right) \\
&\quad \geq 0.
\end{aligned}$$

We conclude that inequality (17) is valid, and thus inequality (16) is valid.

We next prove that the constant $3\sqrt{3}-5$ is best possible in the strong sense.

Consider inequality (16) in a general form as

$$s \leq 2R + (3\sqrt{3}-4)r - k(R-2r)\frac{r}{R}. \tag{18}$$

Putting

$$s = \frac{2}{1-x^2}, \qquad r = x, \qquad R = \frac{(1+x^2)^2}{4x(1-x^2)}$$

and

$$x \to 1$$

in (18), we get

$$k \leq \lim_{x\to 1}\left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right)\frac{(x + \frac{2}{\sqrt{3}} + 1)(x^2 + 1)^2}{(x + 1)x^2(x + \frac{1}{\sqrt{3}})^2} = 3\sqrt{3} - 5.$$

Therefore, the best possible value for k in (18) is $k_{\max} = 3\sqrt{3} - 5$. This completes the proof of Proposition 1. □

Half a century ago, F Leuenberger proved the following inequality (see [4]):

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{\sqrt{3}}{2r}. \tag{19}$$

Huang [9] considered the improved version of (19) and proposed the following.

Open problem Find the largest constant k such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{\sqrt{3}}\left(k \cdot \frac{1}{R} + \frac{3 - k}{2} \cdot \frac{1}{r}\right) \tag{20}$$

holds for any triangle ΔABC.

Some results related to the above Open problem were given by Shi [12], Chen [13], and Chen [14], respectively, as follows:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{\sqrt{3}}\left(\frac{1}{R} + \frac{1}{r}\right), \tag{21}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{\sqrt{3}}\left(\frac{5}{4} \cdot \frac{1}{R} + \frac{7}{8} \cdot \frac{1}{r}\right), \tag{22}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{\sqrt{3}}\left(\frac{97}{77} \cdot \frac{1}{R} + \frac{67}{77} \cdot \frac{1}{r}\right). \tag{23}$$

The above results show that inequality (20) is valid for $k \leq 97/77$. This prompts us to ask a natural question: What is the largest constant k such that inequality (20) holds true? The following proposition gives a perfect answer to this question.

Proposition 2 *In any ΔABC we have the inequality*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{\sqrt{3}}\left(\sqrt[3]{2} \cdot \frac{1}{R} + \frac{3 - \sqrt[3]{2}}{2} \cdot \frac{1}{r}\right), \tag{24}$$

where the constant $\sqrt[3]{2}$ is best possible, that is, it cannot be replaced by larger numbers.

Proof By using the identity (see [3])

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{s^2 + r^2 + 4Rr}{4Rrs},$$

it follows that inequality (24) is equivalent to the following inequality:

$$s^2 - \frac{4s}{\sqrt{3}}\left(\sqrt[3]{2}r + \frac{3 - \sqrt[3]{2}}{2}R\right) + r^2 + 4Rr \leq 0. \tag{25}$$

It is obvious that inequality (25) can be transformed to the form

$$s \geq f(R, r).$$

By using Theorem 1, in order to prove that inequality (25) holds for any triangle, it is enough to prove that inequality (25) holds for the isosceles triangle. Note that inequality (25) is homogeneous with respect to R, r, and s, we may assume the side lengths of the triangle as

$$a = 2, \qquad b = \frac{1+x^2}{1-x^2}, \qquad c = \frac{1+x^2}{1-x^2},$$

where $0 < x < 1$. Then the semiperimeter s, inradius r, and the circumradius R of the triangle can be calculated as follows:

$$s = \frac{2}{1-x^2}, \qquad r = x, \qquad R = \frac{(1+x^2)^2}{4x(1-x^2)}.$$

Substituting x for s, r, R in (25) gives

$$\begin{aligned} & s^2 - \frac{4s}{\sqrt{3}}\left(\sqrt[3]{2}r + \frac{3-\sqrt[3]{2}}{2}R\right) + r^2 + 4Rr \\ &= \frac{4}{(1-x^2)^2} - \frac{8}{\sqrt{3}(1-x^2)}\left(\sqrt[3]{2}x + \frac{(3-\sqrt[3]{2})(1+x^2)^2}{8x(1-x^2)}\right) + x^2 + \frac{(1+x^2)^2}{1-x^2} \\ &= -\frac{3}{x(x^2-1)^2}\left(x - \frac{1}{\sqrt{3}}\right)^2\left(x + \frac{1-2\sqrt[3]{2}}{\sqrt{3}}\right)^2\left(x + \frac{1+\sqrt[3]{2}}{\sqrt{3}}\right) \\ &\leq 0. \end{aligned}$$

The inequality (25) is proved, we thus conclude that inequality (24) is valid.

Next, we need to show that the constant $\sqrt[3]{2}$ is best possible in the strong sense.

Consider inequality (24) in a general form as

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{\sqrt{3}}\left(k \cdot \frac{1}{R} + \frac{3-k}{2} \cdot \frac{1}{r}\right). \tag{26}$$

Choosing

$$a = 2, \qquad b = c = 2\sqrt[3]{2} + 2\sqrt[3]{4} + 2$$

in (26), one has

$$\begin{aligned} & \sqrt[3]{2} - \frac{1}{2} \leq \frac{1}{\sqrt{3}}\left(k \cdot \frac{1}{\sqrt{3}(\frac{8}{15}\sqrt[3]{2} + \frac{2}{5}\sqrt[3]{4} + \frac{14}{15})} + \frac{3-k}{2} \cdot \frac{\sqrt{3}}{2\sqrt[3]{2}-1}\right) \\ & \iff \quad \left(\frac{2}{5}\sqrt[3]{2} + \frac{3}{10}\sqrt[3]{4} - \frac{4}{5}\right)k + \frac{4}{5}\sqrt[3]{2} - \frac{2}{5}\sqrt[3]{4} - \frac{3}{5} \leq 0 \\ & \implies \quad k \leq \sqrt[3]{2}. \end{aligned}$$

Thus, the best possible values for k in (26) is $k_{\max} = \sqrt[3]{2}$. The proof of Proposition 2 is completed. □

In [15], Walker presented a celebrated inequality for non-obtuse triangles, *i.e.*,

$$s^2 \geq 2R^2 + 8Rr + 3r^2. \tag{27}$$

Inequality (27) is known in the literature as Walker's inequality. We establish a sharpened version of inequality (27), as follows.

Proposition 3 *In any non-obtuse* $\triangle ABC$ *we have the inequality*

$$s^2 \geq 2R^2 + 8Rr + 3r^2 + \frac{2(R-2r)(R-(\sqrt{2}+1)r)^2}{R}, \tag{28}$$

where the constant 2 *is best possible, that is, it cannot be replaced by larger numbers.*

Proof By making use of Theorem 2, in order to prove the validity of inequality (28) for any non-obtuse triangle, it is sufficient to prove that inequality (28) is valid for the isosceles triangles in the case when $2r \leq R < (\sqrt{2}+1)r$, and inequality (28) is valid for the right triangles in the case when $R \geq (\sqrt{2}+1)r$.

We rewrite inequality (28) in an equivalent form, by transferring all the terms to the left

$$H(s,R,r) = s^2 - 2R^2 - 8Rr - 3r^2 - \frac{2(R-2r)(R-(\sqrt{2}+1)r)^2}{R} \geq 0. \tag{29}$$

Let us consider the following two cases.

Case 1. $2r \leq R < (\sqrt{2}+1)r$.

By the homogeneity of inequality (29) with respect to R, r, and s, we may assume the side lengths of the isosceles triangle as

$$a = 2, \qquad b = \frac{1+x^2}{1-x^2}, \qquad c = \frac{1+x^2}{1-x^2}.$$

The semiperimeter s, inradius r, and circumradius R of the triangle can be expressed by

$$s = \frac{2}{1-x^2}, \qquad r = x, \qquad R = \frac{(1+x^2)^2}{4x(1-x^2)}.$$

Moreover, the assumption $2r \leq R < (\sqrt{2}+1)r$ implies that

$$\sqrt{2} - 1 < x < \sqrt{\frac{2\sqrt{2}}{7} + \frac{1}{7}}.$$

Direct computation gives

$$\begin{aligned} H(s,R,r) &= \frac{4}{(1-x^2)^2} - \frac{(1+x^2)^4}{8x^2(1-x^2)^2} - \frac{2(1+x^2)^2}{1-x^2} - 3x^2 \\ &\quad - 2\left(\frac{(1+x^2)^2}{4x(1-x^2)} - 2x\right)\left(\frac{(1+x^2)^2}{4x(1-x^2)} - (\sqrt{2}+1)x\right)^2 \frac{4x(1-x^2)}{(1+x^2)^2} \end{aligned}$$

$$\begin{aligned}
&= \left(-45\sqrt{2} - \frac{261}{4}\right)\frac{(x+\sqrt{2}-1)(x-\sqrt{2}+1)(x^2-\frac{1}{3})^2}{x^2(x^2-1)(x^2+1)^2} \\
&\quad \times \left(x^4 - \left(\frac{14}{41}\sqrt{2} - \frac{8}{41}\right)x^2 - \frac{2}{41}\sqrt{2} + \frac{7}{41}\right) \\
&= \left(-45\sqrt{2} - \frac{261}{4}\right)\frac{(x+\sqrt{2}-1)(x-\sqrt{2}+1)(x^2-\frac{1}{3})^2}{x^2(x^2-1)(x^2+1)^2} \\
&\quad \times \left(\left(x^2 - \frac{7}{41}\sqrt{2} + \frac{4}{41}\right)^2 + \frac{173}{1{,}681} - \frac{26}{1{,}681}\sqrt{2}\right) \\
&\geq 0.
\end{aligned}$$

Therefore, inequality (29) is valid for the isosceles triangles under the assumption of $2r \leq R < (\sqrt{2}+1)r$.

Case 2. $R \geq (\sqrt{2}+1)r$.

In view of the homogeneity of inequality (29) with respect to R, r, and s, we may assume the side lengths of the right triangle as

$$c = 1, \qquad a = \frac{1-x^2}{1+x^2}, \qquad b = \frac{2x}{1+x^2},$$

where $0 < x < 1$. The semiperimeter s, inradius r, and circumradius R of the triangle can be expressed by

$$s = \frac{x+1}{x^2+1}, \qquad r = \frac{x(1-x)}{1+x^2}, \qquad R = \frac{1}{2}.$$

Thus, we have

$$\begin{aligned}
H(s,R,r) &= \frac{(x+1)^2}{(x^2+1)^2} - \frac{1}{2} - \frac{4x(1-x)}{1+x^2} - \frac{3x^2(1-x)^2}{(1+x^2)^2} \\
&\quad - 4\left(\frac{1}{2} - \frac{2x(1-x)}{1+x^2}\right)\left(\frac{1}{2} - (\sqrt{2}+1)\frac{x(1-x)}{1+x^2}\right)^2 \\
&= (-30\sqrt{2} - 42)\frac{x(x-1)(x-\sqrt{2}+1)^2}{(x^2+1)^3}\left(x^2 - \frac{2}{3}x + \frac{1}{3}\right) \\
&= (-30\sqrt{2} - 42)\frac{x(x-1)(x-\sqrt{2}+1)^2}{(x^2+1)^3}\left(\left(x - \frac{1}{3}\right)^2 + \frac{2}{9}\right) \\
&\geq 0.
\end{aligned}$$

Hence, inequality (29) is valid for the right triangles under the assumption of $R \geq (\sqrt{2}+1)r$.

By combining Cases 1 and 2, we deduce from Theorem 2 that inequality (28) holds true for arbitrary non-obtuse triangles.

We next prove that the constant 2 in (28) is best possible in the strong sense.

Consider inequality (28) in a general form as

$$s^2 \geq 2R^2 + 8Rr + 3r^2 + \frac{k(R-2r)(R-(\sqrt{2}+1)r)^2}{R}. \tag{30}$$

Putting

$$s = \frac{2}{1-x^2}, \qquad r = x, \qquad R = \frac{(1+x^2)^2}{4x(1-x^2)}$$

and

$$x \to 1$$

in (30), we get

$$k \le \lim_{x\to 1} \frac{-2(x-\sqrt{2}-1)(x+\sqrt{2}+1)(x^2+1)^2}{(40\sqrt{2}+57)(x-\sqrt{2}+1)(x+\sqrt{2}-1)(x^2-\frac{2}{7}\sqrt{2}-\frac{1}{7})^2} = 2.$$

Therefore, the best possible values for k in (30) is $k_{\max} = 2$. The proof of Proposition 3 is completed. □

In [16] and [17], Finsler and Hadwiger proved the following inequality:

$$4\sqrt{3}F + Q \le a^2 + b^2 + c^2 \le 4\sqrt{3}F + 3Q, \tag{31}$$

where a, b, c are the side lengths of a triangle, F is the area, and

$$Q = (a-b)^2 + (b-c)^2 + (c-a)^2.$$

Here, we establish an improved Finsler-Hadwiger inequality for non-obtuse triangles.

Proposition 4 *In any non-obtuse* $\triangle ABC$ *we have the inequality*

$$a^2 + b^2 + c^2 \le 4\sqrt{3}F + (2-\sqrt{3})(3+2\sqrt{2})Q, \tag{32}$$

where a, b, c *are the sides lengths of a triangle,* F *is the area of the triangle. The constant* $(2-\sqrt{3})(3+2\sqrt{2})$ *is best possible, that is, it cannot be replaced by smaller numbers.*

Proof By using the identities (see [3])

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 2\bigl(s^2 - 3r^2 - 12Rr\bigr),$$
$$a^2 + b^2 + c^2 = 2\bigl(s^2 - r^2 - 4Rr\bigr),$$
$$F = sr,$$

it follows that inequality (32) is equivalent to

$$\begin{aligned} H(s,R,r) = {}& \bigl(2(2-\sqrt{3})(3+2\sqrt{2})-2\bigr)s^2 + 4\sqrt{3}rs + 8Rr + 2r^2 \\ & - (2-\sqrt{3})(3+2\sqrt{2})\bigl(6r^2+24Rr\bigr) \ge 0. \end{aligned} \tag{33}$$

It is easy to see that inequality (33) can be equivalently transformed to the form of

$$s \ge f(R,r).$$

By making use of Theorem 2, in order to prove the validity of inequality (33) for any non-obtuse triangle, it is sufficient to prove that inequality (33) is valid for the isosceles triangles in the case when $2r \le R < (\sqrt{2}+1)r$, and inequality (33) is valid for the right triangles in the case when $R \ge (\sqrt{2}+1)r$.

We consider the following two cases.

Case 1. $2r \le R < (\sqrt{2}+1)r$.

By the homogeneity of inequality (33) with respect to R, r, and s, we may assume the side lengths of the isosceles triangle as

$$a = 2, \qquad b = \frac{1+x^2}{1-x^2}, \qquad c = \frac{1+x^2}{1-x^2}.$$

The semiperimeter s, inradius r, and circumradius R of the triangle can be expressed by

$$s = \frac{2}{1-x^2}, \qquad r = x, \qquad R = \frac{(1+x^2)^2}{4x(1-x^2)}.$$

Moreover, the assumption $2r \le R < (\sqrt{2}+1)r$ implies that

$$\sqrt{2}-1 < x < \sqrt{\frac{2\sqrt{2}}{7}+\frac{1}{7}}.$$

Direct computation gives

$$\begin{aligned} H(s,R,r) &= \bigl(2(2-\sqrt{3})(3+2\sqrt{2})-2\bigr)\left(\frac{2}{1-x^2}\right)^2 + 8\sqrt{3}\frac{x}{1-x^2} \\ &\quad + \frac{2(1+x^2)^2}{(1-x^2)} + 2x^2 - (2-\sqrt{3})(3+2\sqrt{2})\left(6x^2 + \frac{6(1+x^2)^2}{(1-x^2)}\right) \\ &= (54\sqrt{3}-72\sqrt{2}+36\sqrt{6}-102)(x-\sqrt{2}+1) \\ &\quad \times \left(x + \frac{(7-\sqrt{2})(4\sqrt{3}+1)}{47}\right)\frac{(x-\frac{1}{\sqrt{3}})^2}{(x^2-1)^2} \\ &\ge 0. \end{aligned}$$

Hence, inequality (33) is valid for the isosceles triangles under the assumption of $2r \le R < (\sqrt{2}+1)r$.

Case 2. $R \ge (\sqrt{2}+1)r$.

In view of the homogeneity of inequality (33) with respect to R, r, and s, we may assume the side lengths of the right triangle as

$$c = 1, \qquad a = \frac{1-x^2}{1+x^2}, \qquad b = \frac{2x}{1+x^2},$$

where $0 < x < 1$. The semiperimeter s, inradius r, and circumradius R of the triangle can be expressed by

$$s = \frac{x+1}{x^2+1}, \qquad r = \frac{x(1-x)}{1+x^2}, \qquad R = \frac{1}{2}.$$

Thus, we have

$$\begin{aligned} H(s,R,r) &= \big(2(2-\sqrt{3})(3+2\sqrt{2})-2\big)\frac{(x+1)^2}{(x^2+1)^2} \\ &\quad + 4\sqrt{3}\frac{x(1-x^2)}{(1+x^2)^2} + \frac{4x(1-x)}{1+x^2} + \frac{2x^2(1-x)^2}{(1+x^2)^2} \\ &\quad - (2-\sqrt{3})(3+2\sqrt{2})\left(\frac{6x^2(1-x)^2}{(1+x^2)^2} + \frac{12x(1-x)}{1+x^2}\right) \\ &= (24\sqrt{2}-18\sqrt{3}-12\sqrt{6}+34)\left(x^2 - \left(\frac{5\sqrt{3}}{47} - \frac{22\sqrt{2}}{47} + \frac{6\sqrt{6}}{47} + \frac{13}{47}\right)x \right. \\ &\quad \left. + \frac{22}{47}\sqrt{2} - \frac{5}{47}\sqrt{3} - \frac{6}{47}\sqrt{6} + \frac{34}{47}\right)\frac{(x-\sqrt{2}+1)^2}{(x^2+1)^2} \\ &\geq 0. \end{aligned}$$

We conclude that $H(s,R,r) \geq 0$, that is, inequality (33) is valid for the right triangles under the assumption of $R \geq (\sqrt{2}+1)r$.

By combining Cases 1 and 2, we deduce from Theorem 2 that inequality (32) holds true for arbitrary non-obtuse triangles.

Finally, we need to prove that the constant $(2-\sqrt{3})(3+2\sqrt{2})$ in (32) is best possible in the strong sense.

Consider inequality (32) in a general form as

$$a^2+b^2+c^2 \leq 4\sqrt{3}F + k(a-b)^2 + (b-c)^2 + (c-a)^2. \tag{34}$$

Putting

$$a = \frac{\sqrt{2}}{2}, \qquad b = \frac{\sqrt{2}}{2}, \qquad c = 1, \qquad F = \frac{1}{4}$$

into (34), we get

$$k \geq (2-\sqrt{3})(3+2\sqrt{2}).$$

Therefore, the best possible values for k in (34) is that $k_{\min} = (2-\sqrt{3})(3+2\sqrt{2})$. This completes the proof of Proposition 4. □

Competing interests
The authors declare that they have no conflicts of interest to this work.

Authors' contributions
SW finished the proof and the writing work. YC gave SW some advice on the proof and writing. All authors read and approved the final manuscript.

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