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Norm inequalities for the conjugate operator in two-weighted Lebesgue spaces

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Abstract

In this article, first, we prove that weighted-norm inequalities for the M-harmonic conjugate operator on the unit sphere whenever the pair (u, v) of weights satisfies the A_p-condition, and $ud\sigma$, $vd\sigma$ are doubling measures, where $d\sigma$ is the rotation-invariant positive Borel measure on the unit sphere with total measure 1. Then, we drive cross-weighted norm inequalities between the Hardy-Littlewood maximal function and the sharp maximal function whenever (u, v) satisfies the A_p-condition, and $vd\sigma$ does a certain regular condition.

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Keywords: two-weighted norm inequality, non-isotropic metric, maximal function, sharp maximal function, M-harmonic conjugate operator, Hilbert transform

1 Introduction

Let B be the unit ball of $\mathbb{C}^n$ with norm $|z| = \langle z, z\rangle^{1/2}$ where $\langle , \rangle$ is the Hermitian inner product, S be the unit sphere and σ be the rotation-invariant probability measure on S.

For $z \in B$, $\xi \in S$, we define the $\mathcal{M}$-harmonic conjugate kernel $K(z, \xi)$ by

$$iK(z, \xi) = 2C(z, \xi) - P(z, \xi) - 1,$$

where $C(z, \xi) = (1 - \langle z, \xi\rangle)^{-n}$ is the Cauchy kernel and $P(z, \xi) = (1 - |z|^2)^n/|1 - \langle z, \xi\rangle|^{2n}$ is the invariant Poisson kernel [1].

For the kernels, C and P, refer to [2]. And for all f - $A(B)$, the ball algebra, such that $f(0)$ is real, the reproducing property of $2C(z, \xi) - 1$ [2, Theorem 3.2.5] gives

$$\int_S K(z, \xi)\mathrm{Re}\, f(\xi)d\sigma(\xi) = -i\left(f(z) - \mathrm{Re}\, f(z)\right) = \mathrm{Im}\, f(z).$$

For $n = 1$, the definition of Kf is the same as the classical harmonic conjugate function and so we can regard Kf as the Hilbert transform on the unit circle. The L^p boundedness property of harmonic conjugate functions on the unit circle for $1 < p < \infty$ was introduced by Riesz in 1924 [3, Theorem 2.3 of Chapter 3]. Later, in 1973, Hunt et al. [4] proved that, for $1 < p < \infty$, conjugate functions are bounded on weighted measured Lebesgue space if and only if the weight satisfies A_p-condition. Most recently, Lee and Rim [5] provided an analogue of that of [4] by proving that, for $1 < p < \infty$, M-harmonic conjugate operator K is bounded on $L^p(\omega)$ if and only if the nonnegative weight ω satisfies the $A_p(S)$-condition on S; i.e., the nonnegative weight ω satisfies

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$$\sup_Q \frac{1}{\sigma(Q)} \int_Q \omega d\sigma \left(\frac{1}{\sigma(Q)} \int_Q \omega^{-1/(p-1)} d\sigma \right)^{p-1} := A_p^\omega < \infty,$$

where $Q = Q(\xi, \delta) = \{\eta \in S : d(\xi, \eta) = |1 - \langle \xi, \eta \rangle|^{1/2} < \delta\}$ is a non-isotropic ball of S.

To define the $A_p(S)$-condition for two weights, we let (u, v) be a pair of two non-negative integrable functions on S. For $p > 1$, we say that (u, v) satisfies two-weighted $A_p(S)$-condition if

$$\sup_Q \frac{1}{\sigma(Q)} \int_Q u d\sigma \left(\frac{1}{\sigma(Q)} \int_Q v^{-1/(p-1)} d\sigma \right)^{p-1} := A_p < \infty, \tag{1.1}$$

where Q is a non-isotropic ball of S. For $p = 1$, the $A_1(S)$-condition can be viewed as a limit case of the $A_p(S)$-condition as $p \searrow 1$, which means that (u, v) satisfies the $A_1(S)$-condition if

$$\sup_Q \frac{1}{\sigma(Q)} \int_Q u d\sigma \left(\mathrm{esssup}_Q v^{-1} \right) := A_1 < \infty,$$

where Q is a non-isotropic ball of S.

In succession of classical weighted-norm inequalities, starting from Muckenhoupt's result in 1975 [6], there have been extensive studies on two-weighted norm inequalities (for textbooks [7-10] and for related topics [11-17]). In [6], Muckenhoupt derives a necessary and sufficient condition on two-weighted norm inequalities for the Poisson integral operator. And then, Sawyer [18,19] obtained characterizations of two-weighted norm inequalities for the Hardy-Littlewood maximal function and for the fractional and Poisson integral operators, respectively. As a result on two-weighted $A_p(S)$-condition itself, Neugebauer [20] proved the existence of an inserting pair of weights. Cruz-Uribe and Pérez [21] give a sufficient condition for Calderón-Zygmund operators to satisfy the weighted weak (p, p) inequality. More recently, Martell et al. [22] provide two-weighted norm inequalities for Calderón-Zygmund operators that are sharp for the Hilbert transform and for the Riesz transforms.

Ding and Lin [23] consider the fractional integral operator and the maximal operator that contain a function homogeneous of degree zero as a part of kernels and the authors prove weighted (L^p, L^q)-boundedness for those operators for two weights.

In [24], Muckenhoupt and Wheeden provided simple examples of a pair that satisfies two-weighted $A_p(\mathbb{R})$-condition but not two-weighted norm inequalities for the Hardy-Littlewood maximal operator and the Hilbert transform. In this article, we prove the converse of the main theorem of [5] by adding a doubling condition for a weight function. And then by adding a suitable regularity condition on a weight function, we derive and prove a cross-weighted norm inequalities between the Hardy-Littlewood maximal function and the sharp maximal function.

Throughout this article, Q denotes a non-isotropic ball of S induced by the non-isotropic metric d on S which is defined by $d(\xi, \eta) = |1 - \langle \xi, \eta \rangle|^{1/2}$. For notational simplicity, we denote $\int_Q f\, d\sigma := f(Q)$ the integral of f over Q, and $\frac{1}{\sigma(Q)} \int_Q f\, d\sigma := f_Q$ the average of f over Q. Also, for a nonnegative integrable function u and a measurable subset E of S, we write $u(E)$ for the integral of u over E. We write $Q(\delta)$ in place of $Q(\xi, \delta)$ whenever the center ξ has no meaning in a context. For a positive constant s, $sQ(\delta)$

means $Q(s\delta)$. We say that a weight v satisfies a doubling condition if there is a constant C independent of Q such that $v(2Q) \leq Cv(Q)$ for all Q.

Theorem 1.1. *Let* $1 < p < \infty$. If (u, v) *satisfies two-weighted* $A_{p'}(S)$*-condition for some* $p' < p$ *and* $ud\sigma$, $vd\sigma$ *are doubling measures, then there exists a constant C which depends on u, v and p, such that for all function f,*

$$\int_S |Kf|^p u\,d\sigma \leq C \int_S |f|^p v\,d\sigma \quad \text{for all } f \in L^p(v). \tag{1.2}$$

To prove the next theorem, we need a regularity condition for v such that for $1 \leq p < \infty$, we assume that for a measurable set $E \subset Q$ and for $\sigma(E) \leq \theta\sigma(Q)$ with $0 \leq \theta \leq 1$, we get

$$v(E) \leq \left(1 - (1-\theta)^p\right) v(Q). \tag{1.3}$$

Let $f \in L^1(S)$ and let $1 < p < \infty$. The (Hardy-Littlewood) maximal and the sharp maximal functions Mf, $f^{\#p}$, resp. on S are defined by

$$Mf(\xi) = \sup_Q \frac{1}{\sigma(Q)} \int_Q |f|\,d\sigma,$$

$$f^{\#p}(\xi) = \sup_Q \left(\frac{1}{\sigma(Q)} \int_Q |f - f_Q|^p d\sigma\right)^{1/p},$$

where each supremum is taken over all balls Q containing ξ. From the definition, the sharp maximal function $f \mapsto f^{\#p}$ is an analogue of the maximal function Mf, which satisfies $f^{\#1}(\xi) \leq 2Mf(\xi)$.

Theorem 1.2. *Let* $1 < p < \infty$. *If* (u, v) *satisfies two-weighted* $A_p(S)$*-condition and* $vd\sigma$ *does* (*1.3*), *then there exists a constant C which depends on u, v and p, such that for all function f,*

$$\int_S (Mf)^p u\,d\sigma \leq C\left(\int_S \left(f^{\#1}\right)^p v\,d\sigma + \int_S |f|^p v\,d\sigma\right).$$

Remark. On the unit circle, we derive a sufficient condition for weighted-norm inequalities for the Hilbert transform for two weights.

The proofs of Theorem 1.1 will be given in Section 3. We start Section 2 by deriving some preliminary properties of (u, v) which satisfies the $A_p(S)$-condition. In Section 4, we prove Theorem 1.2.

2 Two-weight on the unit sphere

Lemma 2.1. *If* (u, v) *satisfies two-weighted* $A_p(S)$*-condition, then for every function* $f \geq 0$ *and for every ball Q,*

$$(f_Q)^p u(Q) \leq A_p \int_Q f^p v\,d\sigma.$$

Proof. If $p = 1$ and (u, v) satisfies two-weighted $A_1(S)$-condition, we get, for every ball Q and every $f \geq 0$,

$$
\begin{aligned}
f_Q u(Q) &= f(Q) u_Q \\
&\le A_1 f(Q) \frac{1}{\operatorname{ess\,sup}_Q v^{-1}} \\
&\le A_1 \int_Q f v \, d\sigma,
\end{aligned}
$$

since $1/\operatorname{ess\,sup}_Q v^{-1} = \operatorname{ess\,inf}_Q v \le (\xi)$ for all $\xi \in Q$.

If $1 < p < \infty$ and (u, v) satisfies two-weighted $A_p(S)$-condition, we have, for every ball Q and every $f \ge 0$, using Holder's inequality with p and its conjugate exponent $p/(p-1)$,

$$
\begin{aligned}
f_Q &= \frac{1}{\sigma(Q)} \int_Q f v^{1/p} v^{-1/p} d\sigma \\
&\le \left(\frac{1}{\sigma(Q)} \int_Q f^p v \, d\sigma \right)^{1/p} \left(\frac{1}{\sigma(Q)} \int_Q v^{-1/(p-1)} d\sigma \right)^{(p-1)/p}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(f_Q)^p u(Q) &= \frac{u(Q)}{\sigma(Q)} \left(\frac{1}{\sigma(Q)} \int_Q v^{-1/(p-1)} d\sigma \right)^{p-1} \int_Q f^p v \, d\sigma \\
&\le A_p \int_Q f^p v \, d\sigma .
\end{aligned}
$$

Therefore, the proof is complete.

Corollary 2.2. *If (u, v) satisfies two-weighted $A_p(S)$-condition, then*

$$
\left(\frac{\sigma(E)}{\sigma(Q)} \right)^p u(Q) \le A_p v(E),
$$

where E is a measurable subset of Q.

Proof. Applying Lemma 2.1 with f replaced by χ_E proves the conclusion.

3 Proof of Theorem 1.1

In this section, we will prove the first main theorem. First, we derive the inequality between two sharp maximal functions of Kf and f.

Lemma 3.1. *Let $f \in L^1(S)$. Then, for $q > p > 1$, there is a constant $C_{p,q}$ such that $(Kf)^{\#p}(\xi) \le C_{p,q} f^{\#q}(\xi)$ for almost every ξ.*

Proof. It suffices to show that for $r \ge 1$ and $q > 1$, there is a constant C_{rq} such that $(Kf)^{\#r}(\xi) \le C_{rq} f^{\#rq}(\xi)$ for almost every ξ,

i.e., for $Q = Q(\xi_Q, \delta)$ a ball of S, we prove that there are constants $\lambda = \lambda(Q, f)$ and C_{rq} such that

$$
\left(\frac{1}{\sigma(Q)} \int_Q \left| K f(\eta) - \lambda \right|^r d\sigma \right)^{1/r} \le C_{r,q} f^{\#^q}(\xi_Q). \tag{3.1}
$$

Fix $Q = Q(\xi_Q, \delta)$ and write

$$
\begin{aligned}
f(\eta) &= \left(f(\eta) - f_Q \right) \chi_{2Q}(\eta) + \left(f(\eta) - f_Q \right) \chi_{S \setminus 2Q}(\eta) + f_Q \\
&:= f_1(\eta) + f_2(\eta) + f_Q .
\end{aligned}
$$

Then, $Kf = Kf_1 + Kf_2$, since $Kf_Q = 0$.

For each $z \in B$, put

$$g(z) = \int_S \left(2C(z,\xi) - 1\right) f_2(\xi) d\sigma(\xi).$$

Then, g is continuous on $B \cup Q$ By setting $\lambda = -ig(\xi_Q)$ in (3.1), we shall drive the conclusion. By Minkowski's inequality, we split the integral in (3.1) into two parts,

$$\begin{aligned}\left(\frac{1}{\sigma(Q)}\int_Q \left|Kf(\eta) + ig(\xi_Q)\right|^r d\sigma(\eta)\right)^{1/r} \\ \leq \left(\frac{1}{\sigma(Q)}\int_Q \left|Kf_1\right|^r d\sigma\right)^{1/r} + \left(\frac{1}{\sigma(Q)}\int_Q \left|Kf_2 + ig(\xi_Q)\right|^r d\sigma\right)^{1/r} := I_1 + I_2.\end{aligned} \tag{3.2}$$

We estimate I_1. By Holder's inequality, it is estimated as

$$\begin{aligned}I_1 &\leq \left(\frac{1}{\sigma(Q)}\int_Q \left|Kf_1\right|^{rq} d\sigma\right)^{1/rq} \\ &\leq \left(\frac{1}{\sigma(Q)}\int_S \left|Kf_1\right|^{rq} d\sigma\right)^{1/rq} \leq \frac{C_{rq}}{\sigma(Q)^{1/rq}} \|f_1\|_{L^{rq}},\end{aligned}$$

since K is bounded on $L^{rq}(S)$ $(rq > 1)$. By replacing f_1 by $f - f_Q$, we get

$$\begin{aligned}\|f_1\|_{L^{rq}} &= \left(\int_{2Q} \left|f - f_Q\right|^{rq} d\sigma\right)^{1/rq} \\ &\leq \left(\int_{2Q} \left|f - f_{2Q}\right|^{rq} d\sigma\right)^{1/rq} + \sigma(2Q)^{1/rq} \left|f_{2Q} - f_Q\right|.\end{aligned}$$

Thus, by applying Hölder's inequality in the last term of the above,

$$\begin{aligned}\sigma(2Q)^{1/rq}\left|f_{2Q} - f_Q\right| &\leq \frac{\sigma(2Q)^{1/rq}}{\sigma(Q)}\int_Q \left|f - f_{2Q}\right| d\sigma \\ &\leq \frac{\sigma(2Q)^{1/rq}\sigma(Q)^{1-1/rq}}{\sigma(Q)}\left(\int_{2Q}\left|f - f_{2Q}\right|^{rq} d\sigma\right)^{1/rq} \\ &= R_2^{1/rq}\left(\int_{2Q}\left|f - f_{2Q}\right|^{rq} d\sigma\right)^{1/rq} \quad \text{(by(4.2))}.\end{aligned}$$

Hence,

$$I_1 \leq C_{rq}\left(1 + R_2^{1/rq}\right) f^{\#^{rq}}(\xi_Q). \tag{3.3}$$

Now, we estimate I_2. Since $f_2 \equiv 0$ on $2Q$, the invariant Poisson integral of f_2 vanishes on Q, i.e., $\lim_{t \nearrow 1} \int_S P(t\eta, \xi) f_2(\eta) d\sigma(\eta) = 0$ whenever $\xi \in Q$. Thus, for almost all $\xi \in Q$,

$$iKf_2(\xi) = \int_{S \setminus 2Q} \left(2C(\xi,\eta) - 1\right) f_2(\eta) d\sigma(\eta) = g(\xi)$$

and then, by Minkowski's inequality for integrals,

$$I_2 = \left(\frac{1}{\sigma(Q)} \int_Q \left| iKf_2 - g(\xi_Q) \right|^r d\sigma \right)^{1/r}$$
$$\leq \int_{S\setminus 2Q} 2\left| f_2(\eta) \right| \left(\frac{1}{\sigma(Q)} \int_Q \left| C(\xi,\eta) - C(\xi_Q,\eta) \right|^r d\sigma(\xi) \right)^{1/r} d\sigma(\eta).$$

By Lemma 6.1.1 of [2], we get an upper bound such that

$$I_2 \leq C\delta \int_{S\setminus 2Q} \frac{\left| f_2(\eta) \right|}{\left| 1 - \langle \eta, \xi_Q \rangle \right|^{n+1/2}} d\sigma(\eta), \tag{3.4}$$

where C is an absolute constant. Write $S\setminus 2Q = \bigcup_{k=1}^{\infty} 2^{k+1}Q\setminus 2^kQ$. Then, the integral of (3.4) is equal to

$$\sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} \frac{\left| f(\eta) - f_Q \right|}{\left| 1 - \langle \eta, \xi_Q \rangle \right|^{n+1/2}} d\sigma(\eta)$$
$$\leq \sum_{k=1}^{\infty} \frac{1}{2^{(2n+1)k}\delta^{2n+1}} \int_{2^{k+1}Q\setminus 2^kQ} \left| f - f_Q \right| d\sigma$$
$$\leq \sum_{k=1}^{\infty} \frac{1}{2^{(2n+1)k}\delta^{2n+1}} \left(\int_{2^{k+1}Q} \left| f - f_{2^{k+1}Q} \right| d\sigma + \sum_{j=0}^{k} \int_{2^{k+1}Q} \left| f_{2^{j+1}Q} - f_{2^jQ} \right| d\sigma \right).$$

By Hölder's inequality, by (4.3),

$$\int_{2^{k+1}Q} \left| f - f_{2^{k+1}Q} \right| d\sigma \leq R_{2^{k+1}\delta} \left(\frac{1}{\sigma(2^{k+1}Q)} \int_{2^{k+1}Q} \left| f - f_{2^{k+1}Q} \right|^{rq} d\sigma \right)^{1/rq}$$
$$\leq R_{2^{k+1}\delta} f^{\#^{rq}}(\xi_Q), \tag{3.5}$$

Similarly, for each j,

$$\int_{2^{k+1}Q} \left| f_{2^{j+1}Q} - f_{2^jQ} \right| d\sigma \leq \frac{\sigma(2^{k+1}Q)}{\sigma(2^jQ)} \int_{2^jQ} \left| f - f_{2^{j+1}Q} \right| d\sigma$$
$$\leq R_{2^{k-j+1}} \int_{2^{j+1}Q} \left| f - f_{2^{j+1}Q} \right| d\sigma \quad \text{(by (4.2))}$$
$$\leq R_{2^{k-j+1}} R_{2^{j+1}\delta} f^{\#^{rq}}(\xi_Q) \quad \text{(from (3.5) with } k = j\text{)}$$
$$= R_1 R_{2^{k+2}\delta} f^{\#^{rq}}(\xi_Q).$$

Thus,

$$\sum_{j=0}^{k} \int_{2^{k+1}Q} \left| f_{2^{j+1}Q} - f_{2^jQ} \right| d\sigma \leq (k+1) R_1 R_{2^{k+2}\delta} f^{\#^{rq}}(\xi_Q). \tag{3.6}$$

Since R_s increases as $s \nearrow \infty$ and $R_1 > 1$, by adding (3.5) to (3.6), we have the upper bound as

$$(k+2) R_1 R_{2^{k+2}\delta} f^{\#^{rq}}(\xi_Q).$$

Eventually, the identity of $R_{2^{k+2}\delta} = R_1 2^{2n(k+2)}\delta^{2n}$ yields that

$$I_2 \le 2^{4n} C R_1^2 \sum_{k=1}^{\infty} \frac{k+2}{2^k} f^{\#^{rq}}(\xi_Q), \tag{3.7}$$

and therefore, combining (3.3) and (3.7), we complete the proof.

The main theorem depends on Marcinkiewicz interpolation theorem between two abstract Lebesgue spaces, which is as follows.

Proposition 3.2. *Suppose* (X, μ) *and* (Y, ν) *are measure spaces;* p_0, p_1, q_0, q_1 *are elements of* $[1, \infty]$ *such that* $p_0 \le q_0$, $p_1 \le q_1$ *and* $q_0 \ne q_1$ *and*

$$\frac{1}{p} = \frac{1+t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1} \quad (0 < t < 1).$$

If T is a sublinear map from $L^{p_0}(\mu) + L^{p_1}(\mu)$ *to the space of measurable functions on Y which is of weak-types* (p_0,q_0) *and* (p_1,q_1), *then T is of type* (p, q).

Now, we prove the main theorem.

Proof of Theorem 1.1. Under the assumption of the main theorem, we will prove that (1.2) holds. We fix $p > 1$ and let $f \in L^p(\nu)$.

By Theorem 1.2, there is a constant C_p such that

$$\begin{aligned} \int_S |Kf|^p u \, d\sigma &\le \int_S |M_u(Kf)|^p u \, d\sigma \\ &\le C_p \int_S \left|(Kf)^{\#^1}\right|^p u \, d\sigma \\ &\le C_p \int_S \left|f^{\#^q}\right|^p u \, d\sigma \quad (\text{by Lemma 3.1 with } q > 1, p/q > 1) \\ &\le 2^p C_p \int_S \left(M|f|^q\right)^{p/q} u \, d\sigma \quad (\text{by the triangle inequality}). \end{aligned} \tag{3.8}$$

where M_u is the maximal operator with respect to $ud\sigma$, the second inequality follows from the doubling condition of $ud\sigma$.

Without loss of generality, we assume $f \ge 0$. By Holder's inequality and by (1.1), we have

$$\begin{aligned} \frac{1}{\sigma(Q)} \int_Q d\sigma &\le \left(\frac{1}{\sigma(Q)} \int_Q f^{p/q} \nu \, d\sigma\right)^{q/p} \left(\frac{1}{\sigma(Q)} \int_Q \nu^{-1/(p/q-1)} \, d\sigma\right)^{1-q/p} \\ &\le A_{p/q}^{q/p} \left(\frac{1}{\sigma(Q)} \int_Q f^{p/q} \nu \, d\sigma\right)^{q/p} \left(\frac{\sigma(Q)}{\nu(Q)}\right)^{q/p} \text{for all } Q. \end{aligned}$$

Thus, if $f_Q > \lambda$, then

$$u(Q) \le A_{p/q} \lambda^{p/q} \int_Q f^{p/q} \nu d\sigma \quad \text{for all } Q. \tag{3.9}$$

Let E be an arbitrary compact subset of $\{\xi \in S: M f(\xi) > \lambda\}$. Since $\nu d\sigma$ is a doubling measure, from (3.9), there exists a constant $C_{p,q}$ such that

$$u(E) \le C_{p,q} \lambda^{p/q} \int_S f^{p/q} \nu d\sigma.$$

Thus, $M f$ is of weak-type $(L^{p/q}(vd\sigma), L^{p/q}(ud\sigma))$. Moreover,

$$\begin{aligned} \|Mf\|_{L^\infty(ud\sigma)} &\le \|Mf\|_{L^\infty} \quad (\text{since } ud\sigma \ll d\sigma) \\ &\le \|f\|_{L^\infty} \\ &= \|f\|_{L^\infty(ud\sigma)} \quad (\text{since } ud\sigma \ll d\sigma, v > 0 \text{ a.e. by } (1.1)). \end{aligned}$$

Now, by Proposition 3.2, $M f$ is of type $(L^r(vd\sigma), L^r(ud\sigma))$ for $f > p/q$.
Hence, the last integral of (3.8) is bounded by some constant times

$$\int_S |f|^{qr} v d\sigma \quad (\text{for all } r > p/q).$$

Since q is arbitrary so that $p/q > 0$, we can replace qr by p with $p > 1$. Therefore, the proof is completed.

4 Proof of Theorem 1.2

Theorem 1.2 can be regarded as cross-weighted norm inequalities for the Hardy-Littlewood maximal function and the sharp maximal function on the unit sphere. For a single A_p-weight in $\mathbb{R}^n$, refer to Theorem 2.20 of [8].

From Proposition 5.1.4 of [2], we conclude that when $n > 1$,

$$\frac{\Gamma^2(n/2+1)}{2^{n-2}\Gamma(n+1)} s^{2n} \le \frac{\sigma(Q(s\delta))}{\sigma(Q(\delta))} \le \frac{2^{n-2}\Gamma(n+1)}{\Gamma^2(n/2+1)} s^{2n},$$

and when $n = 1$,

$$\frac{2}{\pi} s^2 \le \frac{\sigma(Q(s\delta))}{\sigma(Q(\delta))} \le \frac{\pi}{2} s^2$$

for any $s > 0$. Throughout the article, several kinds of constants will appear. To avoid confusion, we define the maximum ratio between sizes of two balls by

$$R_s : R_{s,n} = \max\left(\frac{2^{n-2}\Gamma(n+1)}{\Gamma^2(n/2+1)}, \frac{\pi}{2}\right) s^{2n}, \tag{4.1}$$

and thus, for every $s > 0$, for every $\delta > 0$,

$$\sigma(sQ(\delta)) \le R_s \sigma(Q(\delta)). \tag{4.2}$$

Putting $\delta = 1$ in (4.2), we get

$$\sigma(Q(s)) \le R_s. \tag{4.3}$$

To prove Theorem 1.2, we need some lemmas. The next result is a covering lemma on the unit sphere, related to the maximal function. Let $f \in L^1(S)$ and let $t > \|f\|_{L^1(S)}$. We may assume $\|f\|_{L^1(S)} \neq 0$. Since $\{M f > t\}$ is open, take a ball $Q \subset \{M f > t\}$ with center at each point of $\{M f > t\}$. For such a ball Q,

$$\sigma(Q) \le \frac{1}{t} \int_Q |f| \, d\sigma. \tag{4.4}$$

Thus, to each $\xi \in \{M f > t\}$ corresponds a largest radius δ such that the ball $Q = Q(\xi, \delta) \subset \{M f > t\}$ satisfies (4.4). Hence, we conclude the following simple covering lemma.

Lemma 4.1 (Covering lemma on S). *Let $f \in L^1(S)$ be non-trivial. Then, for $t > \|f\|_{L^1(S)}$, there is a collection of balls $\{Q_{t,j}\}$ such that*

(i) $\left\{\xi \in S : Mf(\xi) > t\right\} \subset \bigcup_j Q_{t,j}$,

(ii) $\sigma(Q_{t,j}) \leq t^{-1} \left|f\right| (Q_{t,j})$,

where each $Q_{t,j}$ has the maximal radius of all the balls that satisfy (ii) in the sense that if Q is a ball that contains some $Q_{t,j}$ as its proper subset, then $\sigma(Q) > t^{-1} \int_Q |f|\, d\sigma$ holds.

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Fix $1 < p < \infty$. We may assume $f^{\#^1} \in L^p(\nu)$ and $f \in L^p(\nu)$, otherwise, Theorem 1.2 holds clearly. Since ν satisfies the doubling condition, we have $||Mf||_{Lp(\nu)} \leq C||f^{\#^1}||_{Lp(\nu)}$. Combining this with $f^{\#^1} \in L^p(\nu)$, we have $||Mf||_{Lp(\nu)} < \infty$.

Suppose that f is non-trivial and we may assume that $f \geq 0$. Let

$$t > \max\left(2, 2R_2^2, \ R_3\right) \|f\|_{L^1(S)}.$$

For $\varepsilon > 0$, E_t be a compact subset of $\{Mf > t\}$ such that $u(\{Mf > t\}) < u(E_t) + e^{-t}\varepsilon$. Indeed, since u is integrable, $u\, d\sigma$ is a regular Borel measure absolutely continuous with respect to σ.

Suppose $\{Q_{t,j}\}$ is a collection of balls having the properties (i) and (ii) of Lemma 4.1. Since $\{Q_{t,j}\}$ is a cover of a compact set E_t, there is a finite subcollection of $\{Q_{t,j}\}$, which covers E_t. By Lemma 5.2.3 of [2], there are pairwise disjoint balls, $Q_{t,j_1}, Q_{t,j_2}, \ldots, Q_{t,j_\ell}$ of the previous subcollection such that

$$E_t \subset \bigcup_{k=1}^{\ell} 3Q_{t,j_k},$$

$$\sigma(E_t) \leq R_3 \sum_{k=1}^{\ell} \sigma(Q_{t,j_k}),$$

where ℓ may depend on t. To avoid the abuse of subindices, we rewrite Q_{t,j_k} as $Q_{t,j}$. Let us note that from the maximality of $Q_{t,j}$,

$$t > \frac{1}{\sigma(2Q_{t,j})} \int_{2Q_{t,j}} f\, d\sigma \geq \frac{\sigma(Q_{t,j})}{\sigma(2Q_{t,j})} t \geq R_2^{-1} t. \tag{4.5}$$

Fix $Q_0 = 2Q_{\kappa t, j_0}$, where $\kappa^{-1} = 2R_2$. (Here, $\kappa < 1/2$, since $R_2 > 1$.) Let $\lambda > 0$ that will be chosen later. From the definition of the sharp maximal function, there are two possibilities: either

$$Q_0 \subset \{f^{\#^1} > \lambda t\} \quad \text{or} \quad Q_0 \not\subset \{f^{\#^1} > \lambda t\}. \tag{4.6}$$

In the first case, since $Q_{t,j}$'s are pairwise disjoint,

$$\sum_{\left\{j : Q_{t,j} \subset Q_0 \subset \{f^{\#^1} > \lambda t\}\right\}} \nu(Q_{t,j}) \leq \nu(\{f^{\#^1} > \lambda t\}),$$

and also,

$$\sum_{\substack{Q_0 \\ Q_0 \subset \{f^{\#^1} > \lambda t\}}} \sum_{\{j: Q_{t,j} \subset Q_0\}} \nu(Q_{t,j}) \leq \nu(\{f^{\#^1} > \lambda t\}). \tag{4.7}$$

In the second case,

$$\frac{1}{\sigma(Q_0)} \int_{Q_0} |f - f_{Q_0}| d\sigma \leq \lambda t. \tag{4.8}$$

Since $2^{-1}t > \|f\|_{L^1(S)}$, by (4.5), taking $f_{Q_0} \leq R_2 kt = 2^{-1}t$ into account, we have

$$\begin{aligned}
\sum_{\left\{j: Q_{t,j} \subset Q_0 \not\subset \{f^{\#^1} > \lambda t\}\right\}} (t - t/2)\sigma(Q_{t,j}) &\leq \sum_{\left\{j: Q_{t,j} \subset Q_0 \not\subset \{f^{\#^1} > \lambda t\}\right\}} \int_{Q_{t,j}} f - f_{Q_0} d\sigma \\
&\leq \sum_{\left\{j: Q_{t,j} \subset Q_0 \not\subset \{f^{\#^1} > \lambda t\}\right\}} \int_{Q_{t,j}} |f - f_{Q_0}| d\sigma \\
&\leq \int_{Q_0} |f - f_{Q_0}| d\sigma \\
&\leq \lambda t \sigma(Q_0) \quad (\text{by } (4.8)).
\end{aligned}$$

Thus,

$$\sum_{\left\{j: Q_{t,j} \subset Q_0 \not\subset \{f^{\#^1} > \lambda t\}\right\}} \sigma(Q_{t,j}) \leq 2\lambda \sigma(Q_0). \tag{4.9}$$

In (4.9), take a small $\lambda > 0$ such that

$$2\lambda < 1. \tag{4.10}$$

(Note that the condition (4.10) enables us to use (1.3).) Thus, (4.9) can be written as

$$\sum_{\left\{j: Q_{t,j} \subset Q_0 \not\subset \{f^{\#^1} > \lambda t\}\right\}} \nu(Q_{t,j}) \leq \left(1 - (1 - 2\lambda)^p\right) \nu(2Q_{\kappa t, j_0}).$$

Adding up all possible Q_0's in the second case of (4.6), we get

$$\sum_{\substack{Q_0 \\ Q_0 \not\subset \{f^{\#^1} > \lambda t\}}} \sum_{\{j: Q_{t,j} \subset Q_0\}} \nu(Q_{t,j}) \leq \left(1 - (1 - 2\lambda)^p\right) \sum_k \nu(2Q_{\kappa t, k}). \tag{4.11}$$

Since $\{Mf > t\} \subset \{Mf > R_2^{-1} t\}$ and $\sigma\left(2Q_{t,j}\right) \leq R_2 t^{-1} \int_{2Q_{t,j}} f\, d\sigma$ holds (4.5), we can construct the collection of balls $\left\{Q_{R_2^{-1} t, j}\right\}$ which covers $\{Mf > R_2^{-1} t\}$ with maximal radius just the same way as Lemma 4.1, so that $2Q_{t,j}$ is contained in $\left\{Q_{R_2^{-1} t, i}\right\}$ for some i. Recall that $R_2^{-1} kt = 2^{-1} R_2^{-2} t > \|f\|_{L^1(S)}$, hence, (4.11) turns into

$$\sum_{\substack{Q_0 \\ Q_0 \not\subset \{f^{\#^1} > \lambda t\}}} \sum_{\{j: Q_{t,j} \subset Q_0\}} \nu(Q_{t,j}) \leq \left(1 - (1 - 2\lambda)^p\right) \sum_k \nu(Q_{R_2^{-1} \kappa t, k}). \tag{4.12}$$

Combining (4.7) and (4.11), we summarize

$$\sum_j v(Q_{t,j}) \le v(\{f^{\#^1} > \lambda t\}) + \sum_k \left(1-(1-2\lambda)^p\right) v(Q_{R_2^{-1}\kappa t,k}). \tag{4.13}$$

Now, put

$$\begin{aligned} \alpha_v(t) &= \sum_j v(Q_{t,j}), \\ \beta_u(t) &= u(E_t). \end{aligned} \tag{4.14}$$

Then,

$$\begin{aligned} \beta_u(t) &\le \int_{\cup_j 3Q_{t,j}} u\,d\sigma \le \sum_j \int_{3Q_{t,j}} u\,d\sigma \\ &\le A_p \sum_j \int_{3Q_{t,j}} v\,d\sigma \quad \text{(by Corollary 2.2 with } E = Q = 3Q_{t,j}) \\ &\le A_p \sum_j \int_{Q_{R_3^{-1}t,j}} v\,d\sigma \\ &= A_p \alpha_v(R_3^{-1}t), \end{aligned} \tag{4.15}$$

where the fourth inequality follows from the fact that $3Q_{t,j} \subset Q_{R_3^{-1}t,i}$ for some i. Indeed, we can construct $Q_{R_3^{-1}t,i}$ as before, since $R_3^{-1}t > \|f\|_{L^1(S)}$.

Eventually, putting the constant $e_p = \int_0^\infty t^p e^{-t} dt$, and $N = \max\left(2, 2R_2^2, R_3\right)$ (which depends only on n), we have

$$\begin{aligned} \int_S |Mf|^p u\,d\sigma &\le \int_0^\infty pt^{p-1}\beta_u(t)dt + e_p\varepsilon \\ &\le \int_0^{N\|f\|_{L^1(S)}} pt^{p-1}\beta_u(t)\,dt + A_p \int_{N\|f\|_{L^1(S)}}^\infty pt^{p-1}\alpha_v(R_3^{-1}t)dt + e_p\varepsilon \quad \text{(by (4.15))} \\ &:= I + II + e_p\varepsilon. \end{aligned}$$

The first term I is dominated by

$$\begin{aligned} N^p \| u\|_{L^1(S)} \| f \|_{L^1(S)}^p &\le N^p \| f \|_{L^p(v)}^p \| u\|_{L^1(S)} \left(\int_S v^{-1/(p-1)} d\sigma\right)^{p-1} \\ &\le N^p A_p \| f \|_{L^p(v)}^p \quad \text{(since } (u,v) \in A_p(S)), \end{aligned}$$

where the first inequality follows from Hölder's inequality for $\|f\|_{L^1(S)}^o$.

On the other hand,

$$\begin{aligned} II &\le A_p C_p \int_0^\infty pt^{p-1} v(\{f^{\#^1} > R_3^{-1}t\})dt \quad \text{(by Lemma 4.3)} \\ &= A_p C_p R_3^p \int_0^\infty pt^{p-1} v(\{f^{\#^1} > t\})dt \quad \text{(by the change of variable)} \\ &= A_p C_p R_3^p \int_S |f^{\#^1}|^p v\,d\sigma. \end{aligned}$$

Hence,

$$\int_S |Mf|^p u\, d\sigma \leq N^p A_p \parallel f \parallel_{L^p(v)}^p + A_p C_p R_3^p \int_S |f^{\#^1}|^p v\, d\sigma + e_p \varepsilon.$$

The first and the last integrals are independent of ε. Letting $\varepsilon \searrow 0$, therefore, the proof is complete after accepting Lemma 4.3.

Lemma 4.2. *Let α_v be defined in (4.14). Then, for every $q \geq p$ and every $r > 0$,*

$$\int_0^r t^{q-1} \alpha_v(t) dt < \infty.$$

Proof. For a positive real number r, we set

$$I_r = \int_0^r q t^{q-1} \alpha_v(t) dt. \tag{4.16}$$

Since $\sum_j v(Q_{t,j}) \leq \int_{\{Mf>t\}} v\, d\sigma$, we have

$$I_r \leq \int_0^r q t^{q-1} \int_{\{Mf>t\}} v\, d\sigma\, dt.$$

We note that I_r is finite, since $p \geq p_0$ and it is bounded by

$$\frac{qr^{q-p}}{p} \int_0^r p t^{p-1} \int_{\{Mf>t\}} v\, d\sigma\, dt \leq \frac{qr^{q-p}}{p} \parallel Mf \parallel_{L^p(v)}^p < \infty,$$

since $M f \in L^p(v)$. Therefore, the proof is complete.

Now, filling up next lemma, we finish the proof of Theorem 1.2.

Lemma 4.3. *Under the same assumption as Theorem 1.2, if α_v is defined in (4.14), then there is a constant C_p such that*

$$\int_0^\infty t^{p-1} \alpha_v(t) dt \leq C_p \int_0^\infty t^{p-1} v(\{f^{\#^1} > t\}) dt.$$

Proof. Recall (4.13), i.e.,

$$\alpha_v(t) \leq v(\{f^{\#^1} > \lambda t\}) + \left(1 - (1 - 2\lambda)^p\right) \alpha_v(R_2^{-1} \kappa t).$$

By integration, it follows that

$$\begin{aligned}
&\int_0^r t^{p-1} \alpha_v(t)\, dt \\
&\leq \int_0^r t^{p-1} v\left(\{f^{\#1} > \lambda t\}\right) dt \\
&\quad + \left(1 - (1 - 2\lambda)^p\right) \int_0^r t^{p-1} \alpha_v\left(R_2^{-1} kt\right) dt \\
&= \int_0^r t^{p-1} v\left(\{f^{\#1} > \lambda t\}\right) dt \\
&\quad + \left(1 - (1 - 2\lambda)^p\right) R_2^p k^{-p} \int_0^{R_2^{-1} kr} t^{p-1} \alpha_v(t)\, dt \\
&\leq \int_0^r t^{p-1} v\left(\{f^{\#1} > \lambda t\}\right) dt \\
&\quad + 2^p R_2^{2p} \left(1 - (1 - 2\lambda)^p\right) \int_0^r t^{p-1} \alpha_v(t)\, dt \quad \left(\text{since } k = 2^{-1} R_2^{-1}, R_2^{-1} k < 1\right),
\end{aligned} \tag{4.17}$$

where the equality is due to the change of variable.

Take a small λ so that

$$\begin{aligned} 2^p R_2^{2p} \left(1 - (1 - 2\lambda)^p\right) &< 1/2, \\ 2\lambda &< 1, \end{aligned}$$

where the second inequality comes from (4.10). Then, by Lemma 4.2, (4.17) can be written as

$$\begin{aligned} \frac{1}{2} \int_0^r t^{p-1} \alpha_v(t) dt &\le \int_0^r t^{p-1} v\left(\left\{f^{\#^1} > \lambda t\right\}\right) dt \\ &= \lambda^{-p} \int_0^{\lambda r} t^{p-1} v\left(\left\{f^{\#^1} > t\right\}\right) dt, \end{aligned}$$

where the equality is caused by the change of variable.

Finally, letting $r \nearrow \infty$, we obtain

$$\int_0^\infty t^{p-1} \alpha_v(t)\, dt \le 2\lambda^{-p} \int_0^\infty t^{p-1} v\left(\left\{f^{\#^1} > t\right\}\right) dt.$$

Therefore, the proof is complete.

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Authors' contributions

KSR drove a sufficient condition for two-weighted norm inequalities for *K*. In proving cross-weighted norm inequalities between the Hardy-Littlewood maximal function and the sharp maximal function on the unit sphere, and JL carried out the study about the covering lemma on the sphere. All authors read and approved the final manuscript.

Competing interests

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