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# Existence and convergence theorems for the new system of generalized mixed variational inequalities in Banach spaces

Nawitcha Onjai-uea[1,2] and Poom Kumam[1,2*]

* Correspondence: poom.kum@kmutt.ac.th
[1]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok 10140, Thailand
Full list of author information is available at the end of the article

**Abstract**

In this article, we introduce and consider the three-step iterative algorithms for solving a new system of generalized mixed variational inequalities involving different three multi-valued operators. In this study, we use a generalized *f*-projection method for finding the solutions of generalized system of mixed variational inequalities in Banach spaces. Our result in this article improves and generalizes some known corresponding results in the literature.

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**Keywords:** a new generalized system mixed variational inequality problem, relaxed cocoercive mappings, fixed point problem, generalized *f*-projection

## 1. Introduction

Let $B$ be a real Banach space which dual space $B^*$ and $C$ be a nonempty closed convex subset of $B$. Let $\langle \cdot, \cdot \rangle$ be the dual pair between $B$ and $B^*$, $J$ denotes the normalized duality mapping and $\mathbb{R}$ be the field of real numbers. Let $T_1, T_2, T_3 : C \to 2^{B^*}$ be nonlinear operators. Let $f_1, f_2, f_3$: $B \to (-\infty, +\infty]$ be three mappings. We consider the following problem:

Find $x^*, y^*, z^* \in C \subset B$ such that there exist $u^* \in T_1(y^*)$, $v^* \in T_2(z^*)$, and $w^* \in T_3(x^*)$ satisfying

$$(\star) \begin{cases} \langle u^* + Jx^* - Jy^*, x - x^* \rangle + f_1(x) - f_1(x^*) \geq 0, & \forall x \in C, \\ \langle v^* + Jy^* - Jz^*, x - y^* \rangle + f_2(x) - f_2(y^*) \geq 0, & \forall x \in C, \\ \langle w^* + Jz^* - Jx^*, x - z^* \rangle + f_3(x) - f_3(z^*) \geq 0, & \forall x \in C. \end{cases}$$

The problem (★) is called *the system of generalized mixed variational inequality problems*, the solution of (★) is denoted by (*SGMVIP*).

Some special cases of the problem (★):

(I) If $f_1(x) = f_2(x) = f_3(x) = 0$, $\forall x \in C$, then problem (★) is equivalent to find $x^*, y^*, z^* \in C$ such that

$$\begin{cases} \langle u^* + Jx^* - Jy^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle v^* + Jy^* - Jz^*, x - y^* \rangle \geq 0, & \forall x \in C, \\ \langle w^* + Jz^* - Jx^*, x - z^* \rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.1}$$

The problem (1.1) is called *the system of generalized variational inequality problems*, the solution of (1.1) is denoted by (*SGVIP*).

(II) If $x^* = z^*$, $T_3(y) = 0$, and $f_3(x) = 0$, $\forall x, y \in C$, then problem (★) is equivalent to *the two step of the system of generalized mixed variational inequality problems:* Find $x^*, y^* \in C \subset B$ such that there exist $u^* \in T_1(y^*)$ and $v^* \in T_2(x^*)$ satisfying

$$\begin{cases} \langle u^* + Jx^* - Jy^*, x - x^* \rangle + f_1(x) - f_1(x^*) \geq 0, \forall x \in C, \\ \langle v^* + Jy^* - Jx^*, x - y^* \rangle + f_2(x) - f_2(y^*) \geq 0, \ \forall x \in C \end{cases} \tag{1.2}$$

which was studied by Zhang and Deng [1].

(III) If $T = T_1 = T_2 = T_3$, $f_2(x) = f_3(x) = 0$, $\forall x \in C$ and $x^* = y^* = z^*$, then problem (★) is equivalent to the *generalized variational inequality problem associated with C, T, and f denoted by GVI(C, T, f)*: find $x^* \in C$ such that there exist $u^* \in T(x^*)$ satisfying

$$\langle u^*, x - x^* \rangle + f_1(x) - f_1(x^*) \geq 0, \quad \forall x \in C \tag{1.3}$$

which was studied by Fan et al. [2].

If $f_1(x) = 0$, $\forall x \in C$ and $T$ is single-valued, then problem (1.3) reduces to the *classical variational inequality problem*, which consists in finding $x \in C$ such that

$$\langle Tx, x - x^* \rangle \geq 0, \quad \forall x \in C \tag{1.4}$$

which is known as the *classical variational inequality* introduced and studied by Stampacchia [3] in 1964. For the recent applications, numerical methods and formulations, (see for example [3-10]) and the references therein. The variational inequalities are equivalent to the fixed point problems. In particular, the solution of the variational inequalities can be computed using the iterative projection method. Alber [11] presented some applications of the generalized projections to approximately solving variational inequalities and von Neumann intersection problem in Banach spaces. In 2005, Li [12] extended the generalized projection operator from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and studied some properties of the generalized projection operator with applications to solving the variational inequality in Banach spaces.

In 2007, Wu and Huang [13], they proved some properties of the generalized $f$-projection operator and proposed iterative method of approximating solutions for a class of generalized variational inequalities in Banach spaces. In 2009, Fan et al. [2] presented some basic results for the generalized $f$-projection operator and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces. In 2010, Petrot [8] used the resolvent operator technique to find the common solutions for a generalized system of relaxed cocoercive mixed variational inequality problems and fixed point problems for Lipschitz mappings in Hilbert spaces.

In 2011, Zhang and Deng [1] introduced and considered the system of mixed variational inequalities in Banach spaces. Using the generalized $f$-projection operator technique, they introduced two-step iterative methods for solving the system of mixed variational inequalities and proved the convergence of the proposed iterative methods under suitable conditions in Banach spaces.

Noor [4] suggested and analyzed several three-step iterative methods, which are also known as Noor iterations, for solving variational inequalities. It has been shown that three-step iterative methods are more efficient than two-step and one-step iterative methods. In addition, it is known that the convergence analysis of three-step can be proved under much weaker conditions.

Motivated and inspired by the recent research studies in this fascinating area, the purpose of this article is to introduce and analyze three-step iterative algorithm for finding a new system of generalized mixed variational inequality problems with three difference multi-valued operators in Banach spaces. Using the generalized *f*-projection method. The results presented in this article extend and improve the results of Zhang and Deng [1] and Fan et al. [2] and some authors.

## 2. Preliminaries

A Banach space $B$ is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in B$ with $||x|| = ||y|| = 1$ and $x \neq y$. Let $U = \{x \in B: ||x|| = 1\}$ be the unit sphere of $B$. then a Banach space $B$ is said to be *smooth* if the $\lim_{t\to 0} \frac{||x+ty|| - ||x||}{t}$ exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in U$. Let $B$ be a Banach space. The *modulus of smoothness* of $B$ is the function $\rho_B$: $[0, \infty) \to [0, \infty)$ defined by $\rho_B(t) = \sup\left\{\frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1, ||y|| \leq t\right\}$. The *modulus of convexity* of $B$ is the function $\eta_B$: $[0, 2] \to [0, 1]$ defined by $\eta_B(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B, ||x|| = ||y|| = 1, ||x-y|| \geq \varepsilon\right\}$. The *normalized duality mapping* $J : B \to 2^{E^*}$ is defined by $J(x) = \{x^* \in B^*: \langle x, x^*\rangle = ||x||^2, ||x^*|| = ||x||\}$. If $B$ is a Hilbert space, then $J = I$, where $I$ is the identity mapping.

If $B$ is a reflexive smooth and strictly convex Banach space and $J^*$: $B^* \to 2^B$ is the normalized duality mapping on $B^*$, then $J^{-1} = J^*, JJ^* = I_{B^*}$ and $J^*J = I_B$, where $I_B$ and $I_B^*$ are the identity mappings on $B$ and $B^*$. If $B$ is a uniformly smooth and uniformly convex Banach space, then $J$ is uniformly norm-to-norm continuous on bounded subsets of $B$ and $J^*$ is also uniformly norm-to-norm continuous on bounded subsets of $B^*$.

Let $B$ and $F$ be Banach spaces, $T$: $D(T) \subset B \to F$, the operator $T$ is said to be *compact* if it is continuous and maps the bounded subsets of $D(T)$ onto the relatively compact subsets of $F$; the operator $T$ is said to be *weak to norm continuous* if it is continuous from the weak topology of $B$ to the strong topology of $F$.

We also need the following lemmas for the proof of our main results.

**Lemma 2.1**. [14]. *Let $q > 1$ and $r > 0$ be two fixed real numbers. Let $B$ be a uniformly convex Banach space if and only if there exists a continuous strictly increasing and convex function $g$: $[0, +\infty) \to [0, +\infty)$, $g(0) = 0$ such that*

$$||\lambda x + (1-\lambda)y||^q \leq \lambda||x||^q + (1-\lambda)||y||^q - \varsigma_q(\lambda)g(||x-y||)$$

*for all $x, y \in B_r = \{x \in B: ||x|| \leq r\}$ and $\lambda \in [0, 1]$, where $\varsigma_q(\lambda) = \lambda(1 - \lambda)^q + \lambda^q(1 - \lambda)$.*

For case $q = 2$, we have

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)g(||x-y||).$$

**Lemma 2.2**. [15]. *Let $B$ be a uniformly convex and uniformly smooth Banach space. We have*

$$||\phi + \Phi||^2 \leq ||\phi||^2 + 2\left\langle \Phi, J^*(\phi + \Phi)\right\rangle, \quad \forall \phi, \Phi \in E^*.$$

Next, we recall the concept of the generalized $f$-projection operator. Let $G: B^* \times C \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, x) = ||\xi||^2 - 2\langle \xi, x\rangle + ||x||^2 + 2\rho f(x), \tag{2.1}$$

where $\xi \in B^*$, $\rho$ is positive number and $f: C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semi-continuous. From definitions of $G$ and $f$, it is easy to see the following properties:

(i) $(||\xi|| - ||x||)^2 + 2\rho f(x) \leq G(\xi, x) \leq (||\xi|| + ||x||)^2 + 2\rho f(x)$;
(ii) $G(\xi, x)$ is convex and continuous with respect to $x$, when $\xi$ is fixed;
(iii) $G(\xi, x)$ is convex and lower semicontinuous with respect to $\xi$, when $x$ is fixed.

**Definition 2.3**. Let $B$ be a real Banach space with its dual $B^*$. Let $C$ be a nonempty closed convex subset of $B$. We say that $\pi_C^f : B^* \to 2^C$ is *a generalized f-projection operator* if

$$\pi_C^F \xi = \{u \in C : G(\xi, u) = \inf_{y \in C} G(\xi, y)\}, \quad \forall \xi \in B^*.$$

In this article, we fixed $\rho = 1$, we have

$$G(\xi, x) = ||\xi||^2 - 2\langle \xi, x\rangle + ||x||^2 + 2f(x).$$

For the generalized $f$-projection operator, Wu and Hung [13] proved the following basic properties.

**Lemma 2.4**. [16]. *Let B be a reflexive Banach space with its dual $B^*$ and let C be a nonempty closed convex subset of B. The following statement holds:*

(i) $\pi_C^f \xi$*is nonempty closed convex subset of C for all* $\xi \in B^*$;
(ii) *if B is smooth, then for all* $\xi \in E^*$, $x \in \pi_C^f \xi$*if and only if*

$$\langle \xi - Jx, x - y\rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C;$$

(iii) *if B is smooth, then for any* $\xi \in B^*$, $\pi_C^f \xi = (J + \rho\partial f)^{-1}\xi$, *where* $\partial f$ *is the subdifferential of the proper convex and lower semi-continuous functional f.*

**Lemma 2.5**. [16]. *If* $f(x) \geq 0$ *for all* $x \in C$, *then for any* $\rho > 0$, *we have*

$$G(Jx, y) \leq G(\xi, y) + 2\rho f(y), \quad \forall \xi \in B^*, y \in C, x \in \pi_C^f \xi.$$

**Lemma 2.6**. [2]. *Let B be a reflexive strictly convex Banach space with its dual $B^*$ and let C be a nonempty closed convex subset of B. If* $f: C \to \mathbb{R} \cup \{+\infty\}$ *is proper, convex, and lower semi-continuous, then*

(i) $\pi_C^f : B^* \to C$*is single valued and norm to weak continuous;*
(ii) *if B has the property (h), that is, for any sequence* $\{x_n\} \subset B$, $x_n \rightharpoonup x \in E$ *and* $||x_n|| \to ||x||$, *implies* $x_n \to x$, *then* $\pi_C^f : B^* \to C$*is continuous.*

Defined the functional $G_2$: $B \times C \to \mathbb{R} \cup \{+\infty\}$ by

$$G_2(x, y) = G(Jx, y), \forall x \in B, y \in C.$$

## 3. Algorithms

First, we establish a useful Lemma for solving the new system of generalized mixed variational inequalities is equivalent to find a fixed point of generalized $f$-projection operator. For this purpose, we recall the following result.

**Lemma 3.1.** *Let C be nonempty subset of a reflexive, strictly convex and smooth Banach space B. If* $f_1, f_2, f_3$: $C \to (-\infty, +\infty]$ *are proper, convex, and lower semi-continuous, then* $(x^*, y^*, z^*)$ *is a solution of problem* (★) *is equivalent to find* $x^*$, $y^*$, *and* $z^*$ *such that* $u^* \in T_1(y^*)$, $v^* \in T_2(z^*)$, $w^* \in T_3(x^*)$ *and*

$$\begin{cases} x^* = \pi_C^{f_1}(Jy^* - u^*), \\ y^* = \pi_C^{f_2}(Jz^* - v^*), \\ z^* = \pi_C^{f_3}(Jx^* - w^*). \end{cases}$$

*Proof.* Since $B$ is a reflexive strictly convex and smooth Banach space, we know that $J$ is single-valued and $\pi_C^{f_i}(i = 1, 2, 3)$ is well defined and single valued. In fact, $(x^*, y^*, z^*)$ is a solution of problem (★) if and only if

$$\begin{aligned} &\langle u^* + Jx^* - Jy^*, x - x^* \rangle + f_1(x) - f_1(x^*) \geq 0, \quad \forall x \in C, \\ &\langle v^* + Jy^* - Jz^*, x - y^* \rangle + f_2(x) - f_2(y^*) \geq 0, \quad \forall x \in C, \\ &\langle w^* + Jz^* - Jx^*, x - z^* \rangle + f_3(x) - f_3(z^*) \geq 0, \quad \forall x \in C \end{aligned}$$

if and only if for all $x \in C$,

$$\begin{aligned} &\langle (Jy^* - u^*) - Jx^*, x - x^* \rangle + f_1(x) - f_1(x^*) \geq 0, \quad \forall x \in C, \\ &\langle (Jz^* - v^*) - Jy^*, y - x \rangle + f_2(x) - f_2(y^*) \geq 0, \quad \forall x \in C, \\ &\langle (Jx^* - w) - Jz^*, z^* - x \rangle + f_3(x) - f_3(z^*) \geq 0, \quad \forall x \in C. \end{aligned}$$

By Lemma 2.4 (ii), if and only if

$$\begin{cases} x* = \pi_C^{f_1}(Jy* - u*), \\ y* = \pi_C^{f_2}(Jz* - v*), \\ z* = \pi_C^{f_3}(Jx* - w*). \end{cases}$$

This complete the proof.

**Algorithm 3.2**. For arbitrarily chosen initial points $x_0$, $y_0$, $z_0 \in C$; compute the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ such that

$$\begin{cases} w_n \in T_3x_n, u_n \in T_1y_n, v_n \in T_2z_n, \\ z_n = (1 - \gamma_n)x_n + \gamma_n\pi_C^{f_3}(Jx_n - w_n), \\ y_n = (1 - \beta_n)x_n + \beta_n\pi_C^{f_2}(Jz_n - v_n), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\pi_C^{f_1}(Jy_n - u_n), \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$, $\forall n \geq 0$.

**Algorithm 3.3**. For arbitrarily chosen initial points $x_0$, $y_0$, $z_0 \in C$; compute the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ such that

$$\begin{cases} w_n \in T_3x_n, u_n \in T_1y_n, v_n \in T_2z_n, \\ z_n = \pi_C^{f_3}(Jx_n - w_n), \\ y_n = \pi_C^{f_2}(Jz_n - v_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n\pi_C^{f_1}(Jy_n - u_n), \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, $\forall n \geq 0$.

If $f_1(x) = f_2(x) = f_3(x) = 0$, $\forall x \in C$, then Algorithm 3.2 reduces to the following iterative method for solving (*SGVIP*) problem (1.1).

**Algorithm 3.4**. For arbitrarily chosen initial points $x_0$, $y_0$, $z_0 \in C$; compute the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ such that

$$\begin{cases} w_n \in T_3x_n, u_n \in T_1y_n, v_n \in T_2z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n\pi_C(Jx_n - w_n), \\ y_n = (1-\beta_n)x_n + \beta_n\pi_C(Jz_n - v_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n\pi_C(Jy_n - u_n), \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$, $\forall n \geq 0$.

If $\gamma_n = 0$, $f_3(x) = 0$ and $T_3(y) = 0$, $\forall x, y \in C$, then Algorithm 3.2 reduces to the following iterative method for solving problem (1.2).

**Algorithm 3.5**. For arbitrarily chosen initial points $x_0$, $y_0$, $\in C$; compute the sequences $\{x_n\}$,$\{y_n\}$ such that

$$\begin{cases} u_n \in T_1y_n, v_n \in T_2x_n, \\ y_n = (1-\beta_n)x_n + \beta_n\pi_C^{f_2}(Jx_n - v_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n\pi_C^{f_1}(Jy_n - u_n), \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$, $\forall n \geq 0$.

If $\beta_n = \gamma_n = 0$, $T = T_1 = T_2 = T_3$, $f_2(x) = f_3(x) = 0$, $\forall x \in C$, then Algorithm 3.2 reduces to the following iterative method for solving problem (1.3).

**Algorithm 3.6**. For arbitrarily chosen initial points $x_0 \in C$; compute the sequence $\{x_n\}$ such that

$$\begin{cases} u_n \in Tx_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n\pi_C^{f_1}(Jx_n - u_n), \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, $\forall n \geq 0$.

## 4. Existence and Convergence Theorems

Now, we state and prove the main results of this study.

**Theorem 4.1**. *Let C be a nonempty, closed convex subset of a uniformly convex and uniformly smooth Banach space B with dual space $B^*$ and $0 \in C$. If the upper semi-continuous set-valued mappings $T_1, T_2, T_3 : C \to 2^{B^*}$ with closed values and the proper convex lower semi-continuous mapping $f_1, f_2, f_3$: $C \to \mathbb{R} \cup \{+\infty\}$ satisfy the following conditions:*

(i) $f_k(x) \geq 0$ *for all* $x \in C$ *and* $f_k(0) = 0$ *for* $k = 1, 2, 3$;

(ii) *for any* $x \in C$ *and any* $z_k \in T_k(x)$, $\langle z_k, J^*(Jx - z_k)\rangle \geq 0$ *for* $k = 1, 2, 3$;

(iii) *the set-valued mappings* $J - T_k$ *are compact for* $k = 1, 2, 3$;

(iv) $0 < a \leq \alpha_n \leq b < 1$, $0 < c \leq \beta_n \leq d < 1$ *and* $0 < e \leq \gamma_n \leq h < 1$, $\exists a, b, c, d, e, h \in (0, 1)$.

*Then problem (★) has a solution $(x^*, y^*, z^*)$ and the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ defined by Algorithm 3.2 have convergent subsequences $\{x_{n_i}\}$, $\{y_{n_i}\}$, and $\{z_{n_i}\}$such that* $x_{n_i} \to x^*, y_{n_i} \to y^*$, and $z_{n_i} \to z^*$*as* $i \to \infty$, *respectively.*

*Proof*. Since $B$ is a uniformly convex and uniformly smooth Banach space, we know that $J$ is a bijection from $B$ onto $B^*$ and uniformly continuous on any bounded subsets of $B$. Hence $\pi_C^{f_k}$ is well defined and single-valued and the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are well defined.

Let $G_2(x, y) = G(Jx, y)$. Then, for any $x \in C$,

$$G_2(x,0) = G(Jx,0) = ||Jx||^2 - 2\langle Jx, 0\rangle + 2f_k(0) = ||x||^2, \quad k = 1, 2, 3.$$

From Lemma 2.5 and the above equation, we have

$$\begin{aligned}\left\|\pi_C^{f_1}(Jy_n - u_n)\right\|^2 &= G_2\left(\pi_C^{f_1}(Jy_n - u_n), 0\right)\\ &= G\left(J\left(\pi_C^{f_1}(Jy_n - u_n)\right), 0\right)\\ &\le G(Jy_n - u_n, 0) = ||Jy_n - u_n||^2.\end{aligned}$$

Similarly proof, we also have

$$\left\|\pi_C^{f_2}(Jz_n - v_n)\right\|^2 \le ||Jz_n - v_n||^2$$

and

$$\left\|\pi_C^{f_3}(Jx_n - w_n)\right\|^2 \le ||Jx_n - w_n||^2.$$

By Lemma 2.2 and condition (ii), we obtain

$$\begin{aligned}&||Jy_n - u_n||^2 \le ||Jy_n||^2 - 2\left\langle u_n, J^*(Jy_n - u_n)\right\rangle \le ||y_n||^2,\\ &||Jz_n - v_n||^2 \le ||Jz_n||^2 - 2\left\langle v_n, J^*(Jz_n - v_n)\right\rangle \le ||z_n||^2,\\ &||Jx_n - w_n||^2 \le ||Jx_n||^2 - 2\left\langle w_n, J^*(Jx_n - w_n)\right\rangle \le ||x_n||^2.\end{aligned} \tag{4.1}$$

It follows that

$$\begin{aligned}||z_n|| &= ||(1-\gamma_n)x_n + \gamma_n\pi_C^{f_3}(Jx_n - w_n)||\\ &\le (1-\gamma_n)||x_n|| + \gamma_n||\pi_C^{f_3}(Jx_n - w_n)||\\ &\le (1-\gamma_n)||x_n|| + \gamma_n||x_n|| = ||x_n||,\end{aligned} \tag{4.2}$$

$$\begin{aligned}||y_n|| &= ||(1-\beta_n)x_n + \beta_n\pi_C^{f_2}(Jz_n - v_n)||\\ &\le (1-\beta_n)||x_n|| + \beta_n||\pi_C^{f_2}(Jz_n - v_n)||\\ &\le (1-\beta_n)||x_n|| + \beta_n||z_n||\\ &\le (1-\beta_n)||x_n|| + \beta_n||x_n|| = ||x_n||\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}||x_{n+1}|| &= ||(1-\alpha_n)x_n + \alpha_n\pi_C^{f_1}(Jy_n - u_n)||\\ &\le (1-\alpha_n)||x_n|| + \alpha_n||\pi_C^{f_1}(Jy_n - u_n)||\\ &\le (1-\alpha_n)||x_n|| + \alpha_n||y_n||\\ &\le (1-\alpha_n)||x_n|| + \alpha_n||x_n|| = ||x_n||.\end{aligned} \tag{4.4}$$

Hence, the sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{\pi_C^{f_1}(Jy_n - u_n)\}, \{\pi_C^{f_2}(Jz_n - v_n)\}$, *and* $\{\pi_C^{f_3}(Jx_n - w_n)\}$ are bounded. So, we take a positive number $r_1, r_2, r_3, r_4, r_5, r_6$ such that $||x_n|| \leq r_1, ||y_n|| \leq r_2, ||z_n|| \leq r_3, ||\pi_C^{f_1}(Jy_n - u_n)|| \leq r_4, ||\pi_C^{f_2}(Jz_n - v_n)|| \leq r_5$ and $||\pi_C^{f_3}(Jx_n - w_n)|| \leq r_6$. We choose a number $r = \max\{r_1, r_2, r_3, r_4, r_5, r_6\}$ such that $\{x_n\}, \{y_n\}, \{z_n\}, \{\pi_C^{f_1}(Jy_n - u_n)\}, \{\pi_C^{f_2}(Jz_n - v_n)\}, \{\pi_C^{f_3}(Jx_n - w_n)\} \subset B_r$, by Lemma 2.1, for $q = 2$ there exists a continuous, strictly increasing, and convex function $g$: $[0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for $\alpha_n, \beta_n, \gamma_n \in [0, 1]$, we have

$$\begin{aligned}||z_n||^2 &= ||(1-\gamma_n)x_n + \gamma_n \pi_C^{f_3}(Jx_n - w_n)||^2 \\ &\leq (1-\gamma_n)||x_n||^2 + \gamma_n||\pi_C^{f_3}(Jx_n - w_n)||^2 - \gamma_n(1-\gamma_n)g(||x_n - \pi_C^{f_3}(Jx_n - w_n)||) \\ &\leq ||x_n||^2 - \gamma_n(1-\gamma_n)g(||x_n - \pi_C^{f_3}(Jx_n - w_n)||)\end{aligned}$$

and

$$\begin{aligned}||y_n||^2 &= ||(1-\beta_n)x_n + \beta_n \pi_C^{f_2}(Jz_n - v_n)||^2 \\ &\leq (1-\beta_n)||x_n||^2 + \beta_n||\pi_C^{f_2}(Jz_n - v_n)||^2 - \beta_n(1-\beta_n)g(||x_n - \pi_C^{f_2}(Jz_n - v_n)||) \\ &\leq (1-\beta_n)||x_n||^2 + \beta_n||z_n||^2 - \beta_n(1-\beta_n)g(||x_n - \pi_C^{f_2}(Jz_n - v_n)||) \\ &\leq (1-\beta_n)||x_n||^2 + \beta_n\left(||x_n||^2 - \gamma_n(1-\gamma_n)g(||x_n - \pi_C^{f_3}(Jx_n - w_n)||)\right) \\ &\quad - \beta_n(1-\beta_n)g(||x_n - \pi_C^{f_2}(Jz_n - v_n)||) \\ &\leq ||x_n||^2 - \beta_n\gamma_n(1-\gamma_n)g(||x_n - \pi_C^{f_3}(Jx_n - w_n)||) \\ &\quad - \beta_n(1-\beta_n)g(||x_n - \pi_C^{f_2}(Jz_n - v_n)||).\end{aligned}$$

We compute

$$\begin{aligned}||x_{n+1}||^2 &= ||(1-\alpha_n)x_n + \alpha_n \pi_C^{f_1}(Jy_n - u_n)||^2 \\ &\leq (1-\alpha_n)||x_n||^2 + \alpha_n||\pi_C^{f_1}(Jy_n - u_n)||^2 - \alpha_n(1-\alpha_n)g(||x_n - \pi_C^{f_1}(Jy_n - u_n)||) \\ &\leq (1-\alpha_n)||x_n||^2 + \alpha_n||y_n||^2 - \alpha_n(1-\alpha_n)g(||x_n - \pi_C^{f_1}(Jy_n - u_n)||) \\ &\leq (1-\alpha_n)||x_n||^2 + \alpha_n\Big(||x_n||^2 - \beta_n\gamma_n(1-\gamma_n)g(||x_n - \pi_C^{f_3}(Jx_n - w_n)||) \\ &\quad -\beta_n(1-\beta_n)g(||x_n - \pi_C^{f_2}(Jz_n - v_n)\Big) - \alpha_n(1-\alpha_n)g(||x_n - \pi_C^{f_1}(Jy_n - u_n)||) \\ &\leq ||x_n||^2 - \alpha_n\beta_n\gamma_n(1-\gamma_n)g(||x_n - \pi_C^{f_3}(Jx_n - w_n)||) \\ &\quad - \alpha_n\beta_n(1-\beta_n)g(||x_n - \pi_C^{f_2}(Jz_n - v_n)||) - \alpha_n(1-\alpha_n)g(||x_n - \pi_C^{f_1}(Jy_n - u_n)||).\end{aligned}$$

From above, we obtain that

$$\begin{aligned}\alpha_n(1-\alpha_n)g(||x_n - \pi_C^{f_1}(Jy_n - u_n)||) &\leq ||x_n||^2 - ||x_{n+1}||^2 \\ \alpha_n\beta_n(1-\beta_n)g(||x_n - \pi_C^{f_2}(Jz_n - v_n)||) &\leq ||x_n||^2 - ||x_{n+1}||^2 \\ \alpha_n\beta_n\gamma_n(1-\gamma_n)g(||x_n - \pi_C^{f_3}(Jx_n - w_n)||) &\leq ||x_n||^2 - ||x_{n+1}||.^2\end{aligned}$$

By the condition (iv), we have

$$\begin{aligned}a(1-b)g(||x_n - \pi_C^{f_1}(Jy_n - u_n)||) &\leq ||x_n||^2 - ||x_{n+1}||^2 \\ ac(1-d)g(||xn - \pi_C^{f_2}(Jz_n - v_n)||) &\leq ||x_n||^2 - ||x_{n+1}||^2 \\ ace(1-h)g(||x_n - \pi_C^{f_3}(Jx_n - w_n)||) &\leq ||x_n||^2 - ||x_{n+1}||.^2\end{aligned}$$

Taking the sum for $n = 0, 1, 2, \ldots, m$ in the above inequality, we get

$$a(1-b)\sum_{n=0}^{m} g(||x_n - \pi_C^{f_1}(Jy_n - u_n)||) \le ||x_0||^2 - ||x_{m+1}||^2$$

and

$$\sum_{n=0}^{\infty} g(||x_n - \pi_C^{f_1}(Jy_n - u_n)||) < \infty. \tag{4.5}$$

It is easy to know that

$$\lim_{n\to\infty} g(||x_n - \pi_C^{f_1}(Jy_n - u_n)||) = 0.$$

Hence, there exist subsequences $\{x_{n_i}\} \subset \{x_n\}$ and $\{y_{n_i}\} \subset \{y_n\}$ such that $u_{n_i} \in T_{1y_{n_i}}$ and

$$\lim_{i\to\infty} ||x_{n_i} - \pi_C^{f_1}(Jy_{n_i} - u_{n_i})|| = 0.$$

By property of functional $g$, we have

$$\lim_{i\to\infty} ||x_{n_i} - \pi_C^{f_1}(Jy_{n_i} - u_{n_i})|| = 0. \tag{4.6}$$

Similarly, we can proof that

$$\begin{aligned} &\lim_{i\to\infty} ||x_{n_i} - \pi_C^{f_2}(Jz_{n_i} - v_{n_i})|| = 0, \\ &\lim_{i\to\infty} ||x_{n_i} - \pi_C^{f_3}(Jx_{n_i} - w_{n_i})|| = 0. \end{aligned} \tag{4.7}$$

Since $\{y_n\}$ is bounded sequence and $(J - T_1)$ is compact on $C$, without loss of generality there exist convergence subsequences say $\{Jy_{n_i} - u_{n_i}\}$ such that

$$\{Jy_{n_i} - u_{n_i}\} \to u_0 \in E^* \text{ as } i \to \infty. \tag{4.8}$$

By the continuity of the $\pi_C^{f_1}$, we have

$$\lim_{i\to\infty} \pi_C^{f_1}(Jy_{n_i} - u_{n_i}) = \pi_C^{f_1}(u_0). \tag{4.9}$$

Again since $\{x_n\}$, $\{z_n\}$ are bounded and $(J - T_2)$, $(J - T_3)$ are compact on $C$, without loss of generality there exist a convergence subsequence say $\{Jx_{n_i} - w_{n_i}\}$ and $\{Jz_{n_i} - v_{n_i}\}$ such that

$$\{Jx_{n_i} - w_{n_i}\} \to w_0 \in E^* \text{ as } i \to \infty, \tag{4.10}$$

and

$$\{Jz_{n_i} - v_{n_i}\} \to v_0 \in E^* \text{ as } i \to \infty. \tag{4.11}$$

By the continuity of $\pi_C^{f_2}$ and $\pi_C^{f_3}$, we have

$$\lim_{i\to\infty} \pi_C^{f_2}(Jz_{n_i} - v_{n_i}) = \pi_C^{f_2}(v_0) \tag{4.12}$$

and

$$\lim_{i\to\infty} \pi_C^{f_3}(Jx_{n_i} - w_{n_i}) = \pi_C^{f_3}(w_0). \tag{4.13}$$

Let $\pi_C^{f_1}(u_0) = x^*, \pi_C^{f_2}(v_0) = y^*$, and $\pi_C^{f_3}(w_0) = z^*$.

By using the triangle inequality, we have

$$||x_{n_i} - x^*|| \le ||x_{n_i} - \pi_C^{f_1}(Jy_{n_i} - u_{n_i})|| + ||\pi_C^{f_1}(Jy_{n_i} - u_{n_i}) - x^*||.$$

From (4.6) and (4.9), we also have

$$\lim_{i\to\infty} x_{n_i} = x^*. \tag{4.14}$$

From Algorithm 3.2 and (4.7), we obtain

$$||y_{n_i} - x_{n_i}|| = \beta_{n_i}||\pi_C^{f_2}(Jz_{n_i} - v_{n_i}) - x_{n_i}|| \to 0, \text{ as } n \to \infty.$$

Since above equations (4.7) and (4.12), we have

$$||y_{n_i} - \pi_C^{f_2}(Jz_{n_i} - v_{n_i})|| \le ||y_{n_i} - x_{n_i}|| + ||x_{n_i} - \pi_C^{f_2}(Jz_{n_i} - v_{n_i})|| \to 0, \text{ as } i \to \infty.$$

So,

$$||y_{n_i} - y^*|| \le ||y_{n_i} - \pi_C^{f_2}(Jz_{n_i} - v_{n_i})|| + ||\pi_C^{f_2}(Jz_{n_i} - v_{n_i}) - y^*|| \to 0, \text{ as } i \to \infty.$$

It follows that

$$\lim_{i\to\infty} y_{n_i} = y^*. \tag{4.15}$$

In the same way, we apply Algorithm 3.2, equations (4.7) and (4.13), we also have

$$\lim_{i\to\infty} z_{n_i} = z^*. \tag{4.16}$$

We can show that

$$\lim_{i\to\infty} u_{n_i} = \lim_{i\to\infty}(Jy_{n_i} - (Jy_{n_i} - u_{n_i})) = Jy^* - u_0. \tag{4.17}$$

Similarly, we have $\lim_{i\to\infty} v_{n_i} = Jz^* - v_0$ and $\lim_{i\to\infty} w_{n_i} = Jx^* - w_0$. Let $u^* = Jy^* - u_0$, $v^* = Jz^* - v_0$, and $w^* = Jx^* - w_0$. Since $T_1$, $T_2$, and $T_3$ are upper semi-continuous with closed values, $T_1$, $T_2$, and $T_3$ are closed, and then $u^* = T_1y^*$, $v^* = T_2z^*$, and $w^* = T_3x^*$. It follows from Algorithm 3.2 and the continuity of the operators $\pi_C^{f_1}, \pi_C^{f_2}$, and $\pi_C^{f_3}$ that

$$x^* = \pi_C^{f_1}(Jy^* - u^*),$$
$$y^* = \pi_C^{f_2}(Jz^* - v^*),$$
$$z^* = \pi_C^{f_3}(Jx^* - w^*).$$

This complete of the proof.

**Theorem 4.2.** *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space B with dual space $B^*$ and $0 \in C$. If the upper semi-continuous set-valued mappings $T_1, T_2, T_3 : C \to 2^{B^*}$with closed values, and the proper convex lower semi-continuous mapping $f_1, f_2, f_3$: $C \to \mathbb{R} \cup \{+\infty\}$ satisfy the following conditions:*

(i) $f_k(x) \ge 0$ *for all* $x \in C$ *and* $f_k(0) = 0$ *for* $k = 1, 2, 3$;
(ii) *for any* $x \in C$ *and any* $z_k \in T_k(x)$, $\langle z_k, J^*(Jx - z_k)\rangle \ge 0$ *for* $k = 1, 2, 3$;
(iii) *the set-valued mappings* $J - T_k$ *are compact for* $k = 1, 2, 3$;

(iv) $0 < a \leq \alpha_n \leq b < 1$, $\exists a, b \in (0, 1)$.

*Then problem (★) has a solution* $(x^*, y^*, z^*)$ *and the sequences* $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ *defined by Algorithm 3.3 have convergent subsequences* $\{x_{n_i}\}$, $\{y_{n_i}\}$, *and* $\{z_{n_i}\}$*such that* $x_{n_i} \to x^*, y_{n_i} \to y^*$, *and* $z_{n_i} \to z^*$*as* $i \to \infty$, *respectively.*

*Proof.* In this instance, (4.2) and (4.3) become

$$||z_n|| \leq ||\pi_C^{f_3}(Jx_n - w_n)|| \leq ||x_n||$$

and

$$||y_n|| \leq ||\pi_C^{f_2}(Jz_n - v_n)|| \leq ||z_n|| \leq ||x_n||.$$

Hence, the sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{\pi_C^{f_1}(Jy_n - u_n)\}, \{\pi_C^{f_2}(Jz_n - v_n)\}$, and $\{\pi_C^{f_3}(Jx_n - w_n)\}$ are bounded. Take a positive number $r_1, r_2, r_3, r_4, r_5, r_6$ such that $||x_n|| \leq r_1, ||y_n|| \leq r_2, ||z_n|| \leq r_3, ||\pi_C^{f_1}(Jy_n - u_n)|| \leq r_4, ||\pi_C^{f_2}(Jz_n - v_n)|| \leq r_5$, and $||\pi_C^{f_3}(Jx_n - w_n)|| \leq r_6$. We choose a number $r = \max\{r_1, r_2, r_3, r_4, r_5, r_6\}$ such that $\{x_n\}, \{y_n\}, \{z_n\}, \{\pi_C^{f_1}(Jy_n - u_n)\}, \{\pi_C^{f_2}(Jz_n - v_n)\}, \{\pi_C^{f_3}(Jx_n - w_n)\}$, by Lemma 2.1, for $q = 2$ there exists a continuous, strictly increasing, and convex function $g$: $[0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for $\alpha_n \in [0, 1]$, we have

$$\begin{aligned} ||x_{n+1}||^2 &= ||(1 - \alpha_n)x_n + \alpha_n \pi_C^{f_1}(Jy_n - u_n)||^2 \\ &\leq (1 - \alpha_n)||x_n||^2 + \alpha_n ||\pi_C^{f_1}(Jy_n - u_n)||^2 - \alpha_n(1 - \alpha_n)g\left(||x_n - \pi_C^{f_1}(Jy_n - u_n)||\right) \\ &\leq (1 - \alpha_n)||x_n||^2 + \alpha_n ||y_n||^2 - \alpha_n(1 - \alpha_n)g\left(||x_n - \pi_C^{f_1}(Jy_n - u_n)||\right) \\ &\leq (1 - \alpha_n)||x_n||^2 + \alpha_n ||x_n||^2 - \alpha_n(1 - \alpha_n)g\left(||x_n - \pi_C^{f_1}(Jy_n - u_n)||\right) \\ &\leq ||x_n||^2 - \alpha_n(1 - \alpha_n)g\left(|x_n - \pi_C^{f_1}(Jy_n - u_n)||\right). \end{aligned}$$

Similarly proof of Theorem 4.1, we obtain that

$$\lim_{i\to\infty} ||x_{n_i} - \pi_C^{f_1}(Jy_{n_i} - u_{n_i})|| = 0.$$

Since $\{y_n\}$, $\{z_n\}$, and $\{x_n\}$ are bounded sequences, $(J - T_1)$, $(J - T_2)$, and $(J - T_3)$ are compact on $C$ and by the continuity of the $\pi_C^{f_1}, \pi_C^{f_2}$, and $\pi_C^{f_3}$, we have

$$\begin{aligned} &\lim_{i\to\infty} \pi_c^{f_1}(Jy_{n_i} - u_{n_i}) = \pi_C^{f_1}(u_0) = x^* \\ &\lim_{i\to\infty} \pi_c^{f_2}(Jz_{n_i} - v_{n_i}) = \pi_C^{f_2}(v_0) = y^* \\ &\lim_{i\to\infty} \pi_c^{f_3}(Jx_{n_i} - w_{n_i}) = \pi_C^{f_3}(w_0) = z^*. \end{aligned}$$

Hence, we obtain that $\lim_{i\to\infty} x_{n_i} = x^*$.

By Algorithm 3.3, we get

$$||y_{n_i} - y^*|| \leq ||\pi_C^{f_2}\left(Jz_{n_i} - v_{n_i}\right) - y^*||$$

and

$$||z_{n_i} - z^*|| \leq ||\pi_C^{f_3}\left(Jx_{n_i} - w_{n_i}\right) - z^*||.$$

It follows from above, we obtain that $\lim_{i\to\infty} y_{n_i} = y^*$ and $\lim_{i\to\infty} z_{n_i} = z^*$. Similarly to the proof of Theorem 4.1, we can obtain this Theorem.

If $f_1(x) = f_2(x) = f_3(x) = 0$, $\forall x \in C$, then the following theorem can be obtained from Theorem 4.1 directly.

**Corollary 4.3**. *Let C be a nonempty, closed convex subset of a uniformly convex and uniformly smooth Banach space B with dual space $B^*$ and $0 \in C$. If the upper semi-continuous set-valued mappings $T_1, T_2, T_3 : C \to 2^{B^*}$ with closed values satisfy the following conditions:*

(i) *for any $x \in C$ and any $z_k \in T_k(x)$, $\langle z_k, J^*(Jx - z_k)\rangle \geq 0$ for $k = 1, 2, 3$;*
(ii) *the set-valued mappings $J - T_k$ are compact for $k = 1,2, 3$;*
(iii) $0 < a \leq \alpha_n \leq b < 1$, $0 < c \leq \beta_n \leq d < 1$ and $0 < e \leq \gamma_n \leq h < 1$, $\exists a, b, c, d, e, h \in (0, 1)$.

*Then problem (1.1) has a solution $(x^*, y^*, z^*)$ and the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ defined by Algorithm 3.4 have convergent subsequences $\{x_{n_i}\}$, $\{y_{n_i}\}$, and $\{z_{n_i}\}$such that $x_{n_i} \to x^*, y_{n_i} \to y^*$, and $z_{n_i} \to z^*$as $i \to \infty$, respectively.*

If $x_n = z_n$, $f_3(x) = 0$, and $T_3(y) = 0$, $\forall x, y \in C$, then the following theorem can be obtained from Theorem 4.1 directly.

**Corollary 4.4**. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space B with dual space $B^*$ and $0 \in C$. If the upper semi-continuous set-valued mappings $T_1, T_2 : C \to 2^{B^*}$with closed values and the proper convex lower semi-continuous mapping $f_1, f_2$: $C \to \mathbb{R} \cup \{+\infty\}$ satisfy the following conditions:*

(i) *$f_k(x) \geq 0$ for all $x \in C$ and $f_k(0) = 0$ for $k = 1, 2$;*
(ii) *for any $x \in C$ and any $z_k \in T_k(x)$, $\langle z_k, J^*(Jx - z_k)\rangle \geq 0$ for $k = 1, 2$;*
(iii) *the set-valued mappings $J - T_k$ are compact for $k = 1, 2$;*
(iv) *$0 < a \leq \alpha_n \leq b < 1$ and $0 < c \leq \beta_n \leq d < 1$, $\exists a, b, c, d \in (0, 1)$.*

*Then problem (1.2) has a solution $(x^*, y^*)$ and the sequences $\{x_n\}$ and $\{y_n\}$ defined by Algorithm 3.5 have convergent subsequences $\{x_{n_i}\}$and $\{y_{n_i}\}$such that $x_{n_i} \to x^*$and $y_{n_i} \to y^*$as $i \to \infty$, respectively.*

If $x_n = y_n = z_n$, $T = T_1 = T_2 = T_3$, and $f_2(x) = f_3(x) = 0$, $\forall x \in C$, then the following theorem can be obtained from Theorem 4.1 directly.

**Corollary 4.5**. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space B with dual space $B^*$ and $0 \in C$. If the upper semi-continuous set-valued mappings $T : C \to 2^{B^*}$with closed values and the proper convex lower semi-continuous mapping $f_1$: $C \to \mathbb{R} \cup \{+\infty\}$ satisfy the following conditions:*

(i) *$f_1(x) \geq 0$ for all $x \in C$ and $f_1(0) = 0$;*
(ii) *for any $x \in C$ and any $z \in T(x)$, $\langle z, J^*(Jx - z)\rangle \geq 0$;*
(iii) *the set-valued mappings $J - T$ is compact;*
(iv) $0 < a \leq \alpha_n \leq b < 1$, $\exists a, b \in (0, 1)$.

*Then problem (1.3) has a solution $x^*$ and the sequence $\{x_n\}$ defined by Algorithm 3.6 has a convergent subsequence $\{x_{n_i}\}$such that $x_{n_i} \to x^*$as $i \to \infty$.*

**Remark 4.6**. Theorems 4.1, 4.2, and Corollary 4.3 generalize and improve the main result in [1].

Our results generalize and extend the main result of [8] from a Hilbert space to a Banach space.

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**Author details**
[1]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok 10140, Thailand [2]Centre of Excellence in Mathematics, CHE, Sriayudthaya Rd., Bangkok 10140, Thailand

**Authors' contributions**
All authors contributed equally and significantly in this research work. All authors read and approved the final manuscript.

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