# SYMMETRY AND CONCENTRATION BEHAVIOR OF GROUND STATE IN AXIALLY SYMMETRIC DOMAINS

TSUNG-FANG WU

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We let $\Omega(r)$ be the axially symmetric bounded domains which satisfy some suitable conditions, then the ground-state solutions of the semilinear elliptic equation in $\Omega(r)$ are nonaxially symmetric and concentrative on one side. Furthermore, we prove the necessary and sufficient condition for the symmetry of ground-state solutions.

## 1. Introduction

Let $N \geq 2$ and $2 < p < 2^*$, where $2^* = 2N/(N-2)$ for $N \geq 3$ and $2^* = \infty$ for $N = 2$. Consider the semilinear elliptic equation

$$\begin{aligned} -\Delta u + u &= |u|^{p-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega$ is a domain in $\mathbb{R}^N$. When $\Omega$ is a bounded domain in $\mathbb{R}^N$ being convex in the $z_i$ direction and symmetric with respect to the hyperplane $\{z_i = 0\}$, the famous theorem by Gidas, Ni, and Nirenberg [6] (or see Han and Lin [7]): if $u$ is a positive solution of (1.1) belonging to $C^2(\Omega) \cap C(\overline{\Omega})$, then $u$ is axial symmetric in $z_i$. However, the axially symmetry of positive solution generally fails if $\Omega$ is not convex in the $z_i$ direction. For instance, Dancer [5], Byeon [2, 3], and Jimbo [8] proved that (1.1) in axially symmetric dumbbell-type domain has nonaxially symmetric positive solutions. Wang and Wu [13] and Wu [15] showed the same result in a finite strip with hole. In this paper, we want to show that the symmetry and concentration behavior of ground-state solutions in axially symmetric bounded domains $\Omega(r)$ (will be defined later), where the domains $\Omega(r)$ are different from those of Dancer [5], Byeon [2, 3], Jimbo [8], and are extensions of Wang and Wu [13] and Wu [15]. The definition of ground-state solution of (1.1) is stated as follows. Consider the energy functionals $a$, $b$, and $J$ in $H_0^1(\Omega)$,

$$a(u) = \int_\Omega \left(|\nabla u|^2 + u^2\right), \qquad b(u) = \int_\Omega |u|^p, \qquad J(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u). \tag{1.2}$$


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It is well known that the solutions of (1.1) are the critical points of the energy functional $J$. Consider the minimax problem

$$\alpha_\Gamma(\Omega)=\inf_{\gamma\in\Gamma(\Omega)}\max_{t\in[0,1]}J(\gamma(t)),\tag{1.3}$$

where

$$\Gamma(\Omega)=\{\gamma\in C([0,1],H_0^1(\Omega))\mid\gamma(0)=0,\gamma(1)=e\},\tag{1.4}$$

$J(e)=0$ and $e\neq 0$. We call a non zero critical point $u$ of $J$ in $H_0^1(\Omega)$ with $J(u)=\alpha_\Gamma(\Omega)$ a ground-state solution. It follows easily from the mountain pass theorem of Ambrosetti and Rabinowitz [1] that such a ground-state exists. We remark that the ground-state solutions of (1.1) can also be obtained by the Nehari minimization problem

$$\alpha_0(\Omega)=\inf_{v\in\mathbf{M}_0(\Omega)}J(v),\tag{1.5}$$

where $\mathbf{M}_0(\Omega)=\{u\in H_0^1(\Omega)\backslash\{0\}\,|\,a(u)=b(u)\}$. Note that $\mathbf{M}_0(\Omega)$ contains every nonzero solution of (1.1) and $\alpha_\Gamma(\Omega)=\alpha_0(\Omega)$ (see Willem [14] and Wang [12]).

Now, we consider the following assumptions of an axially symmetric unbounded domain $\Omega$. For the generic point $z=(x,y)\in\mathbb{R}^{N-1}\times\mathbb{R}$,

($\Omega$1) $\Omega$ is a $y$-symmetric (axially symmetric) domain of $\mathbb{R}^N$, that is, $(x,y)\in\Omega$ if and only if $(x,-y)\in\Omega$;

($\Omega$2) $\Omega$ is separated by a $y$-symmetric bounded domain $D$, that is, there exist two disjoint subdomains $\Omega_1$ and $\Omega_2$ of $\Omega$ such that

$$\begin{gathered}(x,y)\in\Omega_2\text{ if and only if }(x,-y)\in\Omega_1,\\ \Omega\setminus\overline{D}=\Omega_1\cup\Omega_2;\end{gathered}\tag{1.6}$$

($\Omega$3) equation (1.1) in $\Omega$ does not admit any solution $u\in H_0^1(\Omega)$ such that $J(u)=\alpha_0(\Omega)$.

Now, we give some examples. The infinite strip with hole: $\Omega'=\mathbf{A}^r\backslash\omega$, where $\mathbf{A}^r=B^{N-1}(0;r)\times\mathbb{R}$ and $\omega\subset\mathbf{A}^r$ is a $y$-symmetric bounded domain, and $\Omega''=\{(x,y)\in\mathbb{R}^{N-1}\times\mathbb{R}\,|\,|x|^2<|y|+1\}$. Clearly, $\Omega'$ and $\Omega''$ satisfy ($\Omega$1) and ($\Omega$2). Furthermore, by Lien, Tzeng, and Wang [9, Lemma 2.5], if $\Omega$ is a ball-up domain in $\mathbb{R}^N$, then (1.1) in $\Omega$ does not admit any solution $u\in H_0^1(\Omega)$ such that $J(u)=\alpha_0(\Omega)$. Thus, the domain $\Omega''$ satisfies ($\Omega$3). Moreover, along the same line of the proof of Lien, Tzeng, and Wang [9, Lemma 2.5], we obtain $\alpha_0(\Omega')=\alpha_0(\mathbf{A}^r)$. By Lemma 2.8, the domain $\Omega'$ satisfies ($\Omega$3) (or see Wang [12, Example 2.13 and Proposition 2.14]).

Let $\Omega(r)=\Omega\cap B^N(0;r)$ be a $y$-symmetric bounded domain and let $\Omega_t^+=\{(x,y)\in\Omega\mid y>t\}$ and $\Omega_t^-=\{(x,y)\in\Omega\mid y<t\}$, then our first main result is the following theorem.

Theorem 1.1. *Suppose that $\Omega$ satisfies ($\Omega$1), ($\Omega$2), and ($\Omega$3). Then, for each $\varepsilon>0$ and $l\geq 0$ there exists an $\tilde{r}(\varepsilon,l)>0$ such that for $r>\tilde{r}(\varepsilon,l)$, if $v$ is a ground-state solution of (1.1) in $\Omega(r)$, then either $\int_{\Omega_l^+}|v|^p<\varepsilon$ or $\int_{\Omega_l^-}|v|^p<\varepsilon$.*

Note that, if we take $\varepsilon = (p/(p-2))\alpha_0(\Omega)$ and $l = 0$, then there exists an $r_0 > 0$ such that for $r > r_0$, every ground-state solution of (1.1) in $\Omega(r)$ is not $y$-symmetric. Then, we have the following result.

Corollary 1.2. *Let $\varepsilon = (p/(p-2))\alpha_0(\Omega)$ and $l = 0$, then there exists an $r_0 > 0$ such that for $r > r_0$, (1.1) in $\Omega(r)$ has at least three positive solutions of which one is $y$-symmetric and the other two are not $y$-symmetric.*

By Theorem 1.1, for each $\varepsilon > 0$ and $l \geq 0$ there exists an $m_0 \in \mathbb{N}$ such that for each $m \geq m_0$, (1.1) in $\Omega(m)$ has a ground-state solution $v_m$ that satisfies $\int_{\Omega_l^+} |v_m|^p < \varepsilon$ or $\int_{\Omega_{-l}^-} |v_m|^p < \varepsilon$. Then, we have the following results.

Theorem 1.3. (i) *The sequence $\{v_m\}$ is a $(PS)_{\alpha_0(\Omega)}$-sequence in $H_0^1(\Omega)$ for $J$;*
(ii) *$v_m \rightharpoonup 0$ weakly in $L^p(\Omega)$ and in $H_0^1(\Omega)$ as $m \to \infty$.*

By Theorem 1.1, the ground-state solutions of (1.1) in $\Omega(r)$ are not $y$-symmetric for large $r$. In this motivation, we consider the positive ground-state solutions of the following equation:

$$\begin{aligned} -\Delta u + u &= f(u) \quad \text{in } \Theta, \\ u &= 0 \quad \text{on } \partial\Theta, \end{aligned} \tag{1.7}$$

where $\Theta$ is a $y$-symmetric bounded domain and the nonlinear term $f$ is usually assumed to satisfy the following conditions:

($f1$) $f(-t) = -f(t)$ and $f(t) = o(|t|)$ near $t = 0$;
($f2$) there exist two constants $\theta \in (0, 1/2)$ and $C_0 > 0$ such that $0 < F(u) \equiv \int_0^u f(s)ds \leq \theta u f(u)$ for all $u \geq C_0$;
($f3$) $|f(t)| \leq C|t|^q$ for some $1 < q < (N+2)/(N-2)$ if $N > 2$, $1 < q < \infty$ if $N = 2$ and for large $t$;
($f4$) $\partial^2 f/\partial t^2(t) \geq 0$ for $t \neq 0$.

$f(t) = |t|^{p-2}t$ is a typical example. Under the conditions ($f1$) through ($f3$), the definition of ground-state solutions of (1.7) is similar to the minimax problem (1.3). Here, we modify the proof of Chern and Lin [4] to get the following results.

Theorem 1.4. *Let $v \in C^2(\Theta) \cap C(\overline{\Theta})$ be a positive ground-state solutions of (1.7) in $\Theta$. Then, there exists a $z_0 \in \{y = 0\} \cap \Theta$ such that $(\partial v/\partial y)(z_0) = 0$ if and only if $v$ is $y$-symmetric.*

Corollary 1.5. *If $v$ is a positive ground-state solution of (1.1) in $\Omega(r)$ as in Corollary 1.2 and $z_c$ is a critical point of $v$, then $z_c \notin \{y = 0\} \cap \Omega$. In particular, either $(\partial v/\partial y)(z) < 0$ or $(\partial v/\partial y)(z) > 0$ for all $z \in \{y = 0\} \cap \Omega$.*

## 2. Preliminaries

We define the $y$-symmetric domains and $y$-symmetric functions as follows.

*Definition 2.1.* (i) $\Omega$ is $y$-symmetric provided that $z = (x, y) \in \Omega$ if and only if $(x, -y) \in \Omega$;
(ii) let $\Omega$ be a $y$-symmetric domain in $\mathbb{R}^N$. A function $u : \Omega \to \mathbb{R}$ is $y$-symmetric (axially symmetric) if $u(x, y) = u(x, -y)$ for $(x, y) \in \Omega$.

Throughout this paper, let $\Omega$ be a $y$-symmetric domain in $\mathbb{R}^N$, $H_s(\Omega)$ the $H^1$- closure of the space $\{u \in C_0^\infty(\Omega) \mid u \text{ is } y\text{-symmetric}\}$ and let $X(\Omega)$ be either the whole space $H_0^1(\Omega)$ or the $y$-symmetric Sobolev space $H_s(\Omega)$. Then, $H_s(\Omega)$ is a closed linear subspace of $H_0^1(\Omega)$. Let $H_s^{-1}(\Omega)$ be the dual space of $H_s(\Omega)$.

We define the Palais-Smale (PS) sequences, (PS)-values and (PS)-conditions in $X(\Omega)$ for $J$ as follows.

*Definition 2.2.* We define the following:

(i) for $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\text{PS})_\beta$-sequence in $X(\Omega)$ for $J$ if $J(u_n) = \beta + o(1)$ and $J'(u_n) = o(1)$ strongly in $X^{-1}(\Omega)$ as $n \to \infty$;

(ii) $\beta \in \mathbb{R}$ is a (PS)-value in $X(\Omega)$ for $J$ if there is a $(\text{PS})_\beta$-sequence in $X(\Omega)$ for $J$;

(iii) $J$ satisfies the $(\text{PS})_\beta$-condition in $X(\Omega)$ if every $(\text{PS})_\beta$-sequence in $X(\Omega)$ for $J$ contains a convergent subsequence.

By Willem [14], for any $\beta \in \mathbb{R}$, a $(\text{PS})_\beta$-sequence in $X(\Omega)$ for $J$ is bounded. Moreover, a (PS)-value $\beta$ should be nonnegative.

Lemma 2.3. *Let $\beta \in \mathbb{R}$ and $\{u_n\}$ be a $(PS)_\beta$-sequence in $X(\Omega)$ for $J$, then there exists a positive number $c(\beta)$ such that $\|u_n\|_{H^1} \le c(\beta)$ for large $n$. Furthermore,*

$$a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2}\beta + o(1) \tag{2.1}$$

*and $\beta \ge 0$. Moreover, $c(\beta)$ can be chosen so that $c(\beta) \to 0$ as $\beta \to 0$.*

Now, we consider the Nehari minimization problem

$$\alpha_X(\Omega) = \inf_{u \in \mathbf{M}(\Omega)} J(u), \tag{2.2}$$

where $\mathbf{M}(\Omega) = \{u \in X(\Omega) \backslash \{0\} \mid a(u) = b(u)\}$. Note that $\mathbf{M}(\Omega)$ contains every nonzero solution of (1.1) in $\Omega$, $\alpha_X(\Omega) > 0$ and if $u_0 \in \mathbf{M}(\Omega)$ achieves $\alpha_X(\Omega)$, then $u_0$ is a positive (or negative) solution of (1.1) in $\Omega$ (see [13, 14]). Moreover, we have the following useful lemma, whose proof can be found in [13, Lemma 7].

Lemma 2.4. *Let $\{u_n\}$ be in $X(\Omega)$. Then, $\{u_n\}$ is a $(PS)_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for $J$ if and only if $J(u_n) = \alpha_X(\Omega) + \mathrm{o}(1)$ and $a(u_n) = b(u_n) + \mathrm{o}(1)$.*

We denote

(i) $\alpha_X(\Omega)$ by $\alpha_0(\Omega)$ for $X(\Omega) = H_0^1(\Omega)$ and $\alpha_X(\Omega)$ by $\alpha_s(\Omega)$ for $X(\Omega) = H_s(\Omega)$,

(ii) $\mathbf{M}(\Omega)$ by $\mathbf{M}_0(\Omega)$ for $X(\Omega) = H_0^1(\Omega)$ and $\mathbf{M}(\Omega)$ by $\mathbf{M}_s(\Omega)$ for $X(\Omega) = H_s(\Omega)$.

*Remark 2.5.* By the principle of symmetric criticality (see [11]), we have every $(\text{PS})_\beta$-sequence in $X(\Omega)$ for $J$ is a $(\text{PS})_\beta$-sequence in $H_0^1(\Omega)$ for $J$.

Let $\Omega$ be any unbounded domain and $\xi \in C^\infty([0,\infty))$ such that $0 \le \xi \le 1$ and

$$\xi(t) = \begin{cases} 0 & \text{for } t \in [0,1], \\ 1 & \text{for } t \in [2,\infty). \end{cases} \tag{2.3}$$

Let

$$\xi_n(z) = \xi\left(\frac{2|z|}{n}\right). \tag{2.4}$$

Then, we have the following results whose proof can be found in [15].

Proposition 2.6. *Equation (1.1) in $\Omega$ does not admit any solution $u_0$ such that $J(u_0) = \alpha_X(\Omega)$ if and only if for each $(PS)_{\alpha_X(\Omega)}$-sequence $\{u_n\}$ in $X(\Omega)$ for $J$, there exists a subsequence $\{u_n\}$ such that $\{\xi_n u_n\}$ is also a $(PS)_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for $J$.*

Proposition 2.7. *$J$ does not satisfy the $(PS)_{\alpha_X(\Omega)}$-condition in $X(\Omega)$ for $J$ if and only if there exists a $(PS)_{\alpha_X(\Omega)}$-sequence $\{u_n\}$ in $X(\Omega)$ for $J$ such that $\{\xi_n u_n\}$ is also a $(PS)_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for $J$.*

Let $\Omega_1 \subsetneqq \Omega_2$, clearly $\alpha_X(\Omega_1) \geq \alpha_X(\Omega_2)$. Then, we have the following useful results.

Lemma 2.8. *Let $\Omega_1 \subsetneqq \Omega_2$ and $J : X(\Omega_2) \to \mathbb{R}$ be the energy functional. Suppose that $\alpha_X(\Omega_1) = \alpha_X(\Omega_2)$. Then, the following hold:*

(i) *equation (1.1) in $\Omega_1$ does not admit any solution $u_0 \in X(\Omega_1)$ such that $J(u_0) = \alpha_X(\Omega_1)$;*

(ii) *$J$ does not satisfy the $(PS)_{\alpha_X(\Omega_2)}$-condition.*

The proof is given by Wang and Wu [13, Lemma 13].

By the Rellich compact theorem, $J$ satisfies the $(PS)_{\alpha_X(\Omega)}$-condition in $X(\Omega)$ if $\Omega$ is a bounded domain.

Lemma 2.9. *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. Then, the $(PS)_{\alpha_X(\Omega)}$-condition holds in $X(\Omega)$ for $J$. Furthermore, (1.1) in $\Omega$ has a positive solution $u_0$ such that $J(u_0) = \alpha_X(\Omega)$.*

## 3. Concentration behavior

We need the following results.

Lemma 3.1. *Let $\Omega$ be an unbounded domain. Then,*

$$\alpha_X(\Omega(r)) \searrow \alpha_X(\Omega) \quad \text{as } r \nearrow \infty. \tag{3.1}$$

*Proof.* Since $\Omega(r)$ is a bounded domain for all $r > 0$, by Lemmas 2.8 and 2.9, we have $\alpha_X(\Omega(r))$ is monotone decreasing as $r$ is monotone increasing and $\alpha_X(\Omega(r)) > \alpha_X(\Omega)$. Thus, there exists a $d_0 \geq \alpha_X(\Omega)$ such that

$$\alpha_X(\Omega(r)) \searrow d_0 \quad \text{as } r \nearrow \infty. \tag{3.2}$$

Claim that $d_0 \leq \alpha_X(\Omega)$. Let $\{u_n\}$ be a $(PS)_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for $J$. By Lemma 2.3, there exists a $c > 0$ such that

$$\int_\Omega |\nabla u_n|^2 + u_n^2 \leq c, \qquad \int_\Omega |u_n|^p \leq c \tag{3.3}$$

for all $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, there exists a sequence $\{r_n\}$ such that $r_n > 0$ with $r_n \nearrow \infty$ as $n \to \infty$ and

$$\int_{\Omega \cap \{|z| \geq r_n\}} |\nabla u_n|^2 + u_n^2 < \frac{1}{n}, \qquad \int_{\Omega \cap \{|z| \geq r_n\}} |u_n|^p < \frac{1}{n}. \tag{3.4}$$

Now, define $\eta_{r_n}(z) = \eta(2|z|/r_n)$, where $\eta \in C_c^\infty([0,\infty))$ satisfies $0 \leq \eta \leq 1$ and

$$\eta(t) = \begin{cases} 1 & \text{for } t \in [0,1], \\ 0 & \text{for } t \in [2,\infty). \end{cases} \tag{3.5}$$

Then, $\eta_{r_n} u_n \in X(\Omega)$. From (3.4), we obtain

$$\begin{aligned} a(\eta_{r_n} u_n) &= a(u_n) + o(1), \\ b(\eta_{r_n} u_n) &= b(u_n) + o(1). \end{aligned} \tag{3.6}$$

By the routine computations, there exists a sequence $\{s_n\} \subset \mathbb{R}^+$ such that $a(s_n \eta_{r_n} u_n) = b(s_n \eta_{r_n} u_n)$, $s_n = 1 + o(1)$ and

$$J(s_n \eta_{r_n} u_n) = J(\eta_{r_n} u_n) + o(1) = \alpha_X(\Omega) + o(1), \tag{3.7}$$

that is, $s_n \eta_{r_n} u_n \in \mathbf{M}(\Omega(r_n))$ and $J(s_n \eta_{r_n} u_n) \geq \alpha_X(\Omega(r_n)) = d_0 + o(1)$. Taking $n \to \infty$ , we get $\alpha_X(\Omega) \geq d_0$. Therefore, $\alpha_X(\Omega) = d_0$. □

Let $\Omega_t^+ = \{(x,y) \in \Omega \mid y > t\}$ and $\Omega_t^- = \{(x,y) \in \Omega \mid y < t\}$. Then, we have the following result.

Lemma 3.2. *Suppose that the domain $\Omega$ satisfies* ($\Omega$1), ($\Omega$2), *and* ($\Omega$3). *Then, for each $\varepsilon > 0$ and $l \geq 0$, there exists a $\delta(\varepsilon, l) > 0$ such that if $u \in \mathbf{M}_0(\Omega)$ and $J(u) < \alpha_0(\Omega) + \delta(\varepsilon, l)$, then either $\int_{\Omega_l^+} |u|^p < \varepsilon$ or $\int_{\Omega_{-l}^-} |u|^p < \varepsilon$.*

*Proof.* If not, there exist $c > 0$, $l_0 \geq 0$, and $\{u_n\} \subset \mathbf{M}_0(\Omega)$ such that $J(u_n) = \alpha_0(\Omega) + o(1)$,

$$\int_{\Omega_{l_0}^+} |u_n|^p \geq c, \qquad \int_{\Omega_{-l_0}^-} |u_n|^p \geq c. \tag{3.8}$$

By Lemma 2.4, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha_0(\Omega)}$-sequence in $H_0^1(\Omega)$ for $J$. Now, $\Omega$ satisfies condition ($\Omega$3). By Proposition 2.6, there exists a subsequence $\{u_n\}$ such that $\{\xi_n u_n\}$ is also a $(\mathrm{PS})_{\alpha_0(\Omega)}$-sequence in $H_0^1(\Omega)$ for $J$, where $\xi_n$ is as in (2.4). Let $v_n = \xi_n u_n$. We obtain

$$\begin{aligned} J(v_n) &= \alpha_0(\Omega) + \mathrm{o}(1), \\ J'(v_n) &= \mathrm{o}(1) \quad \text{in } H^{-1}(\Omega). \end{aligned} \tag{3.9}$$

Since $\Omega$ is a $y$-symmetric domain in $\mathbb{R}^N$ separated by a bounded domain, there exists a $n_0 > l_0$ such that $v_n = 0$ in $\overline{\Omega(n_0)}$ for $n > 2n_0$, and $\Omega \backslash \overline{\Omega(n_0)} = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \Omega_{n_0}^+$ and $\Omega_2 = \Omega_{-n_0}^-$. Moreover, $v_n = v_n^1 + v_n^2$, where

$$v_n^i(z) = \begin{cases} v_n(z) & \text{for } z \in \Omega_i \\ 0 & \text{for } z \notin \Omega_i \end{cases} \quad \text{for } i = 1,2. \tag{3.10}$$

Then, $v_n^i \in H_0^1(\Omega_i)$ and $a(v_n^i) = b(v_n^i) + o(1)$. By (3.9), we obtain

$$J'(v_n^i) = \mathrm{o}(1) \text{ strongly in } H^{-1}(\Omega_i) \qquad \text{for } i = 1,2. \tag{3.11}$$

Assume that

$$J(v_n^i) = c_i + \mathrm{o}(1) \quad \text{for } i = 1,2. \tag{3.12}$$

Since $J(v_n) = J(v_n^1) + J(v_n^2) = \alpha_0(\Omega) + \mathrm{o}(1)$, we have $c_1 + c_2 = \alpha_0(\Omega)$. Since $c_i$ are (PS)-values in $H_0^1(\Omega_i)$ for $J$, by Lemma 2.3, $c_i \geq 0$ and

$$\begin{aligned} c_1\left(\frac{2p}{p-2}\right) &= \int_{\Omega_{l_0}^+} |v_n^1|^p + o(1) = \int_{\Omega_{l_0}^+} |u_n|^p + o(1), \\ c_2\left(\frac{2p}{p-2}\right) &= \int_{\Omega_{-l_0}^-} |v_n^2|^p + o(1) = \int_{\Omega_{-l_0}^-} |u_n|^p + o(1). \end{aligned} \tag{3.13}$$

By (3.8), we have $c_i > 0$ for $i = 1,2$. We have that

$$\alpha_0(\Omega) = c_1 + c_2 \geq \alpha_0(\Omega_1) + \alpha_0(\Omega_2), \tag{3.14}$$

which contradicts the fact that $\alpha_0(\Omega) \leq \alpha_0(\Omega_i)$ for $i = 1,2$. □

Now, we begin to show the proof of Theorem 1.1. By Lemma 3.1, for each $\varepsilon > 0$ and $l \geq 0$, there exists a $\delta(\varepsilon, l) > 0$ such that if $u \in \mathbf{M}_0(\Omega)$ and $J(u) < \alpha_0(\Omega) + \delta(\varepsilon, l)$, then $\int_{\Omega_l^+} |u|^p < \varepsilon$ or $\int_{\Omega_{-l}^-} |u|^p < \varepsilon$. Moreover, by Lemma 3.2, there exists an $\tilde{r} > 0$ such that

$\alpha_0(\Omega(r)) < \alpha_0(\Omega) + \delta(\varepsilon)$ for all $r > \tilde{r}$. Thus, if $v$ is a ground-state solution of (1.1) in $H_0^1(\Omega(r))$ for $r > \tilde{r}$, then $v \in \mathbf{M}_0(\Omega(r)) \subset \mathbf{M}_0(\Omega)$, $J(v) < \alpha_0(\Omega) + \delta(\varepsilon)$ and either $\int_{\Omega_l^+} |v|^p < \varepsilon$ or $\int_{\Omega_{-l}^-} |v|^p < \varepsilon$.

Now, we begin to show the proof of Theorem 1.3.

(i) By Lemma 3.1, we have $J(v_m) = \alpha_0(\Omega(m)) = \alpha_0(\Omega) + o(1)$. Since $v_m \in \mathbf{M}_0(\Omega(m)) \subset \mathbf{M}_0(\Omega)$, from Lemma 2.4 we can conclude that $\{v_m\}$ is a $(\mathrm{PS})_{\alpha_0(\Omega)}$-sequence in $H_0^1(\Omega)$ for $J$.

(ii) Let $v \in L^q(\Omega)$, where $1/p + 1/q = 1$. Then, for each $\varepsilon > 0$ there exists an $l > 0$ such that

$$\int_{(\Omega(l))^c} |v|^q < \varepsilon^q. \tag{3.15}$$

By Theorem 1.1, there exists an $m_0 > l$ such that

$$\int_{\Omega(l)} |v_m|^q < \varepsilon^p \quad \forall m > m_0. \tag{3.16}$$

Thus, for each $\varepsilon > 0$ there exists an $m_0$ such that

$$\begin{aligned} \int_\Omega v_m v = \int_{(\Omega(l))^c} v_m v + \int_{\Omega(l)} v_m v &\le \left(\int_{(\Omega(l))^c} |v_m|^p\right)^{1/p} \left(\int_{(\Omega(l))^c} |v|^q\right)^{1/q} \\ &+ \left(\int_{\Omega(l)} |v_m|^p\right)^{1/p} \left(\int_{\Omega(l)} |v|^q\right)^{1/q} \le (c_1 + c_2)\varepsilon \quad \forall m > m_0, \end{aligned} \tag{3.17}$$

where $c_1 = ((2p/(p-2))\alpha_0(\Omega))$ and $c_2 = \|v\|_{L^q}$. This implies that $v_m \rightharpoonup 0$ weakly in $L^p(\Omega)$ as $m \to \infty$. Since $v_m$ is a solution of (1.1) in $\Omega(m)$, we have

$$\int_{\Omega(m)} \nabla v_m \nabla \varphi + v_m \varphi = \int_{\Omega(m)} |v_m|^{p-2} v_m \varphi \quad \forall \varphi \in H_0^1(\Omega(m)). \tag{3.18}$$

First, we need to show for each $\varepsilon > 0$ and $\varphi \in C_c^1(\mathbf{S})$ there exists an $m_0$ such that

$$\int_{\Omega(m)} \nabla v_m \nabla \varphi + v_m \varphi < \varepsilon \quad \forall m > m_0 \tag{3.19}$$

for $\varphi \in C_c^1(\Omega)$. Let $K = \operatorname{supp} \varphi$, then $K \subset \Omega$ is compact and there exists an $m_1$ such that $K \subset \Omega(m)$ for all $m \ge m_1$. From Theorem 1.4, for each $\varepsilon > 0$ there exist $l_0 > 0$ and $m_0$ such that $\varphi \in H_0^1(\Omega(m))$,

$$\int_{(\Omega(l_0))^c} |\varphi|^p = 0, \qquad \int_{\Omega(l_0)} |v_m|^p < \varepsilon^{(p-1)/p} \quad \forall m > m_0. \tag{3.20}$$

We obtain

$$\begin{aligned}\int_{\Omega(m)} |v_m|^{p-2} v_m \varphi &= \int_{(\Omega(l_0))^c} |v_m|^{p-2} v_m \varphi + \int_{\Omega(l_0)} |v_m|^{p-2} u_m^1 \varphi \\ &\le \left(\int_{(\Omega(l_0))^c} |v_m|^p\right)^{(p-1)/p} \left(\int_{(\Omega(l_0))^c} |\varphi|^p\right)^{1/p} \\ &\quad + \left(\int_{\Omega(l_0)} |v_m|^p\right)^{(p-1)/p} \left(\int_{\Omega(l_0)} |\varphi|^p\right)^{1/p} \\ &\le c\varepsilon,\end{aligned} \tag{3.21}$$

$$\begin{aligned}\int_{\Omega} \nabla v_m \nabla \varphi + \int_{\Omega} v_m \varphi &= \int_{\Omega(m)} \nabla v_m \nabla \varphi + \int_{\Omega(m)} v_m \varphi \\ &= \int_{\Omega(m)} |v_m|^{p-2} v_m \varphi \quad \forall m > m_0.\end{aligned} \tag{3.22}$$

We have that

$$\int_{\Omega} \nabla v_m \nabla \varphi + \int_{\Omega} v_m \varphi \le c\varepsilon \quad \forall m > m_0. \tag{3.23}$$

Since $\alpha_0(\Omega(m+1)) < \alpha_0(\Omega)$, there exists a $C > 0$ such that $\|v_m\|_{H^1} \le C$. Thus, for each $\varepsilon > 0$ and $\psi \in H_0^1(\Omega)$, there exists a $\varphi \in C_c^1(\Omega)$ such that

$$\|\psi - \varphi\|_{H^1} < \frac{\varepsilon}{C}. \tag{3.24}$$

From (3.23) and (3.24), we can conclude that for each $\varepsilon > 0$ and $\psi \in H_0^1(\Omega)$ there exists an $m_0 > 0$ such that

$$\begin{aligned}\langle v_m, \psi\rangle_{H^1} &= \langle v_m, \psi - \varphi\rangle_{H^1} + \langle v_m, \varphi\rangle_{H^1} \\ &\le C\|\psi - \varphi\|_{H^1} + \langle v_m, \varphi\rangle_{H^1} \\ &< \varepsilon + c\varepsilon \quad \text{for } m > m_0.\end{aligned} \tag{3.25}$$

This implies that $v_m \rightharpoonup 0$ weakly in $H_0^1(\Omega)$.

## 4. Symmetry

Now, we begin to show the proof of Theorem 1.4. Let $v$ be a ground-state solution of (1.7) in $\Theta$ and let $z^* = (x, -y)$ be the reflection point of $z = (x, y)$ with respect to the hyperplane $T := \{y = 0\}$. First, we claim that either

$$v(z) \ge v(z^*) \quad \forall z \in \Theta^+ \tag{4.1}$$

or

$$v(z) \le v(z^*) \quad \forall z \in \Theta^+, \tag{4.2}$$

where $\Theta^+$ is one of half domain $\Theta\backslash T$. If not, then the following two sets

$$A_+ = \{z \in \Theta^+ \mid v(z) > v(z^*)\}, \tag{4.3}$$

$$A_- = \{z \in \Theta^+ \mid v(z) < v(z^*)\}, \tag{4.4}$$

are nonempty. Let $w(z) = v(z) - v(z^*)$ for $z \in \Theta^+$. Then, $w$ satisfies

$$\begin{gathered} \Delta w - w + f_v(\zeta(z))w = 0, \quad \text{in } \Theta^+, \\ w = 0, \quad \text{in } \partial\Theta^+, \end{gathered} \tag{4.5}$$

where $\zeta(z)$ is between $v(z)$ and $v(z^*)$. Let

$$A_-^* = \{z^* \mid z \in A_-\}. \tag{4.6}$$

For $d > 0$, we define a function

$$u_d(z) = \begin{cases} w(z) & \text{if } z \in A_+, \\ dw(z^*) & \text{if } z \in A_-^*, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Since $\int_{A_+} w\phi_1 > 0$ and $\int_{A_-} w\phi_1 < 0$, there exists a constant $d_0 > 0$ such that

$$\int_\Theta u_{d_0}\phi_1 = \int_{A_+} w\phi_1 + d_0 \int_{A_-} w\phi_1 = 0, \tag{4.8}$$

where $\phi_1$ is the first positive eigenfunction of the following eigenvalue problem:

$$\begin{gathered} (\Delta - 1 + f_v(\zeta(z)))\phi + \lambda\phi = 0 \quad \text{in } \Theta, \\ \phi = 0 \quad \text{on } \partial\Theta. \end{gathered} \tag{4.9}$$

Let $\lambda_2$ be the second eigenvalue of (4.9). Since $v$ is a ground-state solution of (1.7), by the same method of the proof of Theorem 2.11 in [10], we have $\lambda_2$ is nonnegative. Moreover, by (4.3)–(4.7), we have

$$\begin{gathered} \Delta u_d - u_d + f_v(\zeta(z))u_d > 0 \quad \text{for } z \in A_+, \\ \Delta u_d - u_d + f_v(\zeta(z))u_d < 0 \quad \text{for } z \in A_-^*, \\ \Delta u_d - u_d + f_v(\zeta(z))u_d = 0 \quad \text{otherwise.} \end{gathered} \tag{4.10}$$

Therefore, from (4.8) and (4.10), we have

$$\begin{aligned} 0 &> \int_{\Theta} -u_d(z)\left[\Delta u_d(z) - u_d + f_v(\zeta(z))u_d(z)\right]dz \\ &= \int_{\Theta} \left[\left|\nabla u_d(z)\right|^2 + u_d^2 - f_v(\zeta(z))u_d^2(z)\right]dz \\ &\geq \lambda_2 \int_{\Theta} u_d^2(z)dz \geq 0, \end{aligned} \tag{4.11}$$

a contradiction. This proves inequalities (4.1) and (4.2). By (4.1) and (4.2), we may assume $w(z) \geq 0$ for all $z \in \Theta^+$, if $w(z) > 0$ for some $z \in \Theta^+$. Since $w$ satisfies (4.5), by using the strong maximum principle, we have $w > 0$ in $\Theta^+$. Similarly, if $w(z) \leq 0$ and $w(z) < 0$ for some $z \in \Theta^+$, we have $w < 0$ in $\Theta^+$. Suppose that $w(z) > 0$ for all $z \in \Theta^+$. Then, from (4.5) and applying the Hopf Lemma, we have

$$\frac{\partial w}{\partial(-y)}(z_0) = -2\frac{\partial v}{\partial y}(z_0) < 0. \tag{4.12}$$

Similarly, if $w(z) < 0$ for all $z \in \Theta^+$, we have $(\partial v/\partial)y(z_0) < 0$, this contradicts the fact that $(\partial v/\partial)y(z_0) = 0$. Therefore, $w(z) = 0$ for all $z \in \Theta^+$ or $v(x,y) = v(x,-y)$ for all $(x,y) \in \Theta$. The converse is obvious.


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Tsung-Fang Wu: Center for General Education, Southern Taiwan University of Technology, Tainan 71005, Taiwan

*E-mail address*: tfwu@mail.stut.edu.tw


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