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A NECESSARY AND SUFFICIENT CONDITION FOR UNIQUENESS OF SOLUTIONS OF SINGULAR DIFFERENTIAL INEQUALITIES

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ABSTRACT. The author proves that the abstract differential inequality $\|u'(t) - A(t)u(t)\|^2 \le \gamma\left[\omega(t) + \int_0^t \omega(\eta)d\eta\right]$ in which the linear operator $A(t) = M(t) + N(t)$, M symmetric and N antisymmetric, is in general unbounded, $\omega(t) = t^{-2}\psi(t)\|u(t)\|^2 + \|M(t)u(t)\| \, \|u(t)\|$ and γ is a positive constant has a nontrivial solution near t=0 which vanishes at t=0 if and only if $\int_0^1 t^{-1}\psi(t)dt = \infty$. The author also shows that the second order differential inequality $\|u''(t) - A(t)u(t)\|^2 \le \gamma\left[\mu(t) + \int_0^t \mu(\eta)d\eta\right]$ in which $\mu(t) = t^{-4}\psi_0(t)\|u(t)\|^2 + t^{-2}\psi_1(t)\|u'(t)\|^2$ has a nontrivial solution near t=0 such that u(0)=u'(0)=0 if and only if either $\int_0^1 t^{-1}\psi_0(t)dt = \infty$ or $\int_0^1 t^{-1}\psi_1(t)dt = \infty$. Some mild restrictions are placed on the operators M and N. These results extend earlier uniqueness theorems of Hile and Protter.

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1. INTRODUCTION.

Let H be a complex Hilbert space with the usual inner product and norm notation and let A be an linear, in general unbounded, operator defined on a non-trivial domain D in H. Assuming the operator A = M + N where M is symmetric and N is antisymmetric, we consider the differential inequalities

$$\|u'(t) - A(t)u(t)\|^2 \le \gamma\left[\omega(t) + \int_0^t \omega(\eta)d\eta\right] \tag{1.1}$$

where $\omega(t) = \frac{\psi(t)}{t^2}\|u(t)\|^2 + \|M(t)u(t)\| \; \|u(t)\|$ and

$$\|u''(t) - A(t)u(t)\|^2 \le \gamma\left[\mu(t) + \int_0^t \mu(\eta)d\eta\right] \tag{1.2}$$

where $\mu(t) = \frac{\psi_0(t)}{t^4}\|u(t)\|^2 + \frac{\psi_1(t)}{t^2}\|u'(t)\|^2$ and γ is a positive constant. We show, under rather general conditions on M and N, that a necessary and sufficient condition for the existence of an interval (0,T] on which (1.1) will have a nontrivial solution vanishing at t = 0 is

$$\int_0^1 \frac{\psi(t)}{t}dt = \infty. \tag{1.3}$$

Furthermore, we show that a necessary and sufficient condition for the existence of an interval (0,T] on which (1.2) will have a nontrivial solution vanishing at t = 0 is either

$$\int_0^1 \frac{\psi_0(t)}{t}dt = \infty \tag{1.4}$$

or

$$\int_0^1 \frac{\psi_1(t)}{t}dt = \infty. \tag{1.5}$$

Our results extend those of Hile and Protter [1] who prove that the only solution of (1.1) and likewise for (1.2) with homogenous initial conditions is the trivial one provided the functions $t^{-2}\psi(t)$, $t^{-4}\psi_0(t)$ and $t^{-2}\psi_1(t)$ are bounded. Thus our proofs of necessity (See Theorems 2 and 4.) contain the uniqueness theorems of [1] (See Theorems 1 and 3 of [1].) as a special case. Furthermore our results allow for less stringent requirements on the operators M and N in that certain kinds of singularities at t=0 are allowed. Also we show that our results are best in that (1.1) (or (1.2)) will have a nontrivial solution (with zero initial data) on some interval (0,T] for T small if (1.3) (or (1.4) or (1.5)) holds.

Other works considering singular equations abound. (See e.g. [2]-[11] and their references.) Of particular relevance to our results here are [2], [3] and [4]. Lees and Protter [2] show, for A = M = a uniformly elliptic second order partial differential operator (in x), that a differential inequality similar to (1.1) has only the trivial solution vanishing at t = 0 when ψ is unity provided the L_2 norm (in x) of the spatial gradient of u has an infinite order zero initially. Our work confirms the necessity of some such additional information on u in order to obtain their uniqueness. Donaldson and Goldstein [3] and Ames [4] consider specific equations which are special cases of (1.1) and (1.2) and thus obtain sharper results. In particular, Donaldson and Goldstein [3] prove that the only solution of $u' - Au = P(t)u$ vanishing initially is the trivial one provided $P(t)-(1/t + b)I$, for some real b, is dissipative for all positive t and the operator $A = -S^2$ where S is self-adjoint and indpendent of t. They also show that for $P(t) = (1+\varepsilon)/t + b$, for any real b, nontrivial solutions exist. These results are, of course, consistent with ours. Indeed our results show that if ψ is any positive constant, then (1.1) has a nontrivial solution near zero which vanishes at zero. (See Theorem 1.) They also consider the equation

$$v''(t) + \alpha(t)v'(t) = Av(t) \tag{1.6}$$

which is the well known abstract Euler-Poisson-Darboux (EPD) equation if $\alpha(t) = k/t$, k constant, and prove uniqueness for the initial value problem provided $\alpha(t) \geq -1/t$. These results of Donald and Goldstein [3] have been extended by Goldstein [5] as well as Arrate and Garcia [6]. Ames [4] also considers (1.6) with $\alpha(t) = \psi(t)/t$ (where ψ has properties somewhat similar to ours) but requires only that the operator A be symmetric (and independent of t). Furthermore it is known that the solution to the EPD equation (A = the Laplacian) is not unique if $k < 0$ (See e.g., [4].). These results are again consistent with ours. Indeed, for $\alpha(t) = k/t$ corresponds to taking $\psi_1 = 1$, $\psi_0 = 0$ in (1.2) and hence (1.4) holds implying a nontrivial solution exists near zero (See Theorem 3.).

We note that the form of the function α in [4] along with the work of Hile and Protter [1] and Garofalo [7] have been the major motivating factors in this study and especially choosing the form of ω in (1.1) and of μ in (1.2). Finally we note that the extension of the uniqueness theorems of [1] to the n^{th} order time derivative case with A independent of t is contained in [12].

2. THE FIRST ORDER CASE.

Throughtout this section we assume $\psi \in C^2((0,\infty))$ satisfying

$$\psi > 0, \ \psi' \geq 0, \ \psi'' \leq 0. \tag{2.1}$$

Consequently the function $\psi(t)/t$ is nonincreasing and hence

$$t\psi'(t) \leq \psi(t). \tag{2.2}$$

We now give assumptions on the linear operator A which, except for (iii) and (iv), match those of [1] while (iii) and (iv) are more general than the similar conditions given in [1]. It should be noted that not all of these will be needed in the proof of sufficiency.

For $t_0 > 0$, let $C^*([0,t_0];D)$ be the set of $u \in C([0,t_0];D) \cap C^1((0,t_0];H)$ such that $\|u'(t)\|$ is bounded on $(0,t_0)$.

Condition (I). We assume there exists $T > 0$ so that the linear operator $A(t)$, with nontrivial domain D (i.e., $D \neq \{0\}$), satisfies the following:

(i) $A(t) = M(t) + N(t)$, M is symmetric and N is antisymmetric;

(ii) For each $u \in C^*([0,T];D)$, the functions $M(t)u(t)$ and $N(t)u(t)$ are bounded and continuous on $(0,T]$;

(iii) There exists a positive constant γ_1 such that for all $w \in D$ and $t \in (0,T]$

$$\mathrm{Re}(M(t)w,N(t)w) \geq -\gamma_1\left[\|M(t)w\| \ \|w\| + \frac{\psi(t)}{t^2}\|w\|^2\right].$$

(iv) For each $u \in C^*([0,T];D)$ satisfying (1.1), the function $(M(t)u(t),u(t))$ is continuously differentiable on $(0,T]$ and there exists a positive constant γ_2 such that for all $t \in (0,T]$

$$d/dt(M(t)u(t),u(t)) - 2\mathrm{Re}(M(t)u(t),u'(t)) \geq -\gamma_2\left[\|M(t)u(t)\| \ \|u(t)\| + \frac{\psi(t)}{t^2}\|u(t)\|^2\right].$$

Sufficiency. Although the proof of necessity will require that the operator A satsify condition (I), sufficiency will not require properties (iii) and (iv). Furthermore, we show that the nontrivial function satisfying (1.1) actually satisfies a much sharper inequality (See (2.5) below.) than (1.1).

THEOREM 1. (*Sufficiency*) Suppose (1.3) holds and the operator A satisfies condition (I) except possibly for parts (iii) and (iv). Then there exists a $T > 0$ such that inequality (1.1) has a nontrivial solution on $(0,T]$ contained in $C^*([0,T];D)$ which vanishes at $t=0$.

PROOF. Let v be any nonzero element of D. Since (1.3) holds and the function $\psi(t)/t$ is nondecreasing, we have $\lim_{t\downarrow 0} \psi(t)/t = \infty$. Combining this result with part (ii) of condition (I) yields

$$\lim_{t\downarrow 0} \psi(t)t^{-2}\left[1+\|A(t)v\|^2\right]^{-1} = \infty$$

and thus we may choose $T \in (0,T]$ so that $\gamma\psi(t)/t^2 \geq 2\left[1+\|A(t)v\|^2\right]\|v\|^{-2}$ for all $t \in (0,T]$ where γ comes from (1.1). Define $K = \sup\{\|A(t)v\|: 0<t\leq T\}$ which is finite because of condition (I). Then $\gamma\psi(t)/t^2 \geq 2\left[1+\|A(t)v\|^2\right]\|v\|^{-2}$ for all $t \in (0,T]$ and we define

$$\xi(t) = \int_t^T \left[(\gamma/2)\eta^{-2}\psi(\eta) - K^2\|v\|^{-2}\right]^{1/2} d\eta \quad , \quad 0 < t \leq T.$$

Let $u(t) = e^{-\xi(t)}v$. We need to show

$$\lim_{t\downarrow 0} u(t) = 0 \tag{2.3}$$

and that u satisfies (1.1) on $(0,T]$. To determine the initial value of u, note that since ψ is nondecreasing, $\lim_{t\downarrow 0} \psi(t)$ exists. Let $\lim_{t\downarrow 0} \psi(t) = L$, $0 \leq L < \infty$. If $L = 0$, then $\psi^{1/2} \geq \psi$ near zero and thus (1.3) implies $\int_0^T t^{-1}[\psi(t)]^{1/2}dt = \infty$ and hence $\xi(t) \to \infty$ as $t\downarrow 0$ which in turn yields (2.3). On the other hand, if $L \neq 0$, it is clear that $\xi(t) \to \infty$ as $t\downarrow 0$ and thus (2.3) holds.

To show that u satisfies (1.1) on $(0,T]$, note that straightforward calculations give

$$\begin{aligned}\|u'(t) - A(t)u(t)\|^2 &\leq 2\|u'(t)\|^2 + 2\|A(t)u(t)\|^2 \\ &= 2\left[(\gamma/2)t^{-2}\psi(t) - K^2\|v\|^{-2}\right]e^{-2\xi(t)}\|v\|^2 + 2e^{-2\xi(t)}\|A(t)v\|^2\end{aligned} \tag{2.4}$$

Since $\|A(t)v\| \leq K$, inequality (2.4) implies

$$\|u'(t) - A(t)u(t)\|^2 \leq 2\left[(\gamma/2)t^{-2}\psi(t)\|v\|^2\right]e^{-2\xi(t)} = \gamma t^{-2}\psi(t)\|u\|^2 \tag{2.5}$$

and thus (1.1) holds. This completes the proof.

Necessity. Suppose

$$\int_0^1 \frac{\psi(t)}{t}dt < \infty. \tag{2.6}$$

Then the monotonicity of ψ gives $\lim_{t\downarrow 0} \psi(t) = 0$. Also, without loss of generality, we may assume $\lim_{t\downarrow 0} \psi(t)/t = \infty$. Indeed $\lim_{t\downarrow 0} \psi(t)/t$ exists (possibly infinite) since $\psi(t)/t$ is nonincreasing; and furthermore, if $\lim \psi(t)/t < \infty$, inequality (1.1) is still valid on $(0,T]$ if $\psi(t)$ is replaced with $Ct^{1/2}$ for a sufficiently large constant C (depending only on T) and hence $\lim_{t\downarrow 0} \psi(t)/t = \infty$. Additionally, as a consequence of (2.6) and the monotoncity of $\psi(t)/t$, we have

$$t^k\int_t^T \eta^{-k-1}\psi(\eta)d\eta \le t^k[t^{-1}\psi(t)]\int_t^T \eta^{-k}d\eta = t^{k-1}\psi(t)(-T^{-k+1} + t^{-k+1})/(k-1)$$

$$\le \psi(t)/(k-1) \qquad \text{for any } 0 < t \le T,\ k > 1,$$

and hence

$$t^k\int_t^T \eta^{-k-1}\psi(\eta)d\eta \le \psi(t)/(k-1) \quad , k > 1,\ 0 < t \le T. \tag{2.7}$$

Before proving necessity (Theorem 2), we need some preliminary lemmas.

LEMMA 1. Suppose ψ satisfies (2.6). Let $\rho(t) = \psi(t)/t^2$, $\lambda(t) = \int_0^t \psi(\eta)/\eta\, d\eta$, and suppose h and r are nonnegative functions continuous on $(0,T]$ for some $T > 0$. Furthermore, assume $r(t)$ and $h(t)/t$ are bounded near zero. Then, for all $\varepsilon > 0$ and all $T \in [0,T]$, we have

$$2\int_0^T \rho(\xi)\int_0^\xi h(\eta)r(\eta)d\eta \le \varepsilon\int_0^T \rho(\eta)h^2(\eta)d\eta + \varepsilon^{-1}\lambda(T)\int_0^T (r(\eta))^2 d\eta. \tag{2.8}$$

PROOF. Since the result is trivial for T=0, we consider only the case $T > 0$. Thus suppose $0 < t < T$ and use Cauchy-Schwarz along with elementary estimates to get ($\Psi(t) = \int_0^t [\rho(\eta)]^{-1}[r(\eta)]^2 d\eta$)

$$2\int_0^t \rho(\eta)\int_0^\eta h(s)r(s)sd\eta = 2\int_0^t \rho(\eta)\int_0^\eta [\rho(s)]^{1/2}h(s)[\rho(s)]^{-1/2}r(s)dsd\eta \tag{2.9}$$

$$\le 2\int_0^t \rho(\eta)\left[\int_0^\eta \rho h^2 ds\right]^{1/2}[\Psi(\eta)]^{1/2}d\eta \le 2\left[\int_0^t \rho h^2 ds\right]^{1/2}\int_0^t \rho(\eta)[\Psi(\eta)]^{1/2}d\eta$$

$$\le \varepsilon\int_0^t \rho h^2 ds + \varepsilon^{-1}\left[\int_0^t \rho(\eta)[\Psi(\eta)]^{1/2}d\eta\right]^2.$$

The last integral in (2.9) admits the estimate

$$\left[\int_0^t \rho(\eta)[\Psi(\eta)]^{1/2}d\eta\right]^2 \le \left[\int_0^t \eta^{1/2}[\rho(\eta)]^{1/2}\eta^{-1/2}[\rho(\eta)]^{1/2}[\Psi(\eta)]^{1/2}d\eta\right]^2$$

$$\le \left[\int_0^t \eta\rho(\eta)d\eta\right]\left[\int_0^t \eta^{-1}\rho(\eta)\Psi(\eta)d\eta\right] = \lambda(t)\int_0^t R'(\eta)\Psi(\eta)d\eta \tag{2.10}$$

where $R(t) \equiv -\int_t^T \eta^{-1}\rho(\eta)d\eta$ for $t < T$. Since

$$0 \le -R(\eta)\Psi(\eta) \le \left[\int_t^T \eta^{-1}\psi(\eta)d\eta\right]\left[t^{-2}\int_0^t [\rho(\eta)]^{-1}r^2(\eta)d\eta\right]$$

and application of L'Hospital's rule gives

$$\lim_{t\downarrow 0} t^{-2}\int_0^t [\rho(\eta)]^{-1} r^2(\eta)d\eta = \lim_{t\downarrow 0} \frac{\int_0^t \eta^2[\psi(\eta)]^{-1} r^2(\eta)d\eta}{t^2}$$

$$= (1/2) \lim_{t\downarrow 0} r^2(t)t/\psi(t) = 0$$

where the last equality holds because r is bounded near zero and $\psi(t)/t \to \infty$, we get $\lim_{\eta\downarrow 0} R(\eta)\Psi(\eta) = 0$. Using this result, we integrate by parts in the last integral in (2.10) and obtain

$$\left[\int_0^t \rho(\eta)[\Psi(\eta)]^{1/2}d\eta\right]^2 \le \lambda(t)\left[R(t)\Psi(t) - \int_0^t R(\eta)\Psi'(\eta)d\eta\right]. \tag{2.11}$$

Since $\lambda(t)$ and $\Psi(t)$ are nonnegative while R(t) is nonpositive, we may discard the first expression on the right side of (2.11). Also (2.7) with k = 2 gives exactly $-R(\eta)[\rho(\eta)]^{-1} \le 1$ so that $-R(\eta)\Psi'(\eta) \le r^2(\eta)$. Substitution of this into (2.11) and the resulting inequality into (2.9) yields (2.8). This completes the proof.

LEMMA 2. Suppose $z \in C^*([0,T];D)$ such that $z(0) = 0$. Then

$$\int_0^t \rho(\eta)\|z(\eta)\|^2 d\eta \le 4\lambda(t)\int_0^t \|z'(\eta)-N(\eta)z(\eta)\|^2 d\eta \tag{2.12}$$

where the functions ρ and λ are given in Lemma 1.

PROOF. Since z(0) = 0 and the operator N is antisymmetric, we get

$$\|z(\eta)\|^2 = 2\ \mathrm{Re}\int_0^\eta (z(s),z'(s)-N(s)z(s))ds \le 2\int_0^\eta \|z(s)\|\ \|z'(s)-N(s)z(s)\|ds. \tag{2.13}$$

Now multiply (2.13) by $\rho(\eta)$, integrate over [0,t] and apply inequality (2.8) to the resulting right side to get

$$\int_0^t \rho(\eta)\|z(\eta)\|^2 d\eta \le 2\int_0^t \rho(\eta)\int_0^\eta \|z(s)\|\|z'(s)-N(s)z(s)\|ds d\eta$$

$$\le \epsilon\int_0^t \rho(\eta)\|z(\eta)\|^2 d\eta + \epsilon^{-1}\lambda(t)\int_0^t \|z'(s)-N(s)z(s)\|^2 d\eta.$$

Taking $\epsilon = 1/2$ in this expression and simplifying yields (2.12). This completes the proof.

LEMMA 3. Suppose $0 < T < \min\{1,T\}$ and $t_0 > 0$ is such that $t_0 + T < 1$. Also suppose the operator A satisfies condition (I) and $Lu = u' - Au$. Assume that $u \in C^*([0,T];D)$ and $u(0) = u(T) = 0$. Then, for all sufficiently large $\beta > 0$, the size depending only on the constants γ_1 and γ_2 from condition (I), the following holds

$$\beta^2\int_0^T \tau^{-\beta-2}e^{2\tau^{-\beta}}\|u\|^2 dt + C_0[\lambda(T)]^{-1}\int_0^T \rho e^{2\tau^{-\beta}}\|u\|^2 dt + C_1\int_0^T \tau^{\beta}e^{2\tau^{-\beta}}\|Mu\|^2 dt \le C_2\int_0^T e^{2\tau^{-\beta}}\|Lu\|^2 dt \tag{2.14}$$

where $\tau = t+t_0$, $\rho(t) = t^{-2}\psi(t)$ and C_0, C_1 and C_2 are absolute constants.

PROOF. Following [1, p. 61], we set $\varphi(t) = -(t+t_0)^{-\beta}$ and define $v = e^{-\varphi}u$. Then $Lu = e^{\varphi}[v'+\varphi'v-Mv-Nv]$, and defining the function α (See [1, p. 62].) by $\alpha(t) = k_0\tau^{\beta}$, we have $e^{-2\varphi}\|Lu\|^2 = \|v'+\varphi'v-\alpha Mv-(1-\alpha)Mv-Nv\|^2$. Thus, integrating with respect to t from 0 to T, we get

$$\int e^{-2\varphi}\|Lu\|^2 \geq 2\ \mathrm{Re}\int(v'-\alpha Mv-Nv,\varphi'v-(1-\alpha)Mv) + \int\|v'-\alpha Mv-Nv\|^2$$

$$= 2\ \mathrm{Re}\int\varphi'(v',v) + 2\int\alpha(1-\alpha)\|Mv\|^2 - 2\int\alpha\varphi'(Mv,v) - 2\ \mathrm{Re}\int(v',Mv)$$

$$+ 2\ \mathrm{Re}\int(Nv,Mv) + \int\|v'-Nv\|^2$$

$$= I_1 + \dots + I_6.$$

Using estimates for I_1 through I_3 identical to those in [1, proof of Lemma 1] and estimates virtually identical to those of I_4 and I_5 in the same lemma (the only difference is the $1-\alpha$ in [1] is replaced with 1 here) and using (2.12) above to estimate I_6 gives (2.14) and the proof is complete.

We may now prove necessity. It should be noted that Theorem 2 contains the results of [1; Theorem 1] as a special case.

THEOREM 2. (*Necessity*) Suppose the operator A satisfies condition (I) and there exists $T \in (0,T]$ such that $u \in C^*([0,T];D)$ is a solution of (1.1) on $(0,T]$ with $u(0) = 0$. If the function ψ satisfies (2.6), then $u = 0$ on $[0,T)$.

PROOF. Following [1], we show that $u = 0$ on $[0,T']$ for sufficiently small T'. Once this has been done, we may then apply the results of [1, Theorem 1] on the interval $[T',T]$ where $\psi(t)/t^2$ is bounded to get $u = 0$ on $[0,T]$. We choose T' less than one in such a way that $\lambda(T')^{-1}$ is large depending only on known constants (See inequality (2.15) below.) where the function λ is defined in Lemma 1 and by hypothesis $\lambda(t) \downarrow 0$ as $t \downarrow 0$.

Let $\varepsilon > 0$ be given and define the C^∞ function ζ such that $\zeta(t) = 1$ for $0 \leq t \leq T'-\varepsilon$, $= 0$ for $t \geq T'$ and such that $0 < \zeta < 1$ for $T'-\varepsilon < t < T'$. The proof now proceeds as with [1]. (See inequality (2.6) of [1] and note that their T_0 is my T'.) Applying Lemma 3 to ζu we get

$$\beta^2\int_0^{T'-\varepsilon}\tau^{-\beta-2}e^{2\tau^{-\beta}}\|u\|^2dt + C_0[\lambda(T')]^{-1}\int_0^{T'-\varepsilon}\rho e^{2\tau^{-\beta}}\|u\|^2dt + C_1\int_0^{T'-\varepsilon}\tau^{\beta}e^{2\tau^{-\beta}}\|Mu\|^2dt$$

$$\leq C_2\int_0^{T'-\varepsilon}e^{2\tau^{-\beta}}\|Lu\|^2dt + C_2\int_{T'-\varepsilon}^{T'}e^{2\tau^{-\beta}}\|L(\zeta u)\|^2dt.$$

Using nearly identical arguments as in [1] we get, for arbitrary $k_2 > 0$,

$$\int_0^{T'-\varepsilon}e^{2\tau^{-\beta}}\|Lu\|^2dt \leq k_2\int_0^{T'-\varepsilon}e^{2\tau^{-\beta}}\tau^{\beta+1}\|M(t)u(t)\|^2dt$$

$$+\int_0^{T'-\varepsilon}e^{2\tau^{-\beta}}\left[2c(1+\rho) + (k_2)^{-1}\tau^{-\beta-1}c^2\right]\|u(t)\|^2dt.$$

Hence, by choosing k_2 sufficiently small (depending only on C_1 and C_2), β sufficiently large (depending only on t_0, γ and k_2 (and hence C_1 and C_2) and T' sufficiently small (so that $\lambda(T')^{-1} > 2C_2\gamma(\rho(t)^{-1}+1)/C_0$ for $0<t<T$), and doing more estimates as in [1], we get

$$\beta^2\int_0^{T'-\varepsilon}\|u\|^2dt \leq 2C_2\int_{T'-\varepsilon}^{T'}\|L(\zeta u)\|^2dt. \tag{2.15}$$

Letting $\beta \to \infty$, we get $u = 0$ on $[0,T'-\epsilon]$ and hence on $[0,T']$. This completes the proof.

3. THE SECOND ORDER CASE.

Throughout this section we assume $\psi_i \in C^2((0,\infty))$, $i=0,1$, and

$$\psi_i > 0,\ \psi_i' \ge 0,\ \psi_i'' \le 0 \quad \text{on } (0,\infty),\ i = 0,1. \tag{3.1}$$

Consequently the functions $\psi_i(t)/t$ are nonincreasing and hence

$$t\psi_i'(t) \le \psi_i(t) \quad \text{on } (0,\infty),\ i = 0,1. \tag{3.2}$$

We now give assumptions on the operator A which, except for (iii), match those of [1] while (iii) is more general than the similar conditions in [1] in that here the coefficients need not be bounded.

For $t_0 > 0$, let $C_*([0,t_0];D)$ be the set of $u \in C([0,t_0];D) \cap C^1([0,t_0];H) \cap C^2((0,t_0];H)$ such that $\|u''(t)\|$ is bounded on $(0,t_0]$.

Condition (II). We assume there exists $T > 0$ such that the linear operator $A(t)$, with nontrivial domain D (i.e., $D\neq\{0\}$), satisfies the following:

(i) $A(t) = M(t) + N(t)$, M is symmetric and N is antisymmetric;

(ii) For each $u \in C_*([0,T];D)$, the functions $M(t)u(t)$ and $N(t)u(t)$ are bounded and continuous on $(0,T]$;

(iii) For nonnegative constant γ_3, we let

$$F(t) = \gamma_3\left[\frac{\psi_0(t)}{t^3}\|u(t)\|^2 + \frac{\psi_1(t)}{t}\|u'(t)\|^2\right].$$

For funtions $u \in C_*([0,T];D)$, we assume the functions $\mathrm{Re}(N(t)u(t),u'(t))$ and $(M(t)u(t),u(t))$ are continuously differentiable on $(0,T]$ and satisfy the following on $(0,T]$:

$$(d/dt)\mathrm{Re}(N(t)u(t),u'(t)) - \mathrm{Re}(N(t)u(t),u''(t)) \ge -F(t)$$

$$(d/dt)(M(t)u(t),u(t)) - 2\mathrm{Re}(M(t)u(t),u'(t)) \ge -F(t)$$

$$\mathrm{Re}(M(t)u(t),N(t)u(t)) \ge -F(t).$$

Sufficiency. Not all of Condition (II) will be needed to prove sufficiency, and as in the the first order case, we show that our solution actually satisfies a much sharper estimate than (1.2). (See inequalities (3.4) and (3.10).) However, before proving sufficiency, we need a preliminary result.

LEMMA 4. Let $\phi(t) = \min\{\psi_0(t),C\}$ where C is any positive number and suppose (1.4) holds. The function $\phi(t)/t$ is nonincreasing on $(0,\infty)$ and

$$\int_0^1 \phi(t)/t\, dt = \infty. \tag{3.3}$$

PROOF. Clearly $\phi(t)/t$ is nonincreasing since ψ_0 (See inequality (3.2).) has that same property. To prove (3.3), we shall assume, without loss of generality, that there exists a decreasing sequence of numbers $\{a_n\}$ in the open interval (0,1) converging to zero such that $\phi(a_n) = C = \psi_0(a_n)$, $n = 1,2,\ldots$. If this were not the case, it must be that $\phi = \psi_0 < C$ near zero or $\phi = C < \psi_0$ near 0 and in either case the result would hold trivially. Choose a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_1} = a_1$, and $2a_{n_{j+1}} \le a_{n_j}$ for all j. Since $\phi(t)/t$ is nonincreasing and $\phi(a_n)/a_n = C/a_n$, we get

$$\int_0^1 \phi(t)/t\,dt = \sum_{n=1}^{\infty} \int_{a_{n+1}}^{a_n} \phi(t)/t\,dt = \sum_{j=1}^{\infty} \int_{a_{n_{j+1}}}^{a_{n_j}} \phi(t)/t\,dt$$

$$\geq \sum_{j=1}^{\infty} \int_{a_{n_{j+1}}}^{a_{n_j}} \phi(a_{n_j})/a_{n_j}\,dt = \sum_{j=1}^{\infty} C\left[1 - a_{n_{j+1}}/a_{n_j}\right] \geq \sum_{j=1}^{\infty} C/2 = \infty.$$

This completes the proof.

THEOREM 3. *(Sufficiency)* Suppose that either (1.4) or (1.5) holds and the operator A satisfies condition (II) except possibly for part (iii). Then there exists $T > 0$ such that inequality (1.2) has a nontrivial solution on $(0,T]$ contained in $C_*([0,T];D)$ which vanishes at $t = 0$.

PROOF. Suppose (1.5) holds and let v be any nonzero element of D. Using the function ψ_1 in place the function ψ in the proof of Theorem 1, choose the constants K and T and the function ξ as in the proof of Theorem 1. (In addition, we must have $T \leq 1$.) Using analysis similar to that of the first order case, it is easy to show that the function $u(t) = \left[\int_0^t e^{-\xi(s)}ds\right]v$ satisfies $\|u''(t) - A(t)u(t)\|^2 \leq \frac{\gamma\psi_1(t)}{t^2}\|u'(t)\|^2$ on $(0,T]$ with $u(0) = u'(0) = 0$. Hence u satisfies (1.2) and vanishes along with its first derivative at $t = 0$.

Now suppose (1.4) is satisfied. We shall find $T > 0$ and function $u(t)$ which is a nontrivial solution of

$$\|u''(t) - A(t)u(t)\|^2 \leq \frac{\gamma\phi(t)}{t^4}\|u(t)\|^2 \quad \text{on } (0,T] \tag{3.4}$$

$$u(0) = u'(0) = 0. \tag{3.5}$$

where $\phi(t) = \min\{\psi_0(t), 8/\gamma\}$. Thus u will also be a nontrivial solution of (1.2) since $\phi \leq \psi_0$. Let v be any nonzero element of D. Since (1.4) holds and hence (3.3) holds (for $C=8/\gamma$), we may, in a manner similar to that in the proof of Theorem 1, choose $0 < T_0 < T$ so that $\phi(t)/t^2 \geq (8/\gamma)\left[1+\|A(t)v\|^2\right]\|v\|^{-2}$ for all $t\epsilon(0,T_0]$ where γ comes from (1.2). Define $K = \sup\{\|A(t)v\|: 0<t\leq T_0\}$ which is finite because of condition (II). Then $(\gamma/8)t^{-2}\phi(t) - K^2\|v\|^{-2}$ is nonnegative on $(0,T_0]$ and we define

$$\xi(t) = \int_t^{T_0}\left[(\gamma/8)\eta^{-2}\phi(\eta) - K^2\|v\|^{-2}\right]^{1/2}d\eta.$$

Before defining T and u, we make some observations concerning the function ξ. As a result of (3.3) and the boundedness of ϕ, we have $\int_0^1 t^{-1}[\phi(t)]^{1/2}dt = \infty$.

Thus $\lim_{t\downarrow 0} \xi(t) = \infty$ and $\lim_{t\downarrow 0} \phi(t)/t = \infty$. Using L'Hospital's Rule, it is easy to show $\lim_{t\downarrow 0} e^{\xi(t)}\int_0^t e^{-\xi(s)}ds = 0$. Hence we may choose $T\epsilon(0,T_0]$ so that

$$e^{-\xi(t)} \geq \int_0^t e^{-\xi(s)}ds \quad \text{for all } t\epsilon[0,T]. \tag{3.6}$$

Furthermore, if we define the function S by $S(t) = te^{-\xi(t)} - 2\int_0^t e^{-\xi(s)}ds$, then $S'(t) = ([\gamma\phi(t)/8 - K^2\|v\|^{-2}t^2]^{1/2} - 1)e^{-\xi(t)}$ so that $S'(t) \leq 0$ on $(0,T_0]$ since $\phi \leq 8/\gamma$. Thus since $\lim_{t\downarrow 0} S(t) = 0$, we have $S(t) \leq 0$ on $(0,T_0]$ and hence on $(0,T]$. That is,

$$2\int_0^t e^{-\xi(s)}ds \geq te^{-\xi(t)} \quad \text{for all } t\epsilon[0,T]. \tag{3.7}$$

We now let $u(t) = \left[\int_0^t e^{-\xi(s)}ds\right]v$ for $t \epsilon [0,T]$ and show that u, which is obviously nontrivial, satisfies (3.4), and hence also satisfies (1.2) and (3.5). Clearly $u(0) = 0$ and $u'(0) = 0$ since $\lim_{t\downarrow 0} \xi(t) = \infty$. To show that (3.4) holds, notice that on $(0,T]$

$$\|u" - Au\|^2 \leq 2\|u"\|^2 + 2\|Au\|^2 = 2(\xi')^2e^{-2\xi}\|v\|^2 + 2\left[\int_0^t e^{-\xi(s)}ds\right]^2\|Av\|^2. \tag{3.8}$$

Using $\|Av\| \leq K$ and substituting for ξ' in (3.8), we get

$$\begin{aligned}\|u" - Au\|^2 &\leq 2\left[\frac{\gamma\phi(t)}{8t^2} - \frac{K^2}{\|v\|^2}\right]e^{-2\xi}\|v\|^2 + 2\left[\int_0^t e^{-\xi(s)}ds\right]^2K^2 \\ &= (\gamma/4)\phi(t)t^{-2}e^{-2\xi(t)}\|v\|^2 - 2K^2\left\{e^{-2\xi(t)} - \left[\int_0^t e^{-\xi(s)}ds\right]^2\right\}. \\ &\leq (\gamma/4)\phi(t)t^{-2}e^{-2\xi(t)}\|v\|^2\end{aligned} \tag{3.9}$$

where the last inequality is a result of (3.6). We now apply (3.7) to (3.9) to get

$$\begin{aligned}\|u" - Au\|^2 &\leq \gamma\phi(t)t^{-4}\left[\int_0^t e^{-2\xi(s)}ds\right]^2\|v\|^2 \\ &= \gamma\phi(t)t^{-4}\|u(t)\|^2 \leq \gamma\psi_0(t)t^{-4}\|u(t)\|^2.\end{aligned} \tag{3.10}$$

Hence u is a nontrivial solution of (3.4) (and therefore (1.2)) on $(0,T]$. This completes the proof.

Necessity. Suppose

$$\int_0^1 \frac{\psi_0(t)}{t}dt < \infty \quad \text{and} \quad \int_0^1 \frac{\psi_1(t)}{t}dt < \infty. \tag{3.11}$$

We define the function ψ (suppressing its dependence on α since α will be chosen to be 1/2 later (in the proof of Lemma 10)) by

$$\psi(t) = \psi_0(t^\alpha)+\psi_1(t^\alpha)$$

where $0<\alpha<1$. Notice that the function ψ inherits the relevant properties of ψ_0 and ψ_1 along with one additional property. In particular, ψ satisfies the following:

$$\psi>0,\ \psi'\geq 0,\ \psi"\leq 0 \quad \text{on } (0,\infty), \tag{3.12}$$

and

$$\int_0^1 \psi(t)/t\ dt < \infty \quad \text{(as a result of (3.11))}. \tag{3.13}$$

In addition, the monotonicity of ψ_i yields $\psi_i(t) \le \psi_i(t^\alpha)$ for $0\le t\le 1$, $i = 0,1$, so that, for any interval $(0,T_0]$, $T_0 \le 1$, on which (1.2) is satisfied, we get

$$\|u''(t) - A(t)u(t)\|^2 \le \gamma\left[\mu(t) + \int_0^t \mu(\eta)d\eta\right] \qquad 0 < t \le T_0 \tag{3.14}$$

where $\mu(t) = \psi(t)\left[t^{-4}\|u(t)\|^2 + t^{-2}\|u'(t)\|^2\right]$. Also, part (iii) of condition (II) may be restated with ψ_0 and ψ_1 replaced with ψ. Lastly, and very importantly, as a result of (3.2), we get

$$t\psi'(t) \le \alpha\psi(t) \text{ (i.e., } \psi(t)/t^\alpha \text{ is nondecreasing.) on } (0,\infty). \tag{3.15}$$

Hence, using analysis similar to that for getting inequality (2.7), we get

$$t^k\int_t^T \eta^{-k-1}\psi(\eta)d\eta \le \psi(t)/(k-\alpha) \quad , \ k>\alpha>0 \text{ and } 0 < t \le T. \tag{3.16}$$

Before proving necessity, we develop several lemmas.

LEMMA 5. If $u \in C_*([0,T];D)$ for some $T > 0$ and $u(0) = u'(0) = 0$, then

$$\int_0^t e^{-2\varphi(s)}s^{-2}\rho(s)\|u(s)\|^2 ds \le 4(3-\alpha)^{-2}\int_0^t e^{-2\varphi(s)}\rho(s)\|u'(s)\|^2 ds \quad , \quad 0\le t\le T \tag{3.17}$$

where $\rho(t) = \psi(t)/t^2$, $\varphi(t) = -(t+t_0)^{-\beta}$ and $t_0 > 0$.

PROOF. Since $u(0) = u'(0) = 0$, we have $\|u(s)\|^2 = 2\int_0^s (u,u')d\eta \le 2\int_0^s \|u\|\|u'\|d\eta$. Multiply this inequality by $e^{-2\varphi}s^{-2}\rho$ and integrate to get

$$\int_0^t e^{-2\varphi}s^{-2}\rho\|u\|^2 ds \le 2\int_0^t e^{-2\varphi}s^{-2}\rho\int_0^s \|u\|\|u'\|d\eta ds = -2\int_0^t e^{-2\varphi}\Psi'(s)\int_0^s \|u\|\|u'\|d\eta ds \tag{3.18}$$

where $\Psi(s) = \int_s^t \eta^{-2}\rho(\eta)d\eta$ for $0<s\le t$. Now integrate by parts on the right side of (3.18) to get

$$-2\int_0^t e^{-2\varphi}\Psi'\int_0^s \|u\|\|u'\|d\eta ds = \lim_{\varepsilon\downarrow 0} -2e^{-2\varphi}\Psi\int_0^s \|u\|\|u'\|d\eta\Big|_\varepsilon^t + 2\int_0^t \Psi \frac{d}{ds}\left[e^{-2\varphi}\int_0^s \|u\|\|u'\|d\eta\right]ds$$

$$\le \lim_{\varepsilon\downarrow 0} 2e^{-2\varphi(\varepsilon)}\Psi(\varepsilon)\int_0^\varepsilon \|u\|\|u'\|d\eta + 2\int_0^t \Psi\frac{d}{ds}\left[e^{-2\varphi}\int_0^s \|u\|\|u'\|d\eta\right]ds. \tag{3.19}$$

We now observe that the limit on the right side of (3.19) is zero. To prove this, note that (3.13) implies the existence of a positive constant C (depending on t) for which $\int_\varepsilon^t \psi(s)/s\ ds \le C$ which yields $\Psi(\varepsilon) \le \varepsilon^{-3}\int_\varepsilon^t \psi(s)/s\ ds \le C\varepsilon^{-3}$. Now apply L'Hospital's rule to get

$$\lim_{\varepsilon\downarrow 0} \Psi(\varepsilon)\int_0^\varepsilon \|u\|\|u'\|d\eta \le C \lim_{\varepsilon\downarrow 0} \varepsilon^{-3}\int_0^\varepsilon \|u\|\|u'\|d\eta = \lim_{\varepsilon\downarrow 0} -3\varepsilon^{-2}\|u(\varepsilon)\|\|u'(\varepsilon)\| = 0$$

since $u(0) = u'(0) = 0$ and u'' is bounded near zero. Thus, after doing the indicated differentiation, inequality (3.19) becomes

$$-2\int_0^t e^{-2\varphi}\Psi'\int_0^s \|u\|\|u'\|d\eta ds \le -4\int_0^t \Psi\varphi' e^{-2\varphi}\int_0^s \|u\|\|u'\|d\eta ds + 2\int_0^t \Psi e^{-2\varphi}\|u\|\|u'\|ds$$
$$\le 2\int_0^t \Psi e^{-2\varphi}\|u\|\|u'\|ds \tag{3.20}$$

where the last inequality holds since $\varphi'>0$. Inequality (3.16) with $k=3$ yields $\Psi(s) \le s^{-3}\psi(s)/(3-\alpha) = s^{-1}\rho(s)/(3-\alpha)$. Substitution of this into (3.20) and application of Cauchy-Schwarz gives

$$-2\int_0^t e^{-2\varphi}\Psi'\int_0^s \|u\|\|u'\|d\eta ds \le 2(3-\alpha)^{-1}\int_0^t s^{-1}\rho(s)e^{-2\varphi}\|u\|\|u'\|ds$$
$$\le 2(3-\alpha)^{-1}\left[\int_0^t s^{-2}\rho e^{-2\varphi}\|u\|^2 ds\right]^{1/2}\left[\int_0^t \rho e^{-2\varphi}\|u'\|^2 ds\right]^{1/2} \tag{3.21}$$

Substitution of (3.21) into (3.18) and simplification yields (3.17). This completes the proof.

LEMMA 6. Suppose $z \in C_*([0,T_0];D)$ for some $T_0 > 0$ and $z(0)=z'(0)=0$. Then

$$\int_0^t (\varphi')^2\rho\|z\|^2 ds \le \lambda(T_1)\int_0^t \|2\varphi' z'+\varphi'' z-Nz\|^2 ds \quad \text{for any } T \le \min\{T_0,T_1\} \tag{3.22}$$

where φ and ρ are defined as in Lemma 5 and $\lambda(t) = \int_0^t \psi(s)/s\, ds$.

PROOF. Since the function λ is increasing, it suffices to prove (3.22) for $T_1 = t$. The operator N is antisymmetric and hence ($\eta > 0$)

$$\mathrm{Re}\int_0^\eta (\varphi' z, 2\varphi' z'+\varphi'' z-Nz)ds = \mathrm{Re}\int_0^\eta [2(\varphi')^2(z,z') + \varphi'\varphi''\|z\|^2]ds \tag{3.23}$$
$$= \int_0^\eta [(\varphi')^2\|z\|^2]'ds - \int_0^\eta \varphi'\varphi''\|z\|^2 ds = (\varphi'(\eta))^2\|z(\eta)\|^2 - \int_0^\eta \varphi'\varphi''\|z\|^2 ds \ge (\varphi'(\eta))^2\|z(\eta)\|^2$$

since $\varphi'\varphi'' \le 0$. Multiply (3.23) by $\rho(\eta)$ and integrate to get

$$\int_0^t \rho(\varphi')^2\|z\|^2 d\eta \le \mathrm{Re}\int_0^t \rho(\eta)\int_0^\eta (\varphi' z, 2\varphi' z'+\varphi'' z-Nz)ds d\eta$$
$$\le \int_0^t \rho(\eta)\int_0^\eta \|\varphi' z\|\|2\varphi' z'+\varphi'' z-Nz)\|ds d\eta. \tag{3.24}$$

Application of (2.8) to (3.24) (with $h=\|\varphi' z\|$ and $r=\|2\varphi' z'+\varphi'' z-Nz)\|$) yields

$$\int_0^t \rho(\varphi')^2\|z\|^2 d\eta \le (\epsilon/2)\int_0^t \rho\|\varphi' z\|^2 d\eta + (2\epsilon)^{-1}\lambda(t)\int_0^t \|2\varphi' z'+\varphi'' z-Nz\|^2 d\eta. \tag{3.25}$$

Putting $\epsilon = 1$ in (3.25) and simplification yields (3.22) for $T_1 = t$. This completes the proof.

LEMMA 7. Suppose the operator A satisfies condition (II) and $Lu = u'' - Au$. Let φ and ρ be as in Lemma 5 with $t_0 + T < 1$ and suppose $u \in C_*([0,T];D)$. In addition, assume $u(0) = u'(0) = u(T) = u'(T) = 0$. Then, for $\epsilon>0$, we get

$$\int_0^T \rho e^{-2\varphi}(Mu,u)dt \le \left[-1 + 4\{3+2\varepsilon+4\varepsilon^{-1}\psi(T)\}(3-\alpha)^{-2}\right]\int_0^T \rho e^{-2\varphi}\|u'\|^2 dt$$
$$+ (3/\varepsilon)\int_0^T (\varphi')^2 \rho e^{-2\varphi}\|u\|^2 dt + \varepsilon\int_0^T e^{-2\varphi}\|Lu\|^2 dt. \tag{3.26}$$

PROOF. Using the definition of the operator L and the antisymmetry of N, we get (All of the following integrals are taken over [0,T].)

$$\int \rho e^{-2\varphi}(Mu,u)dt = \int \rho e^{-2\varphi}(u''-Lu-Nu,u)dt$$
$$= \mathrm{Re}\int \rho e^{-2\varphi}(u'',u)dt - \mathrm{Re}\int \rho e^{-2\varphi}(Lu,u)dt = J_1 + J_2. \tag{3.27}$$

Integration by parts twice in J_1 and using the fact that u and u′ vanish at both 0 and T yields

$$J_1 = -\int \rho e^{-2\varphi}\|u'\|^2 dt + (1/2)\int (\rho e^{-2\varphi})''\|u\|^2 dt. \tag{3.28}$$

Since $(\rho e^{-2\varphi})'' = e^{-2\varphi}t^{-4}(t^2\psi''-4t\psi'+6\psi-4t^2\varphi'\psi'+8t\varphi'\psi+4t^2\psi(\varphi')^2-2t^2\psi\varphi'')$, $\psi'\ge 0$, $\psi''\le 0$ and $\varphi'>0$, we get

$$(\rho e^{-2\varphi})'' \le e^{-2\varphi}(6t^{-4}\psi + 8t^{-3}\varphi'\psi + 4t^{-2}\psi(\varphi')^2 - 2t^{-2}\psi\varphi'')$$
$$= e^{-2\varphi}(6t^{-2}\rho + 8t^{-1}\varphi'\rho + 4\rho(\varphi')^2 - 2\rho\varphi'').$$

Hence substitution of this into (3.28) yields

$$J_1 \le -\int \rho e^{-2\varphi}\|u'\|^2 dt + \int e^{-2\varphi}(3t^{-2}\rho + 4t^{-1}\varphi'\rho + 2\rho(\varphi')^2 - \rho\varphi'')\|u\|^2 dt. \tag{3.29}$$

To estimate the right side of (3.29), we observe that $-\varphi'' \le 2(\varphi')^2$ for β large since $t_0+T < 1$, and for $\varepsilon > 0$, we get $4t^{-1}\varphi'\rho \le 2\varepsilon t^{-2}\rho + 2\varepsilon^{-1}\rho(\varphi')^2$. Applying these two inequalities to (3.29) produces

$$J_1 \le -\int \rho e^{-2\varphi}\|u'\|^2 dt + (3+2\varepsilon)\int e^{-2\varphi}t^{-2}\rho\|u\|^2 dt + (4+2/\varepsilon)\int \rho(\varphi')^2 e^{-2\varphi}\|u\|^2 dt.$$

Now apply (3.17) to the second integral on the right side of this inequality to get

$$J_1 \le [-1 + 4(3+2\varepsilon)(3-\alpha)^{-2}]\int e^{-2\varphi}\rho\|u'\|^2 dt + (4+2/\varepsilon)\int \rho(\varphi')^2 e^{-2\varphi}\|u\|^2 dt. \tag{3.30}$$

The monotonicity of ψ and application of (3.17) allows the estimate

$$J_2 \le \varepsilon\int e^{-2\varphi}\|Lu\|^2 dt + (4/\varepsilon)\int e^{-2\varphi}\rho^2\|u\|^2 dt$$
$$\le \varepsilon\int e^{-2\varphi}\|Lu\|^2 dt + (4/\varepsilon)\psi(T)\int t^{-2}\rho e^{-2\varphi}\|u\|^2 dt \tag{3.31}$$
$$\le \varepsilon\int e^{-2\varphi}\|Lu\|^2 dt + 4\{4\varepsilon^{-1}(3-\alpha)^{-2}\}\psi(T)\int \rho e^{-2\varphi}\|u'\|^2 dt.$$

Substitution of (3.30) and (3.31) into (3.27) gives (3.26) provided ε is sufficiently small that $4+2/\varepsilon < 3/\varepsilon$. This completes the proof.

LEMMA 8. Let z, u, ρ and φ be as in Lemma 7. Then, for $\varepsilon>0$ small, we get

$$\int_0^T \rho\|z'\|^2 dt \ge [1 - 4\varepsilon(3-\alpha)^{-2}]\int_0^T \rho e^{-2\varphi}\|u'\|^2 dt - 2\varepsilon^{-1}\int_0^T (\varphi')^2\rho e^{-2\varphi}\|u\|^2 dt. \tag{3.32}$$

PROOF. Since $z = e^{-2\varphi}u$, we get (All integrals are taken over [0,T].)

$$\int \rho\|z'\|^2 dt = \int \rho e^{-2\varphi}\|u'-\varphi' u\|^2 dt \tag{3.33}$$
$$= \int \rho e^{-2\varphi}\|u'\|^2 dt - 2\mathrm{Re}\int \rho\varphi' e^{-2\varphi}(u,u')dt + \int \rho(\varphi')^2 e^{-2\varphi}\|u\|^2 dt.$$

Integrating by parts in the second integral on the right side of (3.33) and using $\varphi'' \geq -(\varphi')^2$, for β large, gives

$$-2\mathrm{Re}\int \rho\varphi' e^{-2\varphi}(u,u')dt = \int(\rho\varphi' e^{-2\varphi})'\|u\|^2 dt = \int(\rho'\varphi' + \rho\varphi'' - 2\rho(\varphi')^2)e^{-2\varphi}\|u\|^2 dt$$

$$\geq \int(\rho'\varphi' - 3\rho(\varphi')^2)e^{-2\varphi}\|u\|^2 dt. \tag{3.34}$$

Since $\psi' \geq 0$, we get $\rho' \geq -2\rho/t$ and hence $\rho'\varphi' \geq -2\rho\varphi/t \geq -\varepsilon\rho/t^2 - \rho(\varphi')^2/\varepsilon$. Substitute this into (3.34) and that result into (3.33) to get

$$\int\rho\|z'\|^2 dt \geq \int\rho e^{-2\varphi}\|u'\|^2 dt - \varepsilon\int t^{-2}e^{-2\varphi}\rho\|u\|^2 dt - (2+1/\varepsilon)\int\rho(\varphi')^2 e^{-2\varphi}\|u\|^2 dt. \tag{3.35}$$

Now apply (3.17) to the second integral of the right side of (3.35) and use $2+1/\varepsilon < 2/\varepsilon$ for small ε, we get (3.32). This completes the proof.

LEMMA 9. Suppose the operator A satisfies condition (II) and $z \in C_*([0,T];D)$ such that $z(0) = z'(0) = z(T) = z'(T) = 0$. Then, for $T_0 \geq T$ and $u = e^{-\varphi}z$, we get

$$(2-c_T)\int_0^T \rho e^{-2\varphi}\|u'\|^2 d\eta \leq \varepsilon^{-1}\lambda(T_0)\int_0^T\|z''+(\varphi')^2 z - Mz\|^2 d\eta$$
$$+ (5/\varepsilon)\int_0^T(\varphi')^2\rho e^{-2\varphi}\|u\|^2 d\eta + \varepsilon\int_0^T e^{-2\varphi}\|Lu\|^2 dt. \tag{3.36}$$

where $\varepsilon>0$, $c_T = \varepsilon+\gamma_3(2-\alpha)(1-\alpha)^{-2}\psi(T)+4(3+3\varepsilon+4\varepsilon^{-1}\psi(T))(3-\alpha)^{-2}$, the function λ is defined in Lemma 6, ρ and φ are defined in Lemma 5 and the operator L is defined in Lemma 7.

PROOF. Since $z'(0) = 0$, we get

$$2\int_0^t\|z'\|\|z''+(\varphi')^2 z-Mz\|ds \geq 2\mathrm{Re}\int_0^t(z',z''+(\varphi')^2 z-Mz)ds$$
$$= \|z'(t)\|^2 + 2\mathrm{Re}\int_0^t(\varphi')^2(z',z)ds - 2\mathrm{Re}\int_0^t(z',Mz)ds = \|z'(t)\|^2 + I_1 + I_2. \tag{3.37}$$

We now estimate I_1 and I_2. Integration by parts gives

$$I_1 = 2\mathrm{Re}\int_0^t(\varphi')^2(z',z)ds = \int_0^t(\varphi')^2(\|z\|^2)'ds$$
$$= (\varphi')^2\|z\|^2\Big|_0^t - 2\int_0^t\varphi'\varphi''\|z\|^2 ds = (\varphi'(t))^2\|z(t)\|^2 - 2\int_0^t\varphi'\varphi''\|z\|^2 ds \geq 0. \tag{3.38}$$

This last inequality is true since $\varphi'\varphi'' \leq 0$. To estimate I_2, we use (iii) of condition (II) (using ψ in the expression for F instead of ψ_0 and ψ_1) to get

$$I_2 = -2\int_0^t(z',Mz)ds \geq \int_0^t(-F - (Mz,z)')ds$$
$$\geq -\gamma_3\int_0^t\psi(s)(s^{-3}\|z\|^2 ds + s^{-1}\|z'\|^2)ds - (M(t)z(t),z(t)) \tag{3.39}$$

We now give an estimate for $\int_0^t s^{-3}\psi(s)\|z\|^2 ds$. Since $z(0)=0$, we know

$\|z(t)\|^2 \le t\int_0^t \|z'(s)\|^2 ds$ and apply this to get

$$\int_0^t s^{-3}\psi(s)\|z\|^2 ds \le \int_0^t \rho(s)\int_0^s \|z'(\eta)\|^2 d\eta ds \le -\int_0^t \frac{d}{ds}\left[\int_s^t \xi^{-2}\psi(\xi)d\xi\right]\int_0^s \|z'(\eta)\|^2 d\eta ds. \tag{3.40}$$

Integrating by parts in (3.40) and using (3.16) with k=1, we get

$$\int_0^t s^{-3}\psi(s)\|z\|^2 ds \le \int_0^t \left[\int_s^t \xi^{-2}\psi(\xi)d\xi\right]\|z'(s)\|^2 ds \le (1-\alpha)^{-1}\int_0^t s^{-1}\psi(s)\|z'(s)\|^2 ds. \tag{3.41}$$

Substitution of (3.41) into (3.39) gives

$$I_2 \ge -c_\alpha \int_0^t s^{-1}\psi(s)\|z'(s)\|^2 ds - (M(t)z(t),z(t)) \tag{3.42}$$

where $c_\alpha = \gamma_3(2-\alpha)/(1-\alpha)$ and α comes from the definition of ψ. Combining (3.37), (3.38) and (3.42), we get

$$\begin{aligned}\|z'(t)\|^2 - c_\alpha \int_0^t s^{-1}\psi(s)\|z'(s)\|^2 ds - (M(t)z(t),z(t)) \\ \le 2\int_0^t \|z'\|\|z''+(\varphi')^2 z - Mz\|ds.\end{aligned} \tag{3.43}$$

Multiply (3.43) by $\rho(t)$ and integrate to get

$$\begin{aligned}\int_0^T \rho\|z'\|^2 dt - c_\alpha \int_0^T \rho(t)\int_0^t s^{-1}\psi(s)\|z'(s)\|^2 ds dt - \int_0^T \rho(Mz,z)dt \\ \le 2\int_0^T \rho(t)\int_0^t \|z'\|\|z''+(\varphi')^2 z - Mz\|ds dt.\end{aligned} \tag{3.44}$$

To estimate the second integral in (3.44), we let $P(t) = \int_t^T \rho(\eta)d\eta$ and note that integration by parts produces $(h(t) = t^{-1}\psi(t)\|z'(t)\|^2)$

$$\begin{aligned}\int_0^T \rho(t)\int_0^t h(\eta)d\eta dt &= -\int_0^T P'(t)\int_0^t h(\eta)d\eta dt \\ &= -P(T)\int_0^T h(\eta)d\eta + \lim_{\varepsilon\downarrow 0} P(\varepsilon)\int_0^\varepsilon h(\eta)d\eta + \int_0^T P(\eta)h(\eta)d\eta.\end{aligned} \tag{3.45}$$

But $P(\varepsilon)\int_0^\varepsilon h(s)ds \le \left[\int_\varepsilon^T t^{-1}\psi(t)\right](1/\varepsilon)\int_0^\varepsilon h(s)ds$ and since $z'(0)=0$ (and $\psi(0)=0$ because of (3.13)), we get $\lim_{\varepsilon\downarrow 0} (1/\varepsilon)\int_0^\varepsilon h(s)ds = \lim_{\varepsilon\downarrow 0} h(\varepsilon) = 0$. Hence $\lim_{\varepsilon\downarrow 0} P(\varepsilon)\int_0^\varepsilon h(s)ds = 0$.

Combining this result with the fact that the first term on the right side of (3.45) is nonpositive, we get

$$\int_0^T \rho(\xi)\int_0^\xi h(\eta)d\eta d\xi \le \int_0^T P(\eta)h(\eta)d\eta. \tag{3.46}$$

However, $t^2P(t) = t^2\int_t^T \eta^{-2}\psi(\eta)d\eta \le t\psi(t)/(1-\alpha)$ (We have used (3.16) here with $k = 1$ and $0 < \alpha < 1$ to get the last inequality.) Thus $P(t) \le (1-\alpha)^{-1}t^{-1}\psi(t)$ and hence substitution of this into (3.46) gives

$$\int_0^T \rho(\xi)\int_0^\xi h(\eta)d\eta d\xi \le (1-\alpha)^{-1} \int_0^T \eta^{-1}\psi(\eta)h(\eta)d\eta. \tag{3.47}$$

Substituting $h(t) = t^{-1}\psi(t)\|z'(t)\|^2$ in (3.47) and using the monotonicity of ψ yields

$$\int_0^T \rho(\xi)\int_0^\xi h(\eta)d\eta d\xi \le (1-\alpha)^{-1}\psi(T)\int_0^T \rho\|z'\|d\eta.$$

Substitution of this inequality into (3.44) gives

$$\hat{c} \int_0^T \rho\|z'\|^2 dt - \int_0^T \rho(Mz,z)dt \le 2\int_0^T \rho(t)\int_0^t \|z'\|\|z''+(\varphi')^2 z-Mz\|dsdt. \tag{3.48}$$

where $\hat{c} = 1-(1-\alpha)^{-1}c_\alpha\psi(T)$. Application of (2.8) to the right side of (3.45) gives, for $T_0 \ge T$,

$$\begin{aligned}(\hat{c}-\varepsilon) \int_0^T \rho\|z'\|^2 dt - \int_0^T \rho(Mz,z)dt &\le \varepsilon^{-1}\lambda(T)\int_0^T \|z''+(\varphi')^2 z-Mz\|^2 dt \\ &\le \varepsilon^{-1}\lambda(T_0)\int_0^T \|z''+(\varphi')^2 z-Mz\|^2 dt.\end{aligned} \tag{3.49}$$

To complete the proof, we substitute (3.32) and (3.26) into (3.49) and simplify. This completes the proof.

LEMMA 10. Suppose the hypothesis of Lemma 9 holds. Then

$$\int_0^T (\varphi')^2\rho e^{-2\varphi}\|u\|^2 dt + C(T,T_0) \int_0^T \rho e^{-2\varphi}\|u'\|^2 dt\Big] \le \int_0^T e^{-2\varphi}\|Lu\|^2 dt \tag{3.50}$$

where $C(T,T_0) = [\lambda(T_0)]^{-1}\Big[.02 - (3\gamma_3+23.36)\psi(T)\Big]$.

PROOF. Since $e^{-\varphi}Lu = z''+2\varphi'z'+(\varphi')^2 z+\varphi''z-Mz-Nz$, we get (All integrals are taken over $[0,T]$.)

$$\begin{aligned}\int e^{-2\varphi}\|Lu\|^2 dt &= \int\|z''+2\varphi'z'+(\varphi')^2 z+\varphi''z-Mz-Nz\|^2 dt \\ &= \int\|z''+(\varphi')^2 z-Mz\|^2 dt + 2\,\mathrm{Re}\int(z''+(\varphi')^2 z-Mz,2\varphi'z'+\varphi''z-Nz) + \int\|2\varphi'z'+\varphi''z-Nz\|^2.\end{aligned} \tag{3.51}$$

In [1; pp. 70-72], it is shown (for $\nu_1 = \nu_2 = \nu_3 = 0$) that

$$\mathrm{Re}\int(z''+(\varphi')^2 z-Mz,2\varphi'z'+\varphi''z-Nz) \ge 0.$$

We now apply this result along with (3.22) and (3.36) to (3.51) to obtain

$$\begin{aligned}(1+\varepsilon^2[\lambda(T_0)]^{-1}) \int e^{-2\varphi}\|Lu\|^2 dt &\ge \varepsilon[\lambda(T_0)]^{-1}(2-c_T)\int\rho e^{-2\varphi}\|u'\|^2 dt \\ &+ [1/\lambda(T_1) - 5/\lambda(T_0)]\int(\varphi')^2\rho e^{-2\varphi}\|u\|^2 dt\end{aligned} \tag{3.52}$$

In (3.52), choose $\alpha = 1/2$, $\varepsilon = [\lambda(T_0)]^{1/2}$, and $T_1 > 0$ sufficiently small that

$1/\lambda(T_1) - 5/\lambda(T_0) > 2$ so that (3.50) follows after simplification. This completes the proof.

We may now prove necessity. We note that Theorem 4 contains the results of [1; Theorem 3] as a special case.

THEOREM 4. *(Necessity)* Suppose the operator A satisfies condition (II) and there exists $T \in (0,T]$ such that $u \in C_*([0,T];D)$ is a solution of (1.2) on $(0,T]$ with $u(0) = u'(0) = 0$. If the functions ψ_i, $i=0,1$, satisfy (3.11), then $u = 0$ on $[0,T]$.

PROOF. Proceeding in the same manner as in the proof of Theorem 2, we again use the function ζu, T' to be chosen below, and note that inequality (3.50) yields

$$\beta^2 \int_0^{T'-\varepsilon} \tau^{-2\beta-2} e^{2\tau^{-\beta}} \rho\|u\|^2 dt + C(T',T_0) \int_0^{T'-\varepsilon} e^{2\tau^{-\beta}} \rho\|u'\|^2 dt\Big] \le \int_0^{T'} e^{2\tau^{-\beta}} \|L(\zeta u)\|^2 dt \quad (3.53)$$

Application of inequality (3.14) to the right side of (3.53) gives

$$\beta^2 \int_0^{T'-\varepsilon} \tau^{-2\beta-2} e^{2\tau^{-\beta}} \rho\|u\|^2 dt + c(T',T_0) \int_0^{T'-\varepsilon} e^{2\tau^{-\beta}} \rho\|u'\|^2 dt\Big]$$
$$\le \gamma \int_0^{T'-\varepsilon} e^{2\tau^{-\beta}} \Big[\mu(t) + \int_0^t \mu(s)ds\Big] dt + \int_{T'-\varepsilon}^{T'} e^{2\tau^{-\beta}} \|L(\zeta u)\|^2 dt. \quad (3.54)$$

Using estimates identical to those of [1, p.64], inequality (3.54) may be simplified to get rid of the $\int_0^t \mu(s)ds$ term (and then γ is replaced with 2γ). If we then apply inequality (3.17) to the resulting inequality, we get

$$\beta^2 \int_0^{T'-\varepsilon} \tau^{-2\beta-2} e^{2\tau^{-\beta}} \rho\|u\|^2 dt + C(T',T_0) \int_0^{T'-\varepsilon} e^{2\tau^{-\beta}} \rho\|u'\|^2 dt\Big]$$
$$\le 2\gamma[1+4(3-\alpha)^{-2}] \int_0^{T'-\varepsilon} e^{2\tau^{-\beta}} \rho\|u'\|^2 dt + \int_{T'-\varepsilon}^{T'} e^{2\tau^{-\beta}} \|L(\zeta u)\|^2 dt. \quad (3.55)$$

Thus we choose $T' \in (0,T]$ small and $T_0 = T'$ so that $C(T',T_0) \ge 2\gamma[1 + 4(3-\alpha)^{-2}]$ (with $\alpha = 1/2$) so that (3.55) may be simplified to get

$$\beta^2 \int_0^{T'-\varepsilon} \tau^{-2\beta-2} e^{2\tau^{-\beta}} \rho\|u\|^2 dt \le \int_{T'-\varepsilon}^{T'} e^{2\tau^{-\beta}} \|L(\zeta u)\|^2 dt.$$

As in [1, p.64], for β large, we may now conclude that

$$\beta^2 \int_0^{T'-\varepsilon} \rho\|u\|^2 dt \le \int_{T'-\varepsilon}^{T'} \|L(\zeta u)\|^2 dt.$$

Letting $\beta \to \infty$ we get $u = 0$ on $[0,T']$. This completes the proof.

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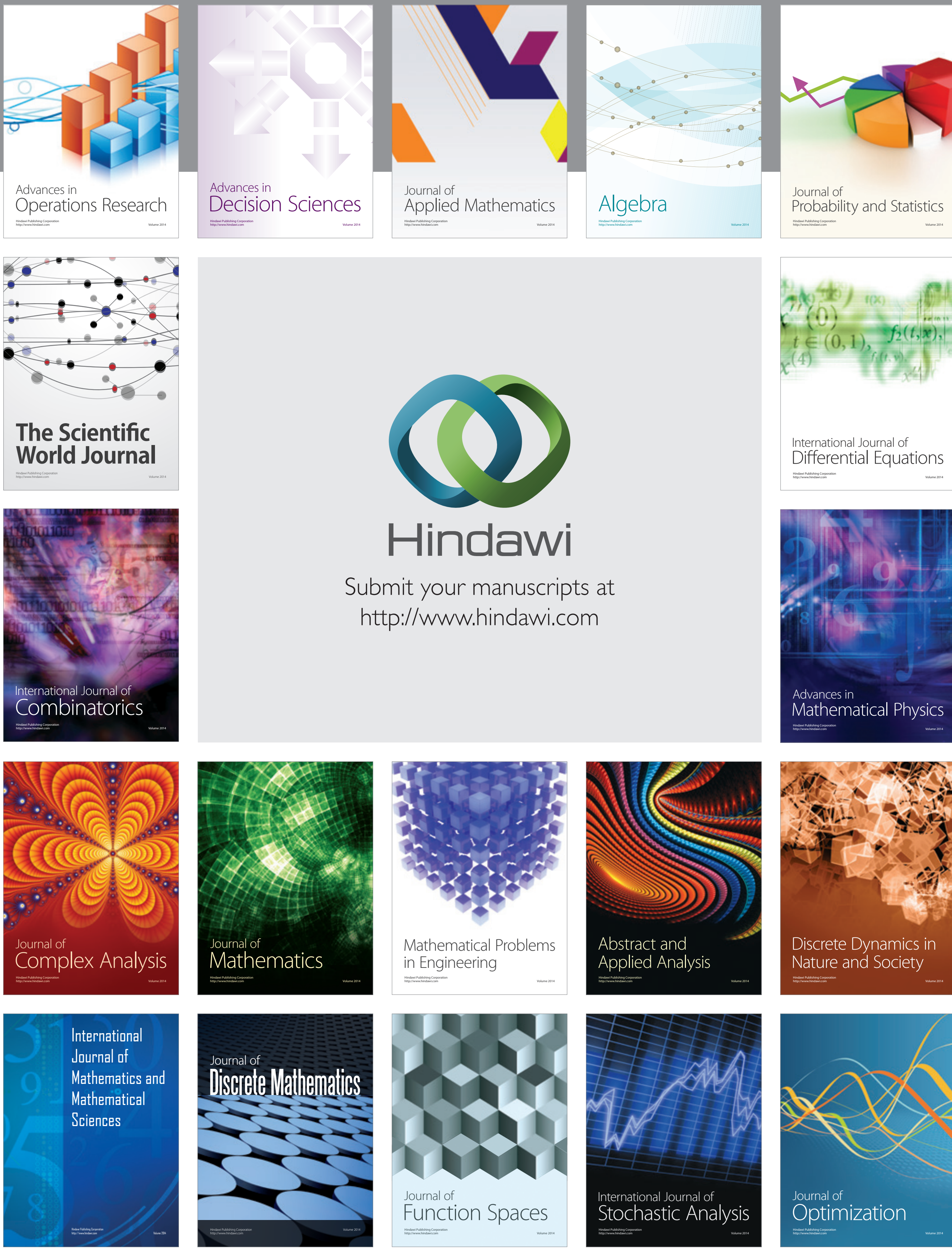
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