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Research Article

Application of BSDE in Standard Inventory Financing Loan

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This paper examines the issue of loans obtained by the small and medium-sized enterprises (SMEs) from banks through the mortgage inventory of goods. And the loan-to-value (LTV) ratio which affects the loan business is a very critical factor. In this paper, we provide a general framework to determine a bank's optimal loan-to-value (LTV) ratio when we consider the collateral value in the financial market with Knightian uncertainty. We assume that the short-term prices of the collateral follow a geometric Brownian motion. We use a set of equivalent martingale measures to build the models about a bank's maximum and minimum levels of risk tolerance in an environment with Knightian uncertainty. The models about the LTV ratios are established with the bank's maximum and minimum risk preferences. Applying backward stochastic differential equations (BSDEs), we get the explicit solutions of the models. Applying the explicit solutions, we can obtain an interval solution for the optimal LTV ratio. Our numerical analysis shows that the LTV ratio in the Knightian uncertainty-neutral environment belongs to the interval solutions derived from the models.

1. Introduction

Because the small and medium enterprises (SMEs) do not have enough credit rating, there is no real estate or a third party to guarantee security. These factors lead to the financing difficulties of SMEs. To solve this problem, we can find an effective solution that is chattel mortgage loans to SMEs. Inventory financing business concerned in this paper is a class of chattel mortgage lending business in the logistics and financial innovation. In this business, the enterprise will provide its production, inventory, and other movable properties to the logistics and warehousing enterprises with legal qualifications in order to receive short-term loans from banks. However, the study about inventory financing business stays in the qualitative stage in many ways. Particularly the LTV ratios (ratio between loan amount and collateral value) are very important in this business. But banks depend largely on the experience of valuation in practice. These valuation methods of banks can not make quantitative analysis about the following factors, such as price volatility of the collateral, probability of default, loan maturity time, and marking to market frequency. Thus banks can not determine accurately LTV ratios of inventory commodities according to the banks' different risk tolerance levels. Therefore, our research about quantitative models of the LTV ratios will provide a scientific basis for the bank decisions and will have significant value of practice.

To get a collateral loan, a bank uses a borrower's own assets as collateral to secure repayment of the loan when it comes to due. The LTV ratio is an important risk factor used by a bank in qualifying a borrower for a collateral loan. This ratio is also closely monitored by a bank after a loan is being made, because the value of collateral could change constantly. For example, if the collateral value decreases, causing an increase of LTV ratio, then the bank will require the borrower to provide more collateral.

To determine an optimal LTV ratio, a critical issue for a bank is to accurately value loan collateral. However, valuing collateral (and any other financial assets as well) is challenging because the capital market involves both risks and uncertainties. Here we follow Chen and Epstein (2002) and Knight (1921) [1, 2] to differentiate risk from uncertainty or Knightian uncertainty (ambiguity) (some studies (e.g., Chen and Epstein (2002); Izhakian (2012) [3]) use ambiguity to refer to Knightian uncertainty). In particular, risk refers to a condition where an outcome is unknown, but the probabilities of the outcome can be measured. In contrast, uncertainty or ambiguity refers to a condition where not only is the event outcome unknown, but also the probabilities associated with the outcome are unknown in the first place. Simply put, risk can be measured, while uncertainty cannot be measured [1] (a possible reason that we cannot measure uncertainty is the lack of all the information we need to estimate probabilities for an event outcome).

This study makes the first effort to introduce Knightian uncertainty into a general framework in determining collateral value when a bank seeks to set up an optimal LTV ratio. Several studies have examined the determinants of bank LTV ratios and the influence of collateral value by considering various types of risks. For example, following the structural method of Merton (1974) [4], Jokivuolle and Peura (2003) [5] present a model of risky debt in which collateral value is correlated with the possibility of default. Cossin and Hricko (2003) [6] determine the discount rate of the collateral by the structured method. But the models based on the structure method assume the endogeneity of default. In fact, other factors not related to a particular debt, such as the company's liquidity problems, are likely to promote corporate's default. So Cossin et al. (2003) [7] assign an exogenous probability of corporate's default. Moreover Cossin et al. (2003) follow the framework and obtain a discount rate of the collateral which is consistent with a bank's risk tolerance. Li et al. (2006) [8] establish a basic model on the determination of LTV ratios based on a reduced-form approach. But none of these studies have explicitly incorporated inherent Knightian uncertainty into their framework (as pointed by Chen and Epstein (2002), "*The Ellsberg Paradox and related evidence have demonstrated that such a distinction (between risk and uncertainty) is behaviorally meaningful*" (p 1403). Recently, some economists have invoked Knightian uncertainty to explain the investor behaviors in times of financial crisis (Dizikes, 2010)).

To appropriately value bank loan collateral, following Chen and Epstein (2002) [1], we consider both the risk and Knightian uncertainty in the financial market. We use a set of probability measures to build a bank's minimum and maximum levels of risk tolerance in an environment with Knightian uncertainty. Under the assumption that the short-term prices of the collateral follow a geometric Brownian motion, we study a borrower's default probability impacted by Knightian uncertainty parameter. Applying BSDEs, we get the explicit solutions of the models about a bank's minimum and maximum levels of risk tolerance. Applying the explicit solutions, we build models of the LTV ratios and obtain an interval solution for the optimal loan-to-value ratios. Finally, our numerical analysis is consistent with the interval solution derived from the model.

The remainder of the paper is organized as follows. In Section 2, we first state the assumptions and the definitions of bank's maximum and minimum risk preference. Then we build the models about the LTV ratios and get the explicit solutions of the models. Applying the explicit solutions, we obtain an interval solution for the optimal LTV ratio. In Section 3, we make numerical analysis of the LTV ratio models. Section 4 draws the concluding remarks.

2. Models

Give a filtered probability space $(\Omega, F, \{F_t\}_{0\le t\le T}, P)$, where the filtration $\{F_t\}_{0\le t\le T}$ is the σ-algebra generated by the Brownian motion $\{W_t\}_{0\le t\le T}$. Suppose that there are two tradable assets in the market. One is a risk-free bond with an interest rate r, and the other is goods used by a borrower as a loan collateral. Their price processes satisfy the following SDEs, where the parameters of r, μ, σ, s are constants, respectively, and T is contractual maturity.

$$dP_t = P_t r dt, \quad P_0 = 1, \tag{1}$$

$$dS_t = S_t\left(\mu dt + \sigma dW_t\right), \quad S_0 = s,\ 0 \le t \le T. \tag{2}$$

At the initial time T_0, the borrower will give a_0 units of the goods with S_0 price to a bank in order to apply to a loan. The bank will give a ω ratio of loan amount for each unit of the collateral.

The model assumptions are listed as follows:

(1) The loan interest rate is a constant R. The loan principal and interest at time t is $v_t = v_0 e^{Rt}$, where v_0 is the underlying asset.

(2) There is a storing cost of loan collateral during the loan period. The bank will hold the cost credited to the loan interest rate.

(3) The loan contract matures at time T, the frequency of covering short positions is M, and the time interval of covering short positions is $\tau(\tau \cdot M = T)$. The trigger level of covering short positions is zero; that is, as long as the loanable value of the collateral (market value of the goods × LTV ratio) is less than the sum of loan principal and interest, the borrower will receive a margin call to restore balance.

(4) We assume the default probability of the loan is Q_0. It is exogenously given by the bank.

In the beginning of the m period, where $m = 1, 2, \ldots, M$, given the interest rate of R, the loan principal and interest is $v_0 e^{R\tau(m-1)}$. The unit number of the collateral is a_{m-1}, with the market value for each unit of S_{m-1}. Thus $v_0 e^{R\tau(m-1)} = \omega a_{m-1} S_{m-1}$.

When the loan contract continues to the end of the m period, the following three cases will appear:

(1) When $v_0 e^{R\tau m} = \omega a_{m-1} S_{m-1}$, the loan principal and interest is equal to the loanable value of the collateral at the end of the m period; the borrower does nothing and the contract simply continues.

(2) When $v_0 e^{R\tau m} < \omega a_{m-1} S_{m-1}$, the borrower can reduce some collateral to a level that $v_0 e^{R\tau m} = \omega a_{m-1} S_{m-1}$; the contract then continues.

(3) When $v_0 e^{R\tau m} > \omega a_{m-1} S_{m-1}$, the borrower must add more collateral so that $v_0 e^{R\tau m} = \omega a_{m-1} S_{m-1}$; the

contract will then continue. Otherwise, the borrower defaults and the contract will be terminated and the liquidation begins. In this case, the bank suffers the loss of $v_0 e^{R\tau m} - a_{m-1} S_m$.

2.1. Knightian Uncertainty and Bank's Maximum and Minimum Risk Preference. To consider the financial market with Knightian uncertainty, we introduce a feasibly controllable set: $\Theta = \{(\theta_t)_{0\le t\le T} \mid |\theta_t| \le k,\ \text{a.e. } t \in [0,T]\}$, where the constant k is nonnegative. Chen and Epstein (2002) call Θ k-ignorance. From SDE (1) and (2), we get

$$S_t = s\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^Q\right\}, \quad 0 \le t \le T, \tag{3}$$

where $W_t^Q = \sigma^{-1}(\mu - r)t + W_t$. Let $\varsigma = \sigma^{-1}(\mu - r)$, $(dQ/dP)|_{F_T} := \exp\{-\varsigma W_T - (1/2)\varsigma^2 T\}$. From the Girsanov theorem, we get that the measure Q is equivalent to the measure P. With the measure Q, the process $\{W_t^Q\}_{0\le t\le T}$ is a Brownian motion. The set Π^θ of equivalent martingale measures is constituted from the set Θ. $\Pi^\theta = \{Q^\theta \mid (dQ^\theta/dQ)|_{F_T} := \exp\{\int_0^T \theta_s dW_s^Q - (1/2)\int_0^T \theta_s{}^2 d_s\},\ (\theta_s)_{0\le s\le T} \in \Theta\}$. The Knightian uncertainty of the financial market is characterized by the set Π^θ. Because the bank does not know which probability measure of Π^θ should be used to calculate the probability of loss, to be conservative, it will calculate the maximum and minimum probability of loss. That is, for any measurable event A, define

$$\begin{aligned} P_{\max}(A) &= \max_{Q^\theta \in \Pi^\theta}\left\{Q^\theta(A)\right\} \\ P_{\min}(A) &= \min_{Q^\theta \in \Pi^\theta}\left\{Q^\theta(A)\right\}. \end{aligned} \tag{4}$$

Clearly, when $k = 0$, there exists a unique equivalent martingale measure Q; thus $P_{\min}(A) = P_{\max}(A) = P(A)$. This indicates that the financial market does not involve the Knightian uncertainty but only risk. All of the previous studies have only considered the situation when $k = 0$, while our study takes the first look when $k \ge 0$ whereas both the risks and Knightian uncertainty in the financial market are being considered.

2.2. Determining LTV Ratio. We build the maximum and minimum risk preferences for a bank to determine the optimal LTV ratio. At the beginning of the m period, $v_0 e^{R\tau(m-1)} = \omega a_{m-1} S_{m-1}$; thus $a_{m-1} = v_0 e^{R\tau(m-1)}/\omega S_{m-1}$. When the borrower defaults, the loss suffered by the bank is loss_m. We have

$$\begin{aligned} \text{loss}_m &= e^{-rm\tau}\left(v_0 e^{R\tau m} - a_{m-1} S_m\right) \\ &= e^{-rm\tau}\left(v_0 e^{R\tau m} - \frac{v_0 e^{R\tau(m-1)}}{\omega}\frac{S_m}{S_{m-1}}\right). \end{aligned} \tag{5}$$

Let L be the maximum loss that the bank is willing to bear, and let L be the function of the underlying asset v_0. For simplicity, we let $L = lv_0$, where l denotes the degree of loan loss determined by the bank. Then we can calculate the probability of loss_m not less than $L = lv_0$ in the m-th time interval. We can also get a bank's minimum and maximum risk preferences in the m-th time interval. Here loss_m is caused by the uncertainty of collateral prices at the m period.

$$\begin{aligned} P_{\max}(\text{loss}_m \ge L) &= \max_{Q^\theta \in \Pi^\theta}\left\{Q^\theta(\text{loss}_m \ge L)\right\} \\ P_{\min}(\text{loss}_m \ge L) &= \min_{Q^\theta \in \Pi^\theta}\left\{Q^\theta(\text{loss}_m \ge L)\right\}. \end{aligned} \tag{6}$$

Next, we will get the explicit solutions of the models. First we give several important lemmas. Let

$L^2(0,T) := \{X : \{X_t\}_{0\le t\le T}$ be $\{F_t\}_{0\le t\le T}$ adapted process and $\|X\|^2 = E\int_0^T |X_s|^2 ds < \infty\}$,

$L^2(\Omega, F, P) := \{\xi : \xi$ be F_T adapted process and $E|\xi|^2 < \infty\}$.

Lemma 1. *For any $\xi \in L^2(\Omega, F, P)$, $(\theta_s)_{0\le s\le T} \in \Theta$, consider the following BSDE:*

$$\begin{aligned} -dy_t^\theta &= -\left(ry_t^\theta + z_t^\theta \cdot \theta_t\right)dt - z_t^\theta dW_t^Q, \\ y_T^\theta &= \xi. \end{aligned} \tag{7}$$

There exist $(\theta_t^1)_{0\le t\le T}$ and $(\theta_t^2)_{0\le t\le T}$ in the set Θ which satisfy

$$\begin{aligned} \max\left\{y_0^\theta : \theta \in \Theta\right\} &= e^{-rT}E^{Q^{\theta^1}}[\xi]; \\ \min\left\{y_0^\theta : \theta \in \Theta\right\} &= e^{-rT}E^{Q^{\theta^2}}[\xi]. \end{aligned} \tag{8}$$

Proof. From Pardoux and Peng (1990) [9], we know that there exists unique adapted solution $(y_t^\theta, z_t^\theta)_{0\le t\le T} \in L^2(0,T) \times L^2(0,T)$ for the BSDE (7), and $y_0^\theta = E^{Q^\theta}(e^{-rT}\xi)$, where $(dQ^\theta/dQ)|_{F_T} := \exp\{-\int_0^T \theta_s dB_s^Q - (1/2)\int_0^T \theta_s^2 ds\}$. From El Karoui et al. (1997) [10], we know that

$$\max_{\theta\in\Theta}\left\{y_0^\theta\right\} = e^{-rT}\max_{Q^\theta\in\mathbb{Q}^\theta}\left[E^{Q^\theta}(\xi)\right] = e^{-rT}E^{Q^{\theta^1}}[\xi], \tag{9}$$

$$\left(\theta_t^1\right)_{0\le t\le T} = \left[-k\cdot\operatorname{sgn}\left(z_t^{\theta^1}\right)\right]_{0\le t\le T} \in \Theta, \tag{10}$$

and $(y_t^{\theta^1}, z_t^{\theta^1})_{0\le t\le T}$ is the solution of the following BSDE.

$$\begin{aligned} -dy_t^{\theta^1} &= \left(-ry_t^{\theta^1} + k\left|z_t^{\theta^1}\right|\right)dt - z_t^{\theta^1} dW_t^Q, \\ y_T^{\theta^1} &= \xi. \end{aligned} \tag{11}$$

Similarly, we can get the formula

$$\min_{\theta\in\Theta}\left\{y_0^\theta\right\} = e^{-rT}\min_{Q^\theta\in\mathbb{Q}^\theta}\left[E^{Q^\theta}(\xi)\right] = e^{-rT}E^{Q^{\theta^2}}[\xi], \tag{12}$$

$$\left(\theta_t^2\right)_{0\le t\le T} = \left[k\cdot\operatorname{sgn}\left(z_t^{\theta^2}\right)\right]_{0\le t\le T} \in \Theta, \tag{13}$$

and $(y_t^{\theta^2}, z_t^{\theta^2})_{0\le t\le T}$ is the solution of the following BSDE.

$$-dy_t^{\theta^2} = \left(-ry_t^{\theta^2} - k\left|z_t^{\theta^2}\right|\right)dt - z_t^{\theta^2}dW_t^Q,$$
$$y_T^{\theta^2} = \xi. \quad (14)$$

□

Lemma 2. *Let ξ be the indicator function $I_{(\text{loss}_m\ge L)}$ and assume that the coefficient of diffusion in the SDE (2) σ is positive; then $(\theta_t^1)_{0\le t\le T} \equiv k$, $(\theta_t^2)_{0\le t\le T} \equiv -k$. Thus*

$$P_{max}(loss_m \ge L) = \max_{Q^\theta\in\Pi^\theta}\left\{Q^\theta(loss_m \ge L)\right\} = E^{Q^{(k)}}\left[I_{(loss_m\ge L)}\right];$$
$$P_{min}(loss_m \ge L) = \min_{Q^\theta\in\Pi^\theta}\left\{Q^\theta(loss_m \ge L)\right\} = E^{Q^{(-k)}}\left[I_{(loss_m\ge L)}\right], \quad (15)$$

where $(dQ^{(k)}/dQ)|_{F_T} := \exp\{-kW_T^Q - (1/2)k^2T\}$, $(dQ^{(-k)}/dQ)|_{F_T} := \exp\{kW_T^Q - (1/2)k^2T\}$.

Proof. Let ξ be the indicator function $I_{(\text{loss}_m\ge L)}$, from formulas (9) and (12), we get

$$\max_{Q^\theta\in\Pi^\theta}\left\{Q^\theta(\text{loss}_m \ge L)\right\} = Q^{\theta^1}(\text{loss}_m \ge L),$$
$$\min_{Q^\theta\in\Pi^\theta}\left\{Q^\theta(\text{loss}_m \ge L)\right\} = Q^{\theta^2}(\text{loss}_m \ge L). \quad (16)$$

From Pardoux and Peng (1994) [11], we get the formula

$$z_t^{\theta^1} = \sigma\cdot\frac{\partial y_t^{\theta^1}}{\partial s},$$
$$z_t^{\theta^2} = \sigma\cdot\frac{\partial y_t^{\theta^2}}{\partial s}. \quad (17)$$

And $\partial y_t^{\theta^1}/\partial s$ and $\partial y_t^{\theta^2}/\partial s$ are the partial derivatives of $y_t^{\theta^1}$ and $y_t^{\theta^2}$ about the tradable goods' price.

We know that the indicator function $I_{(\text{loss}_m\ge L)}$ is a decreasing function about goods' price. According to the comparison theorems of SDE and BSDE, we get that $\partial y_t^{\theta^1}/\partial s$ and $\partial y_t^{\theta^2}/\partial s$ are negative. Thus the processes $z_t^{\theta^1}$ and $z_t^{\theta^2}$ are all negative. From formulas (10) and (13), we get the conclusion obviously. □

Theorem 3.

$$P_{max}(loss_m \ge L) = N\left(\frac{D(\omega) - \left(r - k\sigma - (1/2)\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right),$$
$$P_{min}(loss_m \ge L) = N\left(\frac{D(\omega) - \left(r + k\sigma - (1/2)\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right), \quad (18)$$

where $D(\omega) = \ln\omega(e^{R\tau} - le^{rm\tau - R(m-1)\tau})$.

Proof. From Lemma 2, we get $P_{\max}(\text{loss}_m \ge L) = E^{Q^{(k)}}[I_{(\text{loss}_m\ge L)}] = Q^{(k)}(\text{loss}_m \ge L)$.

For $Q^{(k)}$ in Π^θ, we define the process $W^{Q^{(k)}}(t) = W^Q(t) + kt$. Thus the process $\{W^{Q^{(k)}}(t)\}$ is a Brownian motion about the probability measure $Q^{(k)}$. From SDE (3), we get

$$\ln\frac{S_m}{S_{m-1}} \sim N^{Q^{(k)}}\left(\left(r - k\sigma - \frac{1}{2}\sigma^2\right)\tau, \sigma^2\tau\right),$$
$$Q^{(k)}(\text{loss}_m \ge L)$$
$$= Q^{(k)}\left\{e^{-rm\tau}\left(v_0e^{R\tau m} - \frac{v_0e^{R\tau(m-1)}}{\omega}\frac{S_m}{S_{m-1}}\right) \ge L\right\}$$
$$= Q^{(k)}\left\{0 \le \frac{S_m}{S_{m-1}} \le \omega\left(e^{R\tau} - le^{rm\tau - R(m-1)\tau}\right)\right\}$$
$$= Q^{(k)}\left\{-\infty < \ln\frac{S_m}{S_{m-1}} \le \ln\omega\left(e^{R\tau} - le^{rm\tau - R(m-1)\tau}\right)\right\} \quad (19)$$
$$= \int_{-\infty}^{(\ln\omega(e^{R\tau} - le^{rm\tau - R(m-1)\tau}) - (r - k\sigma - (1/2)\sigma^2)\tau)/\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$
$$= N\left(\frac{\ln\omega\left(e^{R\tau} - le^{rm\tau - R(m-1)\tau}\right) - \left(r - k\sigma - (1/2)\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)$$
$$= N\left(\frac{D(\omega) - \left(r - k\sigma - (1/2)\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right),$$

where $D(\omega) = \ln\omega(e^{R\tau} - le^{rm\tau - R(m-1)\tau})$.

Similarly we can get $P_{\min}(\text{loss}_m \ge L) = N((D(\omega) - (r + k\sigma - (1/2)\sigma^2)\tau)/\sigma\sqrt{\tau})$, which completes the proof. □

If we make a simple assumption that the probability of loan default is exogenous and independent of the particular loan, the bank is concerned with the probability of two events occurring simultaneously. One event is loss_m not less than L, and the other event is the borrower defaults. We assume that the loan's default probability per year is Q_0 with a uniform distribution. Therefore, the default probability of the loan in the m-th period is simply τQ_0. We then get the joint probability of these two events under the bank's maximum and minimum risk preference as follows:

$$P_{\max}(m) \triangleq \tau Q_0 \cdot N\left(\frac{D(\omega) - \left(r - k\sigma - (1/2)\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right),$$

$$P_{\min}(m) \triangleq \tau Q_0 \cdot N\left(\frac{D(\omega) - \left(r + k\sigma - (1/2)\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right). \quad (20)$$

Theorem 4. *One assumes that the loan's default probability is exogenous and independent of the particular loan. One also assumes that the loan's default probability per year is Q_0 with a uniform distribution. During the entire loan period, the probabilities that the bank's losses are not less than L are then estimated as follows:*

$$P_{max}(loss \geq L) = \sum_{m=1}^{M} (1 - \tau Q_0)^{m-1} P_{max}(m) = \sum_{m=1}^{M} (1 - \tau Q_0)^{m-1} \tau Q_0 \cdot N\left(\frac{D(\omega) - \left(r - k\sigma - (1/2)\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right), \quad (21)$$

$$P_{min}(loss \geq L) = \sum_{m=1}^{M} (1 - \tau Q_0)^{m-1} P_{min}(m) = \sum_{m=1}^{M} (1 - \tau Q_0)^{m-1} \tau Q_0 \cdot N\left(\frac{D(\omega) - \left(r + k\sigma - (1/2)\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right). \quad (22)$$

Proof. We know that the borrower can only default once and the probability that the borrower does not default before the $m-1$ periods is $(1-\tau Q_0)^{m-1}$. Thus, with the bank's maximum and minimum risk preference, the probabilities that the bank losses are not less than L in the m-th period are estimated as $(1 - \tau Q_0)^{m-1} P_{\max}(m)$ and $(1 - \tau Q_0)^{m-1} P_{\min}(m)$, respectively. During the entire loan period, the probabilities that the bank's losses are not less than L are then estimated as follows:

$$\begin{aligned} P_{\max}(\text{loss} \geq L) &= P_{\max}(1) + (1 - \tau Q_0) P_{\max}(2) + (1 - \tau Q_0)^2 P_{\max}(3) + \cdots + (1 - \tau Q_0)^{M-1} P_{\max}(M) \\ &= \sum_{m=1}^{M} (1 - \tau Q_0)^{m-1} P_{\max}(m) \\ &= \sum_{m=1}^{M} (1 - \tau Q_0)^{m-1} \tau Q_0 \cdot N\left(\frac{D(\omega) - \left(r - k\sigma - (1/2)\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right), \end{aligned}$$

$$\begin{aligned} P_{\min}(\text{loss} \geq L) &= P_{\min}(1) + (1 - \tau Q_0) P_{\min}(2) + (1 - \tau Q_0)^2 P_{\min}(3) + \cdots + (1 - \tau Q_0)^{M-1} P_{\min}(M) \\ &= \sum_{m=1}^{M} (1 - \tau Q_0)^{m-1} P_{\min}(m) \\ &= \sum_{m=1}^{M} (1 - \tau Q_0)^{m-1} \tau Q_0 \cdot N\left(\frac{D(\omega) - \left(r + k\sigma - (1/2)\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right). \end{aligned} \quad (23)$$

The proof is completed. □

Remark 5. Note that both $P_{\min}(\text{loss} \geq L)$ and $P_{\max}(\text{loss} \geq L)$ reflect a bank's minimum and maximum risk preference for the loan collateral. Given the probabilities of $P_{\min}(\text{loss} \geq L)$ and $P_{\max}(\text{loss} \geq L)$ and the loan's default probability Q_0, mark to market frequency M, loan time T, loan interest rates R, and other parameters, we can solve for the numerical solution of LTV ratios or ω.

3. A Numerical Analysis

We assume that the collateral is copper in the futures market, and its price equation is driven by a Brownian motion, with $\sigma = 0.4118, \mu = 0.255$, and $P = 0.1358$. Following Li et al. (2007) [12], we assume the following parameters in the numerical analysis: $Q_0 = 0.9$ (loan default probability per year), $r = 0.03$ (interest rate), $R = 0.15$ (loan interest rate), $M = 90$ (mark to market frequency), $T = 90$ days (loan time), and k (Knightian uncertainty parameter) is in $(0, 1)$.

We compute the results of formulas (22) and (21) using the Maple software with the minimum and maximum loss probabilities. From formula (22), we get $\omega = 0.57$. From formula (21), we get $\omega = 0.87$. Thus, after considering the Knightian uncertainty, we get the optimal LTV ratio with an interval of $(0.57, 0.87)$. In the Knightian uncertainty-neutral environment, the LTV ratio $\omega = 0.70274$, which is in this interval.

4. Conclusion

We can find that LTV ratios in Knight uncertainty environment are reduced for risk-averse banks, the risk taken is better controlled, and the risk reduction is achieved. Nowadays, because the SMEs encounter the survival bottlenecks, the state has developed a series of measures to ensure the source of funds for SMEs, one of which is to reduce the difficulty of bank loans. Thus the banking institutions were forced to become risk-loving participants. The models of this paper also provide a more reasonable quantitative analysis so as to achieve the tripartite win-win situation among the SMEs and banking institutions and logistics enterprises.

In this paper, we provide a general framework to determine a bank's optimal LTV ratios in the financial market with Knightian uncertainty. In the future, we would extend our models by considering such impacting factors as liquidation delay, liquidity risk, nonzero trigger level, and so on.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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