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m-PARALLELISMS

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Generalizations of the Johnson parallelisms are given using an index m subgroup of a Pappian central collineation group. The parallelisms, called *m-parallelisms*, are constructed and the isomorphisms classes are discussed.

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1. Introduction. In [1], there is a group-theoretic construction of a class of parallelisms in $\mathrm{PG}(3,K)$, where K is a field admitting a quadratic extension. The parallelisms have the property that there is a unique Pappian spread Σ_1 and a collineation group G of the parallelism, which is also the full central collineation group of Σ_1 with fixed axis ℓ. In this case, the group G is transitive on the remaining spreads of the parallelisms and all such spreads are Hall spreads. This construction allows a fairly accurate count of the number of mutually nonisomorphic parallelisms constructed in this manner. This is accomplished by the authors in [5].

Moreover, the following characterization is given.

Theorem 1.1 (see [5]). *Let K be a skewfield, Σ a spread in $\mathrm{PG}(3,K)$, and $\mathcal{P}$ a partial parallelism of $\mathrm{PG}(3,K)$ containing Σ.*

If $\mathcal{P}$ admits, as a collineation group, the full central collineation group G of Σ with a given axis ℓ that acts two-transitive on the remaining spread lines, then
(1) *Σ is Pappian,*
(2) *$\mathcal{P}$ is a parallelism,*
(3) *the spreads of $\mathcal{P}-\{\Sigma\}$ are Hall,*
(4) *G acts transitively on the spreads of $\mathcal{P}-\{\Sigma\}$,*
(5) *$\mathcal{P}$ is one of the parallelisms of the construction of Johnson.*

Hence, using the full central collineation group of an associated Pappian spread forces the remaining spreads of the parallelisms to be Hall spreads. The question is whether such is the case when the parallelism admits only a transitive subgroup; a subgroup which fixes one Pappian spread and acts transitively on the remaining spreads. Are the remaining spreads Hall? In [2], a general construction procedure is given by which several infinite classes of parallelisms are constructed consisting of one Desarguesian spread and the remaining spreads are derived Knuth conical flock spreads.

Theorem 1.2 (see [2]). *Let q be an odd integer equal to $p^{2^b z}$ where z is an odd integer greater than 1. Assume that 2^a is the largest power of 2 that divides $q-1$, then there exists a nonidentity automorphism σ of $\mathrm{GF}(q)$ such that 2^a divides $(\sigma-1)$.*

Let γ_2 and γ_1 be nonsquares of $\mathrm{GF}(q)$ *such that the equation $\gamma_2 t^{\sigma} = \gamma_1 t$ implies that $t = 0$. Then,*

(1) *there exists a parallelism $\mathcal{P}_{\gamma_2,\sigma}$ of derived Knuth type with (q^2+q) derived Knuth planes and one Desarguesian plane;*

(2) *the collineation group of this parallelism contains the central collineation group of the Desarguesian plane with fixed axis ℓ of order $q^2 2^a(q+1)$.*

The basic construction is shown to apply in the infinite case by the authors [4], when K is the field of real numbers.

Theorem 1.3 (see [4]). *Let f be any continuous strictly increasing function on the field of real numbers K such that $\lim_{x\mapsto\pm\infty} f(t) = \pm\infty$. Let Σ_1 be the Pappian spread defined as follows:*

$$x = 0, \qquad y = x\begin{bmatrix} u & -t \\ t & u \end{bmatrix} \quad \forall u,t \in K. \tag{1.1}$$

Let Σ_2 be defined as follows:

$$x = 0, \qquad y = x\begin{bmatrix} u & f(t) \\ t & u \end{bmatrix} \quad \forall u,t \in K, \tag{1.2}$$

where f is a function on K such that $f(t) = t$ implies that $t = 0$, and $f(0) = 0$. Then, Σ_2 is a spread and Σ_1 and Σ_2 share the regulus $\mathcal{R}$ defined by the partial spread $t = 0$.

Assume also that, f is symmetric with respect to the origin in the real Euclidean 2-space and $f(t_0+r) = f(t_0)+r$ for some t_0 and r in the reals implies that $r = 0$. Then $\Sigma_1 \cup \Sigma_2^* g$, for all $g \in G^-$ and where Σ_2^* denotes the derived spread of Σ_2 by derivation of $\mathcal{R}$, is a partial parallelism $\mathcal{P}_f$ in $\mathrm{PG}(3,K)$. *Moreover, $\mathcal{P}_f$ is a parallelism if and only if $f(t)-t$ is an onto function.*

In this paper, we generalize the general group construction in [1], using the full central collineation group G of an associated Pappian spread, but instead of a particular choice of a second Pappian spread sharing a given regulus with the original Pappian spread, we choose m such Pappian spreads. By a choice of cosets of a particular subgroup of G, we are able to construct a tremendous variety of parallelisms. The parallelisms that we obtain are called *m-parallelisms* and, in the finite case, admit a central collineation group (of the original Desarguesian spread in $\mathrm{PG}(3,q)$) of order $q^2(q^2-1)/m$. If $m \neq n$, then an m-parallelism cannot be isomorphic to an n-parallelism.

In a sense, m-parallelisms are generated using particular m Pappian spreads. If n of the m spreads are distinct, we call objects (m,n)-parallelisms. The original construction uses mappings from a particular second Pappian spread for the construction. Such spreads are subject to a choice of coset representations, so further subclasses are obtained.

Actually, we begin our discussion with central collineation groups of finite Desarguesian affine planes acting on parallelisms. However, our arguments apply for a more general class of groups called *parallelism-inducing groups* so our results are ultimately much more general than we initially state.

One of our main construction theorems, using central collineation groups, in the finite case is as follows.

Theorem 1.4. *Let Σ_i, $i = 1,2,\ldots,m+1$, be Desarguesian spreads of* PG$(3,q)$ *containing a regulus $\mathcal{R}$ and assume that the spreads Σ_j for $j \neq 1$ are distinct from Σ_1. Let G denote the full central collineation group of Σ_1 with axis ℓ in $\mathcal{R}$ and assume that m divides $q+1$. Then, there is a normal subgroup G^- of G of order $q^2(q^2-1)/m$, which contains $G_{\mathcal{R}}$.*

Assume that for each Σ_i $i > 2$, there is a line $s_{2,i}$ of $\Sigma_2 - \mathcal{R}$ and an element g_i of $G^- - G_{\mathcal{R}}$ such that $s_{2,i}g_i$ is a line of Σ_i. Choose any coset representative class $\{h_i : i = 2,\ldots,m+1\}$ for G^- in G. Let Σ_i^* denote the spread obtained by the derivation of $\mathcal{R}$. Then $\Sigma_1 \cup_{i=2}^{m+1} \Sigma_i^* h_i k_i$ for all $k_2,\ldots,k_{t+1} \in G^-$ is a parallelism in PG$(3,q)$.

2. The construction in the finite case. The construction in [1] produced parallelisms in PG$(3,q)$ using a Desarguesian spread Σ_1 equipped with a central collineation group of Σ_1, G, with fixed axis ℓ, of order $q^2(q^2-1)$. It turns out that the set of Baer subplanes incident with the origin of Σ_1, which are disjoint from the axis ℓ, are in a single orbit under G and the number of such Baer subplanes is exactly $q^2(q^2-1)$. That is, the group G is regular on this set of Baer subplanes.

The construction also depends on the choice of an initial regulus $\mathcal{R}$ within Σ_1 and containing ℓ. Choose a second Pappian spread Σ_2 containing $\mathcal{R}$ and let G denote the full central collineation group with axis ℓ of Σ_1. If s_2 is any line of $\Sigma_2 - \mathcal{R}$, then we note that $\Sigma_2 = s_2 G_{\mathcal{R}} \cup \mathcal{R}$. Let S be a normal subgroup of G containing $G_{\mathcal{R}}$, and let $h \in S - \mathcal{R}$. We note that $\Sigma_2 S \cup \Sigma_2 hS$ is a partial parallelism. More importantly, if Σ_3 is any Pappian spread distinct from Σ_1 that contains $\mathcal{R}$, and $g \in S - G_{\mathcal{R}}$, then $\Sigma_3 = g s_2 G_{\mathcal{R}} \cup \mathcal{R}$. This says that $\Sigma_3 S = \Sigma_2 S$ as a set and furthermore, it is also true that $\Sigma_3 S \cup \Sigma_3 hS$ is a partial parallelism. Hence, it follows immediately that $\Sigma_2 S \cup \Sigma_3 hS$ is also a partial parallelism. Formally, we list this result below and provide essentially the same argument in a more concrete manner.

Lemma 2.1. *Under the above assumptions, let S denote any normal subgroup of G which contains $G_{\mathcal{R}}$. Let Σ_2 and Σ_3 be Desarguesian spreads distinct from Σ_1 that contain $\mathcal{R}$. Assume that there is an element g of $S - G_{\mathcal{R}}$ which maps an element s_2 of $\Sigma_2 - \mathcal{R}$ onto an element s_3 of Σ_3. Then,*

(1) *s_3 is not in $\mathcal{R}$ and $\Sigma_3 - \mathcal{R} = s_2 g G_{\mathcal{R}} = s_3 G_{\mathcal{R}}$;*

(2) *if $h \in G - S$, then $\Sigma_2 w$ and $\Sigma_3 hu$ share no line for all $w,u \in S$; $\Sigma_2 S \cup \Sigma_3 hS$ is a partial parallelism.*

Proof. We note that $G_{\mathcal{R}}$ acts as a collineation group of any Desarguesian spread which contains $\mathcal{R}$ and acts regularly on the lines (components) of the spread not in $\mathcal{R}$. Hence, this proves (1) (we will see below that s_3 cannot be in $\mathcal{R}$).

Assume that $\Sigma_2 w$ and $\Sigma_3 hu$ share a component. Then, $\Sigma_2 wu^{-1}h^{-1}$ and Σ_3 share a component α. Let $\delta \in \Sigma_2$ such that $wu^{-1}h^{-1}\delta = \alpha$. Suppose that δ is in $\mathcal{R}$. If α is not in $\mathcal{R}$, then $\Sigma_3 = \Sigma_1$ as $wu^{-1}h^{-1}\delta$ is in Σ_1. If $\alpha = \delta$, then $wu^{-1}h^{-1} = 1$ since the group is a central collineation group. But this forces h to be in S, a contradiction. If $\alpha \neq \delta$ then, $wu^{-1}h^{-1}$ leaves $\mathcal{R}$ invariant since $\mathcal{R}$ is a regulus. But, since $G_{\mathcal{R}}$ is in S, again it follows that $wu^{-1}h^{-1}$ is in S, forcing h to be in S.

Hence, δ is not in $\mathcal{R}$. If α is in $\mathcal{R}$, we may use the argument above to conclude that $\Sigma_2 = \Sigma_1$.

So, neither α nor δ is in $\mathcal{R}$. By (1), there exists a unique element z of S which maps α to δ. Hence, $zwu^{-1}h^{-1}\delta = \delta$ so that $zwu^{-1}h^{-1} = 1$ implying that h is in S, again a contradiction. □

Corollary 2.2. *Denote the derived spreads of $\Sigma_i w$ by derivation of $\mathcal{R}w$ by $(\Sigma_i w)^* = \Sigma_i^* w$. Then,*

(1) *$\Sigma_1 \cup \Sigma_2^* w \cup \Sigma_3^* hk$ for h fixed in $G - S$ and for all $w, k \in S$ is a partial parallelism in $\mathrm{PG}(3,q)$;*

(2) *if the order of the S is $q^2(q^2-1)/m$ where m divides $q+1$, then there are $1+2(q(q+1)/m)$ spreads in the partial parallelism. Note that, it is not required that Σ_2 and Σ_3 be distinct.*

Corollary 2.3. *Under the above assumptions, further assume that there are t Desarguesian spreads Σ_i for $i = 2,\dots,t+1$ distinct from Σ_1 and sharing $\mathcal{R}$ with the property that for each Σ_i $i > 2$, there is a line $s_{2,i}$ of $\Sigma_2 - \mathcal{R}$ and an element g_i of S such that $s_{2,i}g_i$ is a line of Σ_i.*

Assume that S is a normal subgroup of G. Let h_i, $i = 2,3,\dots,t+1$, belong to mutually distinct cosets of S. Then,

(1) *$\cup_{i=2}^{t+1}\Sigma_i h_i k_i$, for all $k_2,\dots,k_{t+1} \in S$, is a set of spreads $t|S|$ spreads that share no line of $\mathrm{PG}(3,q)$ not in Σ_1 and disjoint from the axis ℓ of G (of S);*

(2) *$\Sigma_1 \cup_{i=2}^{t+1} \Sigma_i^* h_i k_i$ for all $k_2,\dots,k_{t+1} \in S$ is a partial parallelism in $\mathrm{PG}(3,q)$ of $1 + t(q(q+1)/m)$ spreads provided that the order of S is $q^2(q^2-1)/m$ (we note below that any such group of this order is normal).*

Proof. Suppose that $\Sigma_i h_i k_i$ and $\Sigma_j h_j k_j$ share a component. Then, $\Sigma_i h_i h_i k_j^{-1} h_j^{-1}$ and Σ_j also share a component t_j.

We know that, there exist elements $s_{2,i}$ and $s_{2,j}$ of $\Sigma_2 - \mathcal{R}$ and elements g_i and g_j of S such that $s_{2,i}g_i$ and $s_{2,j}g_j$ are in $\Sigma_i - \mathcal{R}$ and $\Sigma_j - \mathcal{R}$, respectively. Let $\tilde{g} = h_i h_i k_j^{-1} h_j^{-1}$. Let t_i be in Σ_i such that $t_i \tilde{g} = t_j$. It is immediate that t_i and t_j cannot be in $\mathcal{R}$. Hence, there exist elements w_i and w_j of $G_{\mathcal{R}}$ such that $t_i = s_{2,i} g_i w_i$ and $t_j = s_{2,j} g_j w_j$.

Hence, we obtain

$$s_{2,i} g_i w_i \tilde{g} = s_{2,j} g_j w_j. \tag{2.1}$$

Furthermore, since $s_{2,i}$ and $s_{2,j}$ are both in $\Sigma_2 - \mathcal{R}$, it follows that there is an element r of $G_{\mathcal{R}}$ such that $s_{2,i} = s_{2,j} r$.

We, in turn, obtain

$$s_{2,j} r g_i w_i \tilde{g} = s_{2,j} g_j w_j. \tag{2.2}$$

Now, since the group G acts regularly on Baer subplanes of Σ_1, which do not intersect the axis, it follows that $r g_i w_i \tilde{g} = g_j w_j$ and thus $r g_i w_i h_i h_i k_j^{-1} h_j^{-1} = g_j w_j$.

Note that all group elements other than h_i and h_j^{-1} involved in the above expression are in S. But, this says that h_i and h_j are in the same coset of S since S is a normal subgroup. Hence, this contradiction completes the proof of the corollary with the exception of the existence of a normal group of order $q^2(q^2-1)/m$ containing $G_{\mathcal{R}}$ provided that m divides $q+1$.

The full central collineation group $G = EH$, where E is the full elation group of order q^2 and H is a homology group of order q^2-1. Note that E is a normal subgroup and H is cyclic. Let H^- denote the unique cyclic subgroup of order $(q^2-1)/m$ provided that m divides q^2-1.

Then, we assert that EH^- is a normal subgroup of EH and if m divides $q+1$, then it contains $G_{\mathcal{R}}$.

Let $g = eh \in EH = G$, where $e \in E$ and $h \in H$. We recall that $h^{-1}e^{-1}EH^-eh = h^{-1}EH^-eh = Eh^{-1}H^-eh \subseteq Eh^{-1}H^-Eh$ which is $Eh^{-1}EH^{-1}h = Eh^{-1}H^-h = EH$. Hence, EH^- is normal in EH. Since E is in EH^- then $E \cap G_{\mathcal{R}}$ is in EH^-. It remains to show that there is a subgroup of order $q-1$ in $G_{\mathcal{R}} \cap EH^-$. However, H^- has order $(q^2-1)/m$ and is cyclic, so, it contains a group of order $q-1$ if and only if $q-1$ divides $(q^2-1)/m$, if and only if m divides $q+1$. □

Hence, we obtain the following theorem.

Theorem 2.4. *Let Σ_i, for $i = 1,2,\ldots,m+1$, be Desarguesian spreads of* PG$(3,q)$ *containing a regulus $\mathcal{R}$, and assume that the spreads Σ_j for $j \neq 1$ are distinct from Σ_1. Let G denote the full central collineation group of Σ_1 with axis ℓ in $\mathcal{R}$, and assume that m divides $q+1$. Then, there is a normal subgroup G^- of G of order $q^2(q^2-1)/m$ which contains $G_{\mathcal{R}}$.*

Assume that for each Σ_i $i > 2$, there is a line $s_{2,i}$ of $\Sigma_2 - \mathcal{R}$ and an element g_i of $G^- - G_{\mathcal{R}}$ such that $s_{2,i}g_i$ is a line of Σ_i.

Choose any coset representative class $\{h_i : i = 2,\ldots,m+1\}$ for G^- in G. Let Σ_i^* denote the spread obtained by the derivation of $\mathcal{R}$.

Then, $\Sigma_1 \cup_{i=2}^{m+1} \Sigma_i^* h_i k_i$, for all $k_2,\ldots,k_{t+1} \in G^-$, is a parallelism in PG$(3,q)$.

Proof. We merely note that the number of spreads in the partial parallelism is $1+m(q(q+1)/m) = 1+q(q+1) = 1+q+q^2$, so we obtain a parallelism. □

Example 2.5. In order to specify specific instances of the above theorem, assume that q is odd and assume that Σ_i are Desarguesian spreads for $i = 1,2$ of the form $x = 0$, $y = x\left[\begin{smallmatrix} u & \gamma_i t \\ t & u \end{smallmatrix}\right]$ for all $u,t \in \mathrm{GF}(q)$, where γ_i are nonsquares in GF(q) and $\gamma_1 \neq \gamma_2$. Let γ_3 be any nonsquare distinct from γ_1 and γ_2. Let $\theta = (\gamma_2-\gamma_3)/(\gamma_3-\gamma_1)$. Now, consider the mapping of any group G^- of order $q^2(q^2-1)/m$ in E of the form

$$\begin{bmatrix} 1 & 0 & 0 & \theta\gamma_1 \\ 0 & 1 & \theta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.3}$$

Then, $y = x\left[\begin{smallmatrix} 0 & \gamma_2 \\ 1 & 0 \end{smallmatrix}\right]$ maps onto $y = x\left[\begin{smallmatrix} 0 & \theta\gamma_1+\gamma_2 \\ 1+\theta & 0 \end{smallmatrix}\right]$. Now, it follows that

$$\begin{aligned} \gamma_3(1+\theta) &= \gamma_3\left(1+\frac{(\gamma_2-\gamma_3)}{(\gamma_3-\gamma_1)}\right) \\ &= \frac{(\gamma_2-\gamma_3)}{(\gamma_3-\gamma_1)}\gamma_1+\gamma_2 = \theta\gamma_1+\gamma_2. \end{aligned} \tag{2.4}$$

Hence, we may apply the above theorem for any set of nonsquares distinct from γ_1.

We note, however, that the above construction did not actually require finiteness. So, we obtain the following more general result.

Theorem 2.6. *Let Σ_i, $i = 1,2,\dots,m+1$, denote Pappian spreads in* PG$(3,K)$*, for a field K, on the same regulus $\mathcal{R}$ and of the general form,*

$$\Sigma_i : x = 0, \qquad y = x\begin{bmatrix} u & \gamma_i t \\ t & u \end{bmatrix} \quad \forall u,t \in K, \tag{2.5}$$

for any finite set of distinct nonsquares γ_i, $i = 1,2,\dots,m+1$.

Assume that there is an index m-subgroup H^- of the homology group H of Σ_1 with axis $x = 0$. Let E denote the full elation group of Σ_1 with axis $x = 0$ and form EH^- which is a normal subgroup of index m in EH. Further, assume that the full group $(EH)_R \subseteq EH^-$. In the finite case, this is accomplished if and only if m divides $q+1$. Let $H = \cup_{i=2}^{m+1} H^- g_i$, where $g_2 = 1$. Then, $\cup_{i=2}^{m+1}\Sigma_i g_i h_i$, for all $h_2,\dots,h_{m+1} \in EH^-$, is a set of spreads which covers all lines of PG$(3,K)$ *which are disjoint from $x = 0$ and not in Σ_1.*

Proof. More generally, if Σ_i is given by

$$x = 0, \qquad y = x\begin{bmatrix} u+\rho_i t & \gamma_i t \\ t & u \end{bmatrix} \quad \forall u,t \in K, \tag{2.6}$$

then the same elation mapping will work provided that

$$(\gamma_2 - \gamma_i)s = (\gamma_i - \gamma_1), \qquad (\rho_2 - \rho_i)s = (\rho_i - \rho_1) \tag{2.7}$$

have a unique solution for s. Hence, we have at least the solutions when either $\rho_i = \rho_j$ for all i,j or when $\gamma_i = \gamma_j$ for all i,j. □

In particular, we may obtain examples of parallelisms in fields of any characteristic provided that there is a quadratic extension superfield.

Theorem 2.7. *Under the above assumptions, $\Sigma_1 \cup_{i=2}^{m+1} \Sigma_i^* g_i h_i$, for all $h_2,\dots,h_{m+1} \in EH^-$, where Σ_i^* denotes the derived spread by deriving $\mathcal{R} g_i h_i$, for all $h_2,h_3,\dots,h_{m+1} \in EH^-$ (i.e., $\mathcal{R}g$ for all $g \in EH$), is a parallelism of* PG$(3,K)$.

Proof. The only lines which are missing from the previous set $\cup_{i=2}^{m+1}\Sigma_i g_i h_i$ and not in Σ_1 are the lines intersecting $x = 0$ nontrivially. Since these are the Baer subplanes of the regulus nets corresponding to $\mathcal{R}g$ for all $g \in EH$, we have all of the lines covered by q^2+q+1 spreads so we obtain a parallelism. □

Theorem 2.8. *Assume that the set of $(m+1)$ Desarguesian spreads Σ_i are mutually distinct, and $S = EH^-$ is a normal group of index m. Then, the full central collineation group with axis $x = 0$ of the parallelisms constructed above is EH^-.*

Proof. Suppose that there is a central collineation $g \in EH - EH^-$ which acts on the constructed parallelism. Assume, without loss of generality, that $g = g_3$ in the context of the theorem. Then, $\Sigma_3^* g_3 g_3^{-1} = \Sigma_3^*$ is a spread of the parallelism. However, Σ_2^* is also a spread and both spreads cover the Baer subplanes of $\mathcal{R}$ and are distinct, which is a contradiction to the properties of a parallelism. □

Definition 2.9. A parallelism constructed from a group of index m is defined to be an *m-parallelism*.

We remark that, conceivably, different choices of coset representation sets determine nonisomorphic m-parallelisms.

Hence, with $\{g_i\}$ denoting a coset representation set, we denote the associated m-parallelism by $(m,\{g_i\})$.

Corollary 2.10. *An m-parallelism and an n-parallelism for $m \neq n$ are nonisomorphic.*

3. General construction. Let Σ_1 be any Pappian spread in $\mathrm{PG}(3,K)$, and let $\mathcal{R}$ denote a regulus containing a line ℓ. Let G denote the full central collineation group with axis ℓ of Σ_1. Assume that S is a normal subgroup of G of index m, which contains $G_{\mathcal{R}}$, where it is not necessarily assumed that m is finite.

Let Σ_2 denote a Pappian spread in $\mathrm{PG}(3,K)$ containing $\mathcal{R}$ and distinct from Σ_1.

We consider the following set:

$$\mathcal{A} = \{s \in (\Sigma_2 - \mathcal{R})S : \mathcal{R} \cup \{s\} \text{ is a partial spread}\}. \tag{3.1}$$

Then, there is a unique Pappian spread Σ_s containing $\mathcal{R}\cup\{s\}$.

We consider the cardinality of this set of Pappian spreads $\operatorname{card}\{\Sigma_s : s \in \mathcal{A}\}$ and assume that $\operatorname{card}\{\Sigma_s : s \in \mathcal{A}\} \geq m$.

Theorem 3.1. *Under the above assumptions, choose any subset of $\{\Sigma_s : s \in \mathcal{A}\}$ of cardinality m, say $\{\Sigma_s : s \in \mathcal{A}_m\}$, for some subset $\mathcal{A}_m$ of $\mathcal{A}$ of cardinality m. Let $\{g_s$ for $s \in \mathcal{A}_m\}$ denote a coset representation set of the subgroup S. Let $g_2 = 1$ for $2 \in \mathcal{A}_m$. Then $S = \Sigma_1 \cup_{\mathcal{A}_m} \Sigma_s^* g_s h_s$, for all $h_s \in S$ and for all $s \in \mathcal{A}_m$, is a parallelism of $\mathrm{PG}(3,K)$.*

Proof. Clearly, the ideas of the previous sections show that we obtain a partial parallelism. It remains to show that we have a parallelism. Note, it is clear by counting that we have a parallelism in the finite case. We recall that G acts regularly on the set of Baer subplanes of Σ_1 (or rather on the subplanes of the corresponding affine plane) incident with the zero vector of the affine plane, which are disjoint from the axis ℓ of G. We will obtain a parallelism if and only if

$$\cup_{\mathcal{A}_m} \Sigma_s^* g_s h_s, \quad \forall h_s \in S,\ \forall s \in \mathcal{A}_m, \tag{3.2}$$

is a cover of the above-mentioned Baer subplanes. Choose any such subplane π_0. If this subplane is an image of a subplane of $\Sigma_2 - \mathcal{R}$ under S, then the subplane is in $\Sigma_2 h$ for all $h \in S$. Otherwise, the subplane is an image of a subplane of $\Sigma_2 - \mathcal{R}$ by an element of G not in S, and hence equal to $g_{s_1} h_{s_1}$ for some fixed $s_1 \in \mathcal{A}_m$. Let $s_2 \in \Sigma_2 - \mathcal{R}$ such that $s_2 g_{s_1} h_{s_1} = \pi_0$. We know that, there is an element s_2' in $\Sigma_2 - \mathcal{R}$ and an element $m \in S$ such that $s_2' m \in \Sigma_{s_1}$. Moreover, there is an element $n \in G_{\mathcal{R}}$ such that $s_2 n = s_2'$ hence, $s_2' m = s_2 n m \in \Sigma_{s_1}$, where $n, m \in S$. Let $s_2 n m = s_{s_1}$. Thus, $s_2 g_{s_1} h_{s_1} = s_{s_1} m^{-1} n^{-1} g_{s_1} h_{s_1}$. Now, $m^{-1} n^{-1} g_{s_1} h_{s_1} \in g_{s_1} S$ since $g_{s_1} S = S g_{s_1}$. Hence, we have a cover and this completes the proof of the theorem. □

3.1. (m,n)**-parallelisms.** Given an m-parallelism, we assume that there are at least $(m-1)$ Pappian spreads containing a given regulus $\mathcal{R}$, which arise from a given Pappian spread by a mapping from a subgroup G^-. However, this is not necessary for the construction of a parallelism. Given a normal subgroup G^- of G containing $G_{\mathcal{R}}$, assume that we take m Pappian spreads distinct from Σ_1 but of these m, we assume that only n are distinct. Then, we still obtain a parallelism, but now it is not entirely clear what the full central collineation group is that acts on the parallelism. To be clear on this construction, first, assume that m is finite and that we have n Pappian spreads distinct from Σ_1, say Σ_i for $i = 2,3,\dots,n+1$. Assume further, that we have i_j spreads equal to Σ_i for $\sum_{i=2}^{n+1} i_j = n$. Then, we obtain the following parallelism: let $\{g_i : i = 2,\dots,n+1\}$ be a coset representation set, where $g_2 = 1$, then the parallelism is

$$\Sigma_1 \cup_{i=2}^{n+1} \sum_{j=1}^{i_j} \Sigma_i^* g_i h_i, \quad \forall h_i \in G^-, \ \forall i = 2,\dots,n. \tag{3.3}$$

Definition 3.2. Any such parallelism constructed above will be called an (m,n)-parallelism. Since $\{i_j\}$ forms a partition of n, the parallelism depends on the partition. Furthermore, the order is important in this case, so we consider that the partition is ordered. Moreover, the parallelism may depend on the coset representation class $\{g_i\}$. When we want to be clear on the notation, we will refer to the parallelism as a $(m,n,\{i_j\},\{g_i\})$-parallelism. When $n = m$, we use simply the notation of $(m,\{g_i\})$-parallelism.

Furthermore, since each such parallelism depends on a choice of the initial Pappian spreads, the nonisomorphic parallelisms are potentially quite diverse.

4. More examples. Let K be any field. Assume that there is a Pappian spread Σ_1 in $PG(3,K)$, so we may consider the central collineation group EH. We note that H is isomorphic to the multiplicative group of $F-\{0\}$, where F is the field coordinatizing the affine plane defined by Σ_1. Consider EH^-, where H^- is a multiplicative subgroup of H. Then the question becomes does EH^- contain $G_{\mathcal{R}}$? We require H^- to contain a subgroup isomorphic to the multiplicative group of $K-\{0\}$.

We note that, when K has nonsquares and is infinite, a construction of the type mentioned above is possible when H^- is isomorphic to the multiplicative subgroup of $K-\{0\}$, and the cardinality of the set of nonsquares is the cardinality of K. Moreover, it is also possible to take a group H^-, basically, generated by

$$\left\langle \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \gamma_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle, \tag{4.1}$$

if Σ_1 is $x = 0$, $y = x\left[\begin{smallmatrix} u & \gamma_1 t \\ t & u \end{smallmatrix}\right]$ for all $u,t \in K$. Note that the generated group consists of diagonal or off-diagonal type elements.

5. Parallelism-inducing groups. We wish to extend our arguments in the previous sections to more general groups. We recall some of the results of the authors in [3].

Definition 5.1. Let Σ and Σ' be any two distinct spreads in $PG(3,K)$, where K is a field that shares exactly a regulus R and let ℓ be a line of R.

Let G be a collineation group of the affine plane associated with Σ that leaves ℓ invariant and has the following properties:

(i) G is sharply 2-transitive on the set of components of Σ distinct from ℓ,

(ii) G is regular on the set of Baer subplanes of the affine plane associated with Σ which are disjoint from ℓ,

(iii) G_R fixes Σ' and acts regularly on the components of $\Sigma' - R$ (in the finite case, if G_R fixes Σ', then the group is regular on $\Sigma' - R$ by (ii)).

Then, G said to be "parallelism-inducing" with respect to Σ and Σ'.

We justify the above terminology in the following theorem.

Theorem 5.2 (see [3]). *Let G be a parallelism-inducing group with respect to Σ and Σ'. Then, $\Sigma \cup_{g\in G} \Sigma'^* g$ is a parallelism in $PG(3,K)$, where Σ'^* denotes the spread obtained by the derivation of R.*

Our main theorem, essentially, is that the previous results for central collineations hold more generally for parallelism-inducing groups.

Let Σ_1 be any Pappian spread in $PG(3,K)$ and let $\mathcal{R}$ denote a regulus containing a line ℓ. Let G be any parallelism-inducing group for Pappian spreads Σ_1 and Σ_2 fixing the line ℓ of Σ_1. Assume that S is a normal subgroup of G of index m, which contains $G_{\mathcal{R}}$, where it is not necessarily assumed that m is finite.

As noted, Σ_2 will be a Pappian spread in $PG(3,K)$ containing $\mathcal{R}$ and distinct from Σ_1.

We consider the following set defined by (3.1):

$$\mathcal{A} = \{s \in (\Sigma_2 - \mathcal{R})S : \mathcal{R} \cup \{s\} \text{ is a partial spread}\}. \tag{5.1}$$

Then, there is a unique Pappian spread Σ_s containing $\mathcal{R} \cup \{s\}$.

We consider the cardinality of this set of Pappian as spreads $\operatorname{card}\{\Sigma_s : s \in \mathcal{A}\}$ and assume that $\operatorname{card}\{\Sigma_s : s \in \mathcal{A}\} \geq m$.

Theorem 5.3. *Under the above assumptions, choose any subset of $\{\Sigma_s : s \in \mathcal{A}\}$ of cardinality m, say $\{\Sigma_s : s \in \mathcal{A}_m\}$ for some subset $\mathcal{A}_m$ of $\mathcal{A}$ of cardinality m. Let $\{g_s \text{ for } s \in \mathcal{A}_m\}$ denote a coset representation set of the subgroup S. Let $g_2 = 1$ for $2 \in \mathcal{A}_m$. Then, $S = \Sigma_1 \cup_{\mathcal{A}_m} \Sigma_s^* g_s h_s$, for all $h_s \in S$ and for all $s \in \mathcal{A}_m$, is a parallelism of $PG(3,K)$.*

Proof. The previous proofs extend directly to parallelism-inducing groups. Where *central collineation* was used previously, we replace the argument using the assumed *sharply transitive* action of the group in question. □

6. Still more examples.

We consider the following group, which with the full elation group of the associated Desarguesian spread Σ_1, is a putative parallelism-inducing group for Desaguesian spreads

$$nH_y^j : \left\langle (x,y) \longmapsto \left(x^{h^{\lambda(m)}} m^j, y^{h^{\lambda(m)}} m^{j+1}\right) : m \in \mathrm{GF}(q^2) - \{0\}\right\rangle. \tag{6.1}$$

Theorem 6.1 (see [3]). *Any nonlinear parallelism-inducing group for Desarguesian spreads has the form EnH_y^j for some integer j. The group is, in fact, parallelism-inducing provided that the second Desarguesian spread admits*

$$\left\langle (x,y) \longmapsto \left(x^{h^{\lambda(m)}}, y^{h^{\lambda(m)}}\right) : m \in \mathrm{GF}(q) - \{0\} \right\rangle \tag{6.2}$$

as a collineation group and $(qi - i - 1, q^2 - 1) = 1$, where $h^r = q^2$ and $\lambda(m) = 1$, if and only if $m \in \mathrm{GF}(m)$.

Note that in an EnH_y^j group, there is a homology subgroup of the principal Desarguesian spread of order, exactly, $q^2(q^2 - 1)/r$.

Now, take any normal subgroup S of G that contains $G_{\mathcal{R}}$ and apply the previous results. There are a tremendous number of mutually nonisomorphic ways to produce parallelisms. We may extend the definitions of m-parallelisms to include those obtained from the *nearfield parallelism-inducing groups* as well as the (m,n)-parallelisms. Generally speaking, different nearfield groups will produce nonisomorphic parallelisms.

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