THE DYNAMICAL $U(n)$ QUANTUM GROUP

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We study the dynamical analogue of the matrix algebra $M(n)$, constructed from a dynamical R-matrix given by Etingof and Varchenko. A left and a right corepresentation of this algebra, which can be seen as analogues of the exterior algebra representation, are defined and this defines dynamical quantum minor determinants as the matrix elements of these corepresentations. These elements are studied in more detail, especially the action of the comultiplication and Laplace expansions. Using the Laplace expansions we can prove that the dynamical quantum determinant is almost central, and adjoining an inverse the antipode can be defined. This results in the dynamical $GL(n)$ quantum group associated to the dynamical R-matrix. We study a $*$-structure leading to the dynamical $U(n)$ quantum group, and we obtain results for the canonical pairing arising from the R-matrix.

1. Introduction

Dynamical quantum groups have been introduced recently by Etingof and Varchenko [12], see the review paper by Etingof and Schiffmann [10] for an overview and references to the literature, and related algebraic structures have been studied by Lu [26], Xu [38] in the context of deformations of Poisson groupoids, and by Takeuchi [36]. Brzeziński and Militaru [4] compare the various constructions of [26, 36, 38]. In this paper we stick to the definition of Etingof and Varchenko [12] with a slight modification as in [20]. In order to keep the paper self-contained as much as possible we recall the definition in Section 2. We also recall the FRST-construction associated to a solution of the dynamical R-matrix, which gives a wealth of examples, and which we consider explicitly for the trigonometric R-matrix in the $\mathfrak{gl}(n)$-case.

It is well-known that quantum groups have a natural link with special functions of basic hypergeometric type, and in [20] it is shown that this remains valid for the simplest example of a dynamical quantum group associated to the trigonometric R-matrix for SL(2) and in [22], see also [21], for the elliptic R-matrix for SL(2) giving a dynamical

quantum group theoretic interpretation of elliptic hypergeometric series. In particular, [20] gives a dynamical quantum group theoretic interpretation of Askey-Wilson and q-Racah polynomials having many similarities to the interpretation of these polynomials on the (ordinary) quantum SL(2) group using the twisted primitive elements as introduced by Koornwinder [24], see also [18, 30]. This naturally suggests a link between these two approaches, and the link is established by Stokman [35] using the coboundary element of Babelon et al. [2], a universal element in the tensor product of the quantized universal algebra. This element also has a natural interpretation in the context of twisted primitive elements as shown by Rosengren [33]. However, the coboundary element is only known for the $\mathfrak{sl}(2)$-case, but there are conjectures about its form for the $\mathfrak{sl}(n)$ case; see Buffenoir and Roche [5]. The notion of twisted primitive elements of Koornwinder [24], and especially its generalization to coideals, has turned out to be enormously fruitful for the interpretation of special functions of one or many variables as spherical functions on quantum groups or quantum symmetric spaces; see, for example, [7, 9, 25, 29, 32].

As one of the highlights of the application of Lie theory to special functions we mention the group theoretic derivation of the addition formula for Jacobi polynomials as obtained by Koornwinder [23] by working on the symmetric space $U(n)/U(n-1)$ and establishing the spherical and associated spherical elements in terms of Jacobi polynomials; see also Askey [1, Lecture 4] for a nice introduction. In the case $n = 2$ this gives the addition formula for Legendre polynomials. For q-analogues of addition formulas for the Legendre polynomials see the overview [19]. In the quantum group setting, Noumi et al. [31] established the little q-Jacobi polynomials as spherical functions, and Floris [14] calculated the associated spherical elements in terms of little q-Jacobi polynomials and derived an addition formula. On the other hand, using the notion of coideals, Dijkhuizen and Noumi [9] established Askey-Wilson polynomials as spherical functions on a quantum analogue of $U(n)/U(n-1)$.

In light of the above it is natural to ask for the spherical (and associated spherical) elements on the dynamical quantum group analogue of $U(n)/U(n-1)$, and if a precise link to special functions can be established. For this we need to study the dynamical $U(n)$ quantum group more closely, and in a previous paper [21] we have studied general aspects of dynamical quantum groups for this purpose. In case $n = 2$ the authors in [20] show that the algebraic approach to quantum groups as discussed by Dijkhuizen and Koornwinder [8] is applicable, and we expect this to hold true for general n. This paper serves as a first step in this specific programme by defining the dynamical $U(n)$ quantum group and studying some of its elementary properties. In a future paper its corepresentation theory and (associated) spherical functions have to be studied.

The general theory provides us with a dynamical analogue of the algebra of functions on the space of $n \times n$-matrices. In Section 2 we recall the algebraic notions and FRST-construction of Etingof and Varchenko [11, 12], see also [10], and this gives an explicit presentation by generators and relations for this dynamical analogue. In order to make this into a dynamical quantum group, we need to equip (a suitable extension) of this algebra with an antipode. For this purpose we study the dynamical analogues of the minor determinants, which are introduced as matrix elements of corepresentations which are analogues of the natural representation in the exterior algebra. There exist a left and right

corepresentation, and we show that the matrix elements, that is, the dynamical quantum minor determinants, are equal using an identity for Hall-Littlewood polynomials. In particular, this gives a dynamical quantum determinant. This is done in Section 3. In Section 4, we continue the study of these dynamical quantum minor elements and we discuss the appropriate analogues of the Laplace expansions. In Section 5 we show how the Laplace expansions imply that the dynamical quantum determinant is almost central, and localizing we find the dynamical $\mathrm{GL}(n)$ quantum group for which we give an explicit expression for the antipode. The treatment of dynamical quantum minor elements, the Laplace expansions, and the extension to an $\mathfrak{h}$-Hopf algebroid is very much motivated by the paper by Noumi et al. [31]. We also introduce a $*$-structure, so that we obtain the dynamical $U(n)$ quantum group in Section 6. Finally, in Section 7 we study the natural pairing, as introduced by Rosengren [34], see also [21], for the case of the dynamical $\mathrm{GL}(n)$ quantum group and the dynamical $U(n)$ quantum group.

2. The dynamical analogue of the matrix algebra $M(n)$

In this section we give the general definitions of the theory of dynamical quantum groups and we recall the generalized FRST-construction. To define the $\mathfrak{h}$-bialgebroid $\mathscr{F}_R(M(n))$ we apply this construction to a solution of the quantum dynamical Yang-Baxter equation (QDYBE).

Let $\mathfrak{h}$ be a finite dimensional complex vector space, viewed as a commutative Lie algebra, with dual space $\mathfrak{h}^*$. Let $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ be a diagonalizable $\mathfrak{h}$-module. The quantum dynamical Yang-Baxter equation is given by

$$R^{12}(\lambda - h^{(3)})R^{13}(\lambda)R^{23}(\lambda - h^{(1)}) = R^{23}(\lambda)R^{13}(\lambda - h^{(2)})R^{12}(\lambda). \tag{2.1}$$

Here $R : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ is a meromorphic function, h indicates the action of $\mathfrak{h}$, and the upper indices are leg-numbering notation for the tensor product. For instance, $R^{12}(\lambda - h^{(3)})$ denotes the operator $R^{12}(\lambda - h^{(3)})(u \otimes v \otimes w) = R(\lambda - \mu)(u \otimes v) \otimes w$ for $w \in V_\mu$. An R-matrix is a solution of the QDYBE (2.1) which is $\mathfrak{h}$-invariant.

In the example we study, we identify $\mathfrak{h} \cong \mathfrak{h}^* \cong \mathbb{C}^n$ and take V an n-dimensional vector space with basis $\{v_1, \ldots, v_n\}$. The R-matrix $R : \mathfrak{h}^* \to \mathrm{End}_{\mathfrak{h}}(V \otimes V)$ we consider is given by

$$R(\underline{\lambda}) = q \sum_{a=1}^{n} E_{aa} \otimes E_{aa} + \sum_{a<b} E_{aa} \otimes E_{bb} + \sum_{a>b} g(\lambda_a - \lambda_b) E_{aa} \otimes E_{bb} + \sum_{a \neq b} h_0(\lambda_a - \lambda_b) E_{ba} \otimes E_{ab}, \tag{2.2}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $E_{ab} \in \mathrm{End}(V)$ such that $E_{ab} v_c = \delta_{bc} v_a$ and the meromorphic functions h_0 and g are given by

$$h_0(\lambda) = \frac{q^{-1} - q}{q^{-2\lambda} - 1},$$

$$g(\lambda) = \frac{(q^{-2\lambda} - q^{-2})(q^{-2\lambda} - q^2)}{(q^{-2\lambda} - 1)^2}. \tag{2.3}$$

Etingof and Varchenko [12] obtain this R-matrix as the exchange matrix for the vector representation of $GL(n)$, and this R-matrix also fits in Isaev [16] and in particular in the discussion in Hadjiivanov et al. [15]. In the last two references, the corresponding matrix $PR(\underline{\lambda})$, where P denotes the flip $a \otimes b \mapsto b \otimes a$, is considered, which then satisfies a dynamical analogue of the braid relation.

2.1. $\mathfrak{h}$-Hopf algebroids and the generalized FRST-construction. We recall the definition of $\mathfrak{h}$-Hopf algebroids, the algebraic notion for a dynamical quantum groups, and the generalized FRST-construction.

Let $\mathfrak{h}$ be a finite dimensional complex vector space, with dual space $\mathfrak{h}^*$. Denote by $M_{\mathfrak{h}^*}$ the field of meromorphic functions on $\mathfrak{h}^*$. For $\alpha \in \mathfrak{h}^*$ we denote by $T_\alpha : M_{\mathfrak{h}^*} \to M_{\mathfrak{h}^*}$ the automorphism $(T_\alpha f)(\lambda) = f(\lambda + \alpha)$ for all $\lambda \in \mathfrak{h}^*$.

Definition 2.1. An $\mathfrak{h}$-algebra is a complex associative algebra $\mathcal{A}$ with 1 which is bigraded over $\mathfrak{h}^*$, $\mathcal{A} = \bigoplus_{\alpha,\beta \in \mathfrak{h}^*} \mathcal{A}_{\alpha\beta}$, with two algebra embeddings $\mu_l, \mu_r : M_{\mathfrak{h}^*} \to \mathcal{A}_{00}$ (the left and right moment map) such that $\mu_l(f)a = a\mu_l(T_\alpha f)$, $\mu_r(f)a = a\mu_r(T_\beta f)$, for all $f \in M_{\mathfrak{h}^*}$, $a \in \mathcal{A}_{\alpha\beta}$.

A morphism of $\mathfrak{h}$-algebras is an algebra homomorphism which preserves the bigrading and the moment maps.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathfrak{h}$-algebras. The matrix tensor product $\mathcal{A} \tilde{\otimes} \mathcal{B}$ is the $\mathfrak{h}^*$-bigraded vector space with $(\mathcal{A} \tilde{\otimes} \mathcal{B})_{\alpha\beta} = \bigoplus_{\gamma \in \mathfrak{h}^*} (\mathcal{A}_{\alpha\gamma} \otimes_{M_{\mathfrak{h}^*}} \mathcal{B}_{\gamma\beta})$, where $\otimes_{M_{\mathfrak{h}^*}}$ denotes the usual tensor product modulo the relations

$$\mu_r^{\mathcal{A}}(f)a \otimes b = a \otimes \mu_l^{\mathcal{B}}(f)b, \quad \forall a \in \mathcal{A},\ b \in \mathcal{B},\ f \in M_{\mathfrak{h}^*}. \tag{2.4}$$

The multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ for $a, c \in \mathcal{A}$ and $b, d \in \mathcal{B}$ and the moment maps $\mu_l(f) = \mu_l^{\mathcal{A}}(f) \otimes 1$ and $\mu_r(f) = 1 \otimes \mu_r^{\mathcal{B}}(f)$ make $\mathcal{A} \tilde{\otimes} \mathcal{B}$ into an $\mathfrak{h}$-algebra.

Example 2.2. Let $D_{\mathfrak{h}^*}$ be the algebra of difference operators acting on $M_{\mathfrak{h}^*}$, consisting of the operators $\sum_i f_i T_{\beta_i}$, with $f_i \in M_{\mathfrak{h}^*}$ and $\beta_i \in \mathfrak{h}^*$. This is an $\mathfrak{h}$-algebra with the bigrading defined by $f T_{-\beta} \in (D_{\mathfrak{h}^*})_{\beta\beta}$ and both moment maps equal to the natural embedding.

For any $\mathfrak{h}$-algebra $\mathcal{A}$, there are canonical isomorphisms $\mathcal{A} \cong \mathcal{A} \tilde{\otimes} D_{\mathfrak{h}^*} \cong D_{\mathfrak{h}^*} \tilde{\otimes} \mathcal{A}$, defined by

$$x \cong x \otimes T_{-\beta} \cong T_{-\alpha} \otimes x, \quad \forall x \in \mathcal{A}_{\alpha\beta}. \tag{2.5}$$

The algebra $D_{\mathfrak{h}^*}$ plays the role of the unit object in the category of $\mathfrak{h}$-algebras.

Definition 2.3. An $\mathfrak{h}$-bialgebroid is an $\mathfrak{h}$-algebra $\mathcal{A}$ equipped with two $\mathfrak{h}$-algebra homomorphisms $\Delta : \mathcal{A} \to \mathcal{A} \tilde{\otimes} \mathcal{A}$ (the comultiplication) and $\varepsilon : \mathcal{A} \to D_{\mathfrak{h}^*}$ (the counit) such that $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$ and $(\varepsilon \otimes \mathrm{Id}) \circ \Delta = \mathrm{Id} = (\mathrm{Id} \otimes \varepsilon) \circ \Delta$ (under the identifications (2.5)).

For the definition of the antipode we follow [20].

Definition 2.4. An $\mathfrak{h}$-Hopf algebroid is an $\mathfrak{h}$-bialgebroid $\mathcal{A}$ equipped with a $\mathbb{C}$-linear map $S : \mathcal{A} \to \mathcal{A}$, the antipode, such that $S(\mu_r(f)a) = S(a)\mu_l(f)$ and $S(a\mu_l(f)) = \mu_r(f)S(a)$ for all $a \in \mathcal{A}$, $f \in M_{\mathfrak{h}^*}$, and

$$m \circ (\mathrm{Id} \otimes S) \circ \Delta(a) = \mu_l(\varepsilon(a)1), \quad \forall a \in \mathcal{A},$$
$$m \circ (S \otimes \mathrm{Id}) \circ \Delta(a) = \mu_r(T_\alpha(\varepsilon(a)1)), \quad \forall a \in \mathcal{A}_{\alpha\beta}, \tag{2.6}$$

where $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ denotes the multiplication and $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1 \in M_{\mathfrak{h}^*}$.

If there exists an antipode on an $\mathfrak{h}$-bialgebroid, it is unique. Furthermore, the antipode is antimultiplicative, anticomultiplicative, unital, counital and interchanges the moment maps μ_l and μ_r; see [20, Proposition 2.2]. In Definition 2.4 the maps $m \circ (\mathrm{Id} \otimes S)$ and $m \circ (S \otimes \mathrm{Id})$ are well defined on $\mathcal{A} \tilde{\otimes} \mathcal{A}$; see [21].

Example 2.5. (i) We can equip $D_{\mathfrak{h}^*}$ with an $\mathfrak{h}$-Hopf algebroid structure with comultiplication $\Delta : D_{\mathfrak{h}^*} \to D_{\mathfrak{h}^*} \tilde{\otimes} D_{\mathfrak{h}^*} \cong D_{\mathfrak{h}^*}$ being the canonical isomorphism, counit $\varepsilon : D_{\mathfrak{h}^*} \to D_{\mathfrak{h}^*}$ being the identity, and antipode defined by $S(f T_\alpha) = T_{-\alpha} \circ f$.

(ii) For an $\mathfrak{h}$-Hopf algebroid $\mathcal{A}$ with invertible antipode, the opposite and coopposite are also $\mathfrak{h}$-Hopf algebroids. The opposite algebra $\mathcal{A}^{\mathrm{opp}}$ is the algebra $\mathcal{A}$ with opposite multiplication. Then we equip $\mathcal{A}^{\mathrm{opp}}$ with an $\mathfrak{h}$-Hopf algebroid structure by defining $(\mathcal{A}^{\mathrm{opp}})_{\alpha\beta} = \mathcal{A}_{-\alpha,-\beta}$, $\mu_l^{\mathrm{opp}} = \mu_l^{\mathcal{A}}$, $\mu_r^{\mathrm{opp}} = \mu_r^{\mathcal{A}}$, $\Delta^{\mathrm{opp}} = \Delta^{\mathcal{A}}$, $\varepsilon^{\mathrm{opp}} = S^{D_{\mathfrak{h}^*}} \circ \varepsilon^{\mathcal{A}}$ and $S^{\mathrm{opp}} = (S^{\mathcal{A}})^{-1}$. The coopposite algebra $\mathcal{A}^{\mathrm{cop}}$ has the same algebra structure but $\mu_l^{\mathrm{cop}} = \mu_r^{\mathcal{A}}$, $\mu_r^{\mathrm{cop}} = \mu_l^{\mathcal{A}}$, $(\mathcal{A}^{\mathrm{cop}})_{\alpha\beta} = \mathcal{A}_{\beta\alpha}$ and $\Delta^{\mathrm{cop}} = P \circ \Delta^{\mathcal{A}}$, $\varepsilon^{\mathrm{cop}} = \varepsilon^{\mathcal{A}}$, $S^{\mathrm{cop}} = (S^{\mathcal{A}})^{-1}$, where P is the flip operator.

Let $\lambda \mapsto \bar{\lambda}$ be a complex conjugation on $\mathfrak{h}^*$, and denote $\overline{f}(\lambda) = \overline{f(\bar{\lambda})}$ for all $f \in M_{\mathfrak{h}^*}$.

Definition 2.6. An $\mathfrak{h}$-$*$-bialgebroid $\mathcal{A}$ is an $\mathfrak{h}$-bialgebroid equipped with a $*$-operator, that is, a $\mathbb{C}$-antilinear antimultiplicative involution such that $\mu_l(f) = \mu_l(\bar{f})$ and $\mu_r(f) = \mu_r(\bar{f})$, such that $(* \otimes *) \circ \Delta = \Delta \circ *$ and $\varepsilon \circ * = *_{D_{\mathfrak{h}^*}} \circ \varepsilon$, where $* = *_{D_{\mathfrak{h}^*}}$ on $D_{\mathfrak{h}^*}$ is defined by $(f T_\alpha)^* = T_{-\bar{\alpha}} \circ \bar{f}$.

An $\mathfrak{h}$-Hopf $*$-algebroid is an $\mathfrak{h}$-Hopf algebroid that is an $\mathfrak{h}$-$*$-bialgebroid and has an invertible antipode. Then, see [20], $S \circ *$ is an involution.

Until this point we have seen only the example $D_{\mathfrak{h}^*}$ of an $\mathfrak{h}$-bialgebroid. The generalized FRST-construction provides many examples of $\mathfrak{h}$-bialgebroids from R-matrices; see [10, 11, 13, 20]. We recall the construction and we apply the construction to the R-matrix in (2.2) to obtain the main object of study for this paper.

Let $\mathfrak{h}$ and $M_{\mathfrak{h}^*}$ be as before, let $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ be a finite-dimensional diagonalizable $\mathfrak{h}$-module, and let $R : \mathfrak{h}^* \to \mathrm{End}_{\mathfrak{h}}(V \otimes V)$ be a meromorphic function that commutes with the $\mathfrak{h}$-action on $V \otimes V$. Let $\{v_x\}_{x \in X}$ be a homogeneous basis of V, where X is an index set. Write $R_{xy}^{ab}(\lambda)$ for the matrix elements of R,

$$R(\lambda)(v_a \otimes v_b) = \sum_{x,y \in X} R_{xy}^{ab}(\lambda) v_x \otimes v_y, \tag{2.7}$$

and define $\omega : X \to \mathfrak{h}^*$ by $\nu_x \in V_{\omega(x)}$. Let $\mathcal{A}_R$ be the unital complex associative algebra generated by the elements $\{L_{xy}\}_{x,y \in X}$ together with two copies of $M_{\mathfrak{h}^*}$, embedded as subalgebras. The elements of these two copies will be denoted by $f(\lambda)$ and $f(\mu)$, respectively. The defining relations of $\mathcal{A}_R$ are $f(\lambda)g(\mu) = g(\mu)f(\lambda)$, $f(\lambda)L_{xy} = L_{xy}f(\lambda + \omega(x))$, and $f(\mu)L_{xy} = L_{xy}f(\mu + \omega(y))$ for all $f, g \in M_{\mathfrak{h}^*}$, together with the RLL-relations

$$\sum_{x,y \in X} R_{ac}^{xy}(\lambda)L_{xb}L_{yd} = \sum_{x,y \in X} R_{xy}^{bd}(\mu)L_{cy}L_{ax}, \tag{2.8}$$

for all $a,b,c,d \in X$. The bigrading on $\mathcal{A}_R$ is defined by $L_{xy} \in \mathcal{A}_{\omega(x),\omega(y)}$ and $f(\lambda), f(\mu) \in \mathcal{A}_{00}$. The moment maps defined by $\mu_l(f) = f(\lambda)$, $\mu_r(f) = f(\mu)$ make $\mathcal{A}_R$ into an $\mathfrak{h}$-algebra. The $\mathfrak{h}$-invariance of R ensures that the bigrading is compatible with the RLL-relations (2.8). Finally the counit and comultiplication defined by

$$\varepsilon(L_{ab}) = \delta_{ab}T_{-\omega(a)}, \qquad \varepsilon(f(\lambda)) = \varepsilon(f(\mu)) = f,$$

$$\Delta(L_{ab}) = \sum_{x \in X} L_{ax} \otimes L_{xb}, \qquad \Delta(f(\lambda)) = f(\lambda) \otimes 1, \tag{2.9}$$

$$\Delta(f(\mu)) = 1 \otimes f(\mu)$$

equip $\mathcal{A}_R$ with the structure of an $\mathfrak{h}$-bialgebroid; see [11].

Remark 2.7. Hadjiivanov et al. [15] have introduced an analogue of the FRST-construction involving dynamical R-matrices. They use the analogue of (2.8) in which the R-matrices may be different, and where it is assumed that the R-matrix on the right-hand side is nondynamical. In case the constant R matrix corresponds to a limit case of the dynamical R-matrix by taking the limit $\underline{\lambda} \to \pm\infty$ in the dynamical parameter, we see that we can identify the matrix algebra of [15, 16] with the limiting case $\underline{\mu} \to \pm\infty$. Note that Hadjiivanov et al. [15] do not equip their matrix algebras with a coalgebra type structure.

2.2. The dynamical analogue of the algebra $M(n)$.   Now, we apply the generalized FRST-construction to the R-matrix (2.2) in order to define the $\mathfrak{h}$-bialgebroid $\mathcal{F}_R(M(n))$. Let $X = \{1,\dots,n\}$ and define $\omega : X \to \mathfrak{h}^*$ by $i \mapsto e_i$, where e_i is the ith unit vector of $\mathbb{C}^n$. Let $h(\lambda) = q - h_0(\lambda)$ where h_0 is defined as in (2.3), so

$$h(\lambda) = q \frac{(q^{-2\lambda} - q^{-2})}{(q^{-2\lambda} - 1)}. \tag{2.10}$$

Definition 2.8. The $\mathfrak{h}$-algebra $\mathcal{F}_R(M(n))$ is the algebra generated by the elements t_{ij}, i, $j \in \{1,2,\dots,n\}$ together with two copies of $M_{\mathfrak{h}^*}$, denoted by $f(\underline{\lambda}) = f(\lambda_1,\dots,\lambda_n)$ and $f(\underline{\mu}) = f(\mu_1,\dots,\mu_n)$, embedded as subalgebras. Then the defining relations are given by $f_1(\underline{\lambda})f_2(\underline{\mu}) = f_2(\underline{\mu})f_1(\underline{\lambda})$,

$$f(\underline{\lambda})t_{ij} = t_{ij}f(\underline{\lambda} + \omega(i)), \qquad f(\underline{\mu})t_{ij} = t_{ij}f(\underline{\mu} + \omega(j)), \tag{2.11}$$

with $f, f_1, f_2 \in M_{\mathfrak{h}^*}$, together with the RLL-relations

$$h(\mu_b - \mu_d)t_{ab}t_{ad} = t_{ad}t_{ab}, \qquad \forall b < d,$$

$$h(\lambda_c - \lambda_a)t_{cb}t_{ab} = t_{ab}t_{cb}, \qquad \forall a < c,$$

$$t_{ab}t_{cd} = t_{cd}t_{ab} + [h(\lambda_c - \lambda_a) - h(\mu_b - \mu_d)]t_{cb}t_{ad}, \qquad \forall a < c,\ b < d,$$

$$g(\mu_b - \mu_d)t_{ab}t_{cd} = g(\lambda_a - \lambda_c)t_{cd}t_{ab} + [h(\mu_d - \mu_b) - h(\lambda_a - \lambda_c)]t_{ad}t_{cb}, \qquad \forall a < c,\ b < d.$$

$$(2.12)$$

The bigrading $\mathscr{F}_R(M(n)) = \bigoplus_{m,p \in \mathbb{N}^n} \mathscr{F}_{mp}$ is defined on the generators by $f(\underline{\lambda}), f(\underline{\mu}) \in \mathscr{F}_{00}$, $t_{ij} \in \mathscr{F}_{\omega(i),\omega(j)}$ and the moment maps are given by $\mu_l(f) = f(\underline{\lambda})$, $\mu_r(f) = f(\underline{\mu})$. By defining the comultiplication $\Delta : \mathscr{F}_R(M(n)) \to \mathscr{F}_R(M(n)) \widetilde{\otimes} \mathscr{F}_R(M(n))$ and counit $\varepsilon : \mathscr{F}_R(M(n)) \to D_{\mathfrak{h}^*}$ on the generators by

$$\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}, \qquad \Delta(f(\underline{\lambda})) = f(\underline{\lambda}) \otimes 1,$$

$$\Delta(f(\underline{\mu})) = 1 \otimes f(\underline{\mu}),$$

$$(2.13)$$

and $\varepsilon(t_{ij}) = \delta_{ij}T_{-\omega(i)}$, $\varepsilon(f(\underline{\lambda})) = \varepsilon(f(\underline{\mu})) = f$ and extended as algebra homomorphisms we equip $\mathscr{F}_R(M(n))$ with the structure of an $\mathfrak{h}$-bialgebroid.

Remark 2.9. The case $n = 2$ and restricting to functions depending only on $\lambda_1 - \lambda_2$ gives back the case studied in [20]. In case we take a suitable limit $\underline{\mu} \to \pm\infty$ and restrict to the Cartan subalgebra for $\mathfrak{sl}(n)$, the algebra also occurs in Hadjiivanov et al. [15], Isaev [16]. However, they do not consider the comultiplication and counit.

As in [31] for the quantum case and in [20] for $n = 2$, we can give a linear basis for $\mathscr{F}_R(M(n))$. The proof is more involved since we use relations for the functions h, g. Proposition 2.10 is stated for later reference. See also [11, Section 6] for a related result using the Hecke property.

PROPOSITION 2.10. *For every $n \times n$-matrix A, denote $t^A = t_{11}^{a_{11}} t_{12}^{a_{12}} \cdots t_{1n}^{a_{1n}} t_{21}^{a_{21}} \cdots t_{nn}^{a_{nn}}$. Then $\{t^A : A \in M_n(\mathbb{N})\}$ forms a basis over $M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*}$ for the vector space $\mathscr{F}_R(M(n))$.*

Proof. This follows from the diamond lemma; see [3]. First we introduce a total ordering $\prec$ by $t^A \prec t^B$ if $\sum_{i,j} a_{ij} < \sum_{i,j} b_{ij}$ and in case $\sum_{i,j} a_{ij} = \sum_{i,j} b_{ij}$ we use the lexicographical ordering on $(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, a_{31}, \ldots, a_{nn})$.

We have the following reduction system, which is compatible with the introduced total order. Assume $i < j$, $k < l$,

$$t_{il}t_{ik} \longmapsto h(\mu_k - \mu_l)t_{ik}t_{il}, \qquad t_{jk}t_{ik} \longmapsto h(\lambda_j - \lambda_i)^{-1}t_{ik}t_{jk},$$

$$t_{jl}t_{ik} \longmapsto \left[h(\lambda_j - \lambda_i)^{-1} + h(\mu_l - \mu_k)g(\lambda_i - \lambda_j)^{-1}\right]t_{il}t_{jk} + g(\mu_k - \mu_l)g(\lambda_i - \lambda_j)^{-1}t_{ik}t_{jl},$$

$$t_{jk}t_{il} \longmapsto g(\lambda_i - \lambda_j)^{-1}t_{il}t_{jk} + \left(h(\lambda_j - \lambda_i)^{-1} - h(\mu_k - \mu_l)g(\lambda_i - \lambda_j)^{-1}\right)t_{ik}t_{jl}.$$

$$(2.14)$$

To simplify the coefficients on the right-hand side we use $h(\lambda) - h(-\mu) = h(\mu) - h(-\lambda)$ and $g(\mu) - g(\lambda) = (h(\lambda) - h(-\mu))(h(\lambda) - h(\mu))$. If we prove that the reduction system is resolvable, the lemma follows from [3, Theorem 2.1]. There are 24 types of configuration to be checked. The proof is straightforward using

$$h(-\lambda) = \frac{1}{h(\lambda+1)}, \qquad g(-\lambda) = g(\lambda),$$
$$g(\lambda) = h(\lambda)h(-\lambda),$$

(2.15)

and the identities $h(\lambda) - h(\mu) = h(-\mu) - h(-\lambda)$ and

$$h(\lambda)h(\lambda-1) - h(\mu)h(\nu) + h(\mu-\nu)h(\nu) - h(\mu-\nu)h(\lambda)$$
$$+ h(\nu-\mu)h(\mu) - h(\nu-\mu)h(\lambda) = 0,$$

(2.16)

for all $\lambda, \mu, \nu \in \mathfrak{h}^*$. $\qquad\qquad\square$

3. Exterior corepresentations and dynamical quantum minor determinants

We continue with the study of some elementary corepresentations of $\mathcal{F}_R(M(n))$ analogous to the action of $M(n)$ on the exterior algebra of $\mathbb{C}^n$. Using these corepresentations we find the dynamical determinant in $\mathcal{F}_R(M(n))$. This is analogous as for the case of ordinary quantum groups; see, for example, Chari and Pressley [6, Chapter 7], and references given there. First we recall the general definition of a corepresentation of an $\mathfrak{h}$-bialgebroid on an $\mathfrak{h}$-space; see [20]. We introduce the notion of $\mathfrak{h}$-comodule algebras.

Definition 3.1. An $\mathfrak{h}$-space is a vector space over $M_{\mathfrak{h}^*}$ which is also a diagonalizable $\mathfrak{h}$-module, $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$, with $M_{\mathfrak{h}^*} V_\alpha \subseteq V_\alpha$ for all $\alpha \in \mathfrak{h}^*$. A morphism of $\mathfrak{h}$-spaces is an $\mathfrak{h}$-invariant (i.e., grade preserving) $M_{\mathfrak{h}^*}$-linear map.

In case we want to emphasize the dependence on V we also write $fv = \mu_V(f)v$.

We next define the tensor product of an $\mathfrak{h}$-bialgebroid $\mathcal{A}$ and an $\mathfrak{h}$-space V. Put $V \tilde{\otimes} \mathcal{A} = \bigoplus_{\alpha,\beta \in \mathfrak{h}^*} (V_\alpha \otimes_{M_{\mathfrak{h}^*}} \mathcal{A}_{\alpha\beta})$ where $\otimes_{M_{\mathfrak{h}^*}}$ denotes the usual tensor product modulo the relations $v \otimes \mu_l(f)a = fv \otimes a$. The grading $V_\alpha \otimes_{M_{\mathfrak{h}^*}} \mathcal{A}_{\alpha\beta} \subseteq (V \otimes \mathcal{A})_\beta$ for all α and $f(v \otimes a) = v \otimes \mu_r(f)a$ make $V \tilde{\otimes} \mathcal{A}$ into an $\mathfrak{h}$-space. Analogously $\mathcal{A} \tilde{\otimes} V = \bigoplus_{\alpha,\beta \in \mathfrak{h}^*} (\mathcal{A}_{\alpha\beta} \otimes_{M_{\mathfrak{h}^*}} V_\beta)$ where $\otimes_{M_{\mathfrak{h}^*}}$ denotes the usual tensor product modulo the relations $\mu_r(f)a \otimes v = a \otimes fv$. The grading $\mathcal{A}_{\alpha\beta} \otimes_{M_{\mathfrak{h}^*}} V_\beta \subseteq (\mathcal{A} \tilde{\otimes} V)_\alpha$ and $f(a \otimes v) = \mu_l(f)a \otimes v$, $a \in \mathcal{A}$, $v \in V$, $f \in M_{\mathfrak{h}^*}$, make $\mathcal{A} \tilde{\otimes} V$ into an $\mathfrak{h}$-space.

Definition 3.2. A right corepresentation of an $\mathfrak{h}$-bialgebroid $\mathcal{A}$ on an $\mathfrak{h}$-space V is an $\mathfrak{h}$-space morphism $\rho : V \to V \tilde{\otimes} \mathcal{A}$ such that $(\mathrm{Id} \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{Id}) \circ \rho$, $(\mathrm{Id} \otimes \varepsilon) \circ \rho = \mathrm{Id}$. The first equality is in the sense of the natural isomorphism $(V \tilde{\otimes} \mathcal{A}) \tilde{\otimes} \mathcal{A} \cong V \tilde{\otimes} (\mathcal{A} \tilde{\otimes} \mathcal{A})$ and in the second identity, use the identification $V \cong V \tilde{\otimes} D_{\mathfrak{h}^*}$ defined by $v \otimes fT_{-\alpha} \cong fv$, $f \in M_{\mathfrak{h}^*}$, for all $v \in V_\alpha$.

A left corepresentation of an $\mathfrak{h}$-bialgebroid $\mathcal{A}$ on an $\mathfrak{h}$-space V is an $\mathfrak{h}$-space morphism $\rho : V \to \mathcal{A} \tilde{\otimes} V$ such that $(\Delta \otimes \mathrm{Id}) \circ \rho = (\mathrm{Id} \otimes \rho) \circ \rho$, $(\varepsilon \otimes \mathrm{Id}) \circ \rho = \mathrm{Id}$.

Definition 3.3. Let $\mathcal{A}$ be an $\mathfrak{h}$-bialgebroid and V an $\mathfrak{h}$-space. Then V is a right (left) $\mathfrak{h}$-*comodule algebra* for $\mathcal{A}$ if there exists a right (left) corepresentation $R: V \to V \tilde{\otimes} \mathcal{A}$ ($L: V \to \mathcal{A} \tilde{\otimes} V$) such that

   (i) V is an associative algebra such that $\mu_V(f)vw = v\mu_V(T_\alpha f)w$ for $v \in V_\alpha$, $w \in V$, and $V_\alpha V_\beta \subset V_{\alpha+\beta}$,

   (ii) $R(L)$ is an algebra homomorphism.

If, moreover, V is a unital algebra, we require $R(L)$ to be unital.

Remark 3.4. The algebra structure of $V \tilde{\otimes} \mathcal{A}$ is given by $(v \otimes a)(w \otimes b) = vw \otimes ab$ for $v, w \in V$ and $a, b \in \mathcal{A}$. For $v \in V_\alpha$, $w \in V_\gamma$ and $a \in \mathcal{A}_{\alpha\beta}$, $b \in \mathcal{A}_{\gamma\delta}$, we have $(v \otimes a)(w \otimes b) = vw \otimes ab \in V_{\alpha+\gamma} \otimes \mathcal{A}_{\alpha+\gamma,\beta+\delta}$ using (i) which implies $(V \tilde{\otimes} \mathcal{A})_\beta (V \tilde{\otimes} \mathcal{A})_\delta \subseteq (V \tilde{\otimes} \mathcal{A})_{\beta+\delta}$. For $v \in V_\alpha$,

$$\mu_{V \tilde{\otimes} A}(f)R(v) = (1 \otimes \mu_r(f))R(v) = R(v)(1 \otimes \mu_r(T_\alpha f)) = R(v)\mu_{V \tilde{\otimes} A}(T_\alpha f). \qquad (3.1)$$

So R preserves the relation in (i). Recall that by the $M_{\mathfrak{h}*}$-linearity of a corepresentation we have $R(\mu_V(f)v) = \mu_{V \tilde{\otimes} A}(f)R(v) = (1 \otimes \mu_r(f))R(v)$.

Now we define the $\mathfrak{h}$-space W on which we construct a right corepresentation of $\mathcal{F}_R(M(n))$. W can be seen as the dynamical analogue of the exterior algebra representation.

Definition 3.5. Let W be the unital associative algebra generated by the elements w_i, $i \in \{1, 2, \ldots, n\}$ and a copy of $M_{\mathfrak{h}*}$ embedded as a subalgebra, its elements denoted by $f(\underline{\lambda})$, subject to the relations

$$w_i^2 = 0, \quad \forall i, \qquad w_j w_i = -h(\lambda_j - \lambda_i)w_i w_j, \quad \forall i < j, \qquad (3.2)$$

with h defined by (2.10) and $f(\underline{\lambda})w_i = w_i f(\underline{\lambda} + \omega(i))$ for all $f \in M_{\mathfrak{h}*}$.

For an ordered subset $I = \{i_1, \ldots, i_r\}$, $1 \le i_1 < \cdots < i_r \le n$, of $\{1, \ldots, n\}$ we use the convention $w_I = w_{i_1} \cdots w_{i_r}$, unless mentioned otherwise. Moreover, $\varnothing$ is an ordered subset and $w_\varnothing = 1$ corresponding to the case $r = 0$. The following lemma is easily proved.

LEMMA 3.6. $\dim_{M_{\mathfrak{h}*}} W = 2^n$ and a basis for W is given by $\{w_I : I = \{i_1, \ldots, i_r\}, i_1 < \cdots < i_r, r = 1, \ldots, n\}$. W is an $\mathfrak{h}$-space with $\mu_W(f) = f(\underline{\lambda})$ and $w_I \in W_{\omega(I)}$ for w_I a basis element with $\omega(I) = \sum_{j=1}^r \omega(i_j)$.

Define $W^r = \text{span}_{M_{\mathfrak{h}*}} \{w_I : \#I = r\}$. Then $W = \bigoplus_{r=0}^n W^r$ and $W^r W^s \subset W^{r+s}$, with the convention that $W^r = \{0\}$ if $r > n$.

PROPOSITION 3.7. *Define* $R(1) = 1 \otimes 1$, $R(w_i) = \sum_{j=1}^n w_j \otimes t_{ji}$. *Then R extends uniquely to $R: W \to W \tilde{\otimes} \mathcal{F}_R(M(n))$ such that W is a right $\mathfrak{h}$-comodule algebra for $\mathcal{F}_R(M(n))$.*

Proof. It is clear that W satisfies the conditions of Definition 3.3(i). To see that R can be extended uniquely to an algebra homomorphism we need to verify

$$R(w_i)R(w_i) = 0, \quad \forall i, \qquad R(w_j)R(w_i) = -R(h(\lambda_j - \lambda_i))R(w_i)R(w_j), \quad \forall i < j, \qquad (3.3)$$

and $R(f(\underline{\lambda}))R(w_i) = R(w_i)R(f(\underline{\lambda} + \omega(i)))$ for all $f \in M_{\mathfrak{h}^*}$. By definition of R and the defining relations (3.2) of W we get

$$R(w_i)R(w_i) = \sum_{j,k} w_j w_k \otimes t_{ji} t_{ki} = \sum_{k>j} \left[w_j w_k \otimes t_{ji} t_{ki} - h(\lambda_k - \lambda_j) w_j w_k \otimes t_{ki} t_{ji} \right] \quad (3.4a)$$

$$= \sum_{k>j} w_j w_k \otimes \left(t_{ji} t_{ki} - h(\lambda_k - \lambda_j) t_{ki} t_{ji} \right) = 0, \quad (3.4b)$$

where we use the second relation of (2.12) in the last equality. Let us emphasize that the function h should be interpreted in (3.4a) as $\mu_W(\underline{\lambda} \mapsto h(\lambda_k - \lambda_j))$ and in (3.4b) as $\mu_l(\underline{\lambda} \mapsto h(\lambda_k - \lambda_j))$. Similarly we obtain that the relation $R(w_j)R(w_i) = -R(h(\lambda_j - \lambda_i))R(w_i)R(w_j)$ for $i < j$ is equivalent to

$$\sum_{l>k} w_k w_l \otimes \left(t_{kj} t_{li} - h(\lambda_l - \lambda_k) t_{lj} t_{ki} + h(\mu_j - \mu_i) t_{ki} t_{lj} - h(\mu_j - \mu_i) h(\lambda_l - \lambda_k) t_{li} t_{kj} \right) = 0.$$
$$(3.5)$$

Using Lemma 3.6 it remains to prove that

$$t_{kj} t_{li} - h(\lambda_l - \lambda_k) t_{lj} t_{ki} + h(\mu_j - \mu_i) t_{ki} t_{lj} - h(\mu_j - \mu_i) h(\lambda_l - \lambda_k) t_{li} t_{kj} = 0, \quad (3.6)$$

for $i < j$, $k < l$. To show this we multiply this equation by $h(\lambda_l - \lambda_k) - h(\mu_i - \mu_j)$ and eliminate the products $t_{kj} t_{li}$ and $t_{li} t_{kj}$ using the third and fourth relations in (2.12), respectively. Using $h(\lambda) - h(-\mu) = h(\mu) - h(-\lambda)$ for all λ, μ we obtain that the relation (3.6) holds by (2.15). Using the definition of R and Remark 3.4 the last relation follows analogously.

By the definition of the comultiplication and the counit on the generators of $\mathscr{F}_R(M(n))$ of Definition 2.8 it immediately follows that $(\mathrm{Id} \otimes \Delta) \circ R(w_i) = (R \otimes \mathrm{Id}) \circ R(w_i)$ and $(\mathrm{Id} \otimes \varepsilon) \circ R(w_i) = w_i$. Since R, Δ, and ε are $\mathfrak{h}$-algebra homomorphisms, so are $(\mathrm{Id} \otimes \Delta) \circ R$, $(R \otimes \mathrm{Id}) \circ R$, and $(\mathrm{Id} \otimes \varepsilon) \circ R$. So the equalities $(\mathrm{Id} \otimes \Delta) \circ R = (R \otimes \mathrm{Id}) \circ R$ and $(\mathrm{Id} \otimes \varepsilon) \circ R = \mathrm{Id}$ hold on the generators of W and hence on all of W. R is a corepresentation of $\mathscr{F}_R(M(n))$ on W and W is an $\mathfrak{h}$-comodule algebra for $\mathscr{F}_R(M(n))$. $\qquad \square$

For I and J ordered subsets with $\#I = \#J$ we define the elements ξ_J^I as the corresponding matrix elements;

$$R(w_J) = \sum_{\#I = \#J} w_I \otimes \xi_J^I. \quad (3.7)$$

We use the convention that $\xi_J^I = 0$ for all I, J such that $\#I \neq \#J$. In the remainder of the paper use the convention that a summation over subsets such as $\sum_{\#I = r}$ is a summation over all ordered subsets I such that $\#I = r$.

COROLLARY 3.8. (i) $\Delta(\xi_J^I) = \sum_{\#K = \#I} \xi_K^I \otimes \xi_J^K$ and $\varepsilon(\xi_J^I) = \delta_{IJ} T_{-\omega(I)}$ for all I, J with $\#I = \#J$.
(ii) $R(W^r) \subset W^r \tilde{\otimes} \mathscr{F}_R(M(n))$.

We call the matrix elements ξ_J^I the *dynamical quantum minor determinants* of $\mathscr{F}_R(M(n))$ with respect to the subsets I and J. The element $\xi_{\{1,\ldots,n\}}^{\{1,\ldots,n\}}$ is called the *determinant* of $\mathscr{F}_R(M(n))$, and is also denoted by det.

This right corepresentation has a left analogue, a left $\mathfrak{h}$-comodule algebra V for $\mathscr{F}_R(M(n))$. The proofs are analogous to the ones for the right $\mathfrak{h}$-comodule algebra W, and are skipped.

Definition 3.9. Let V be the unital associative algebra generated by the elements v_i, $i \in \{1,\dots,n\}$ and a copy of $M_{\mathfrak{h}^*}$ embedded as a subalgebra, its elements denoted by $f(\underline{\lambda})$, subject to the relations

$$v_i^2 = 0, \quad \forall i, \qquad v_i v_j = -h(\lambda_i - \lambda_j) v_j v_i, \quad \forall i < j, \tag{3.8}$$

and $f(\underline{\lambda})v_i = v_i f(\underline{\lambda} + \omega(i))$ for all $f \in M_{\mathfrak{h}^*}$.

For an ordered subset $I = \{i_1,\dots,i_r\}$ with $1 \le i_1 < \cdots < i_r \le n$ we denote by v_I the ordered element $v_I = v_{i_r} \cdots v_{i_1} \in V$. Let us emphasize that an element $v_I \in V$ has reversed order compared to $w_I \in W$ by notational convention.

LEMMA 3.10. $\dim_{M_{\mathfrak{h}^*}} V = 2^n$ and a basis for V is given by $\{v_I : I = \{i_1,\dots,i_r\}, i_1 < \cdots < i_r, r = 1,\dots,n\}$. V is an $\mathfrak{h}$-space with $\mu_V(f) = f(\underline{\lambda})$ and $v_I \in V_{\omega(I)}$ for v_I a basis element.

Define $V^r = \text{span}_{M_{\mathfrak{h}^*}} \{v_I : \#I = r\}$. Then $V = \bigoplus_{r=0}^n V^r$ and $V^r V^s \subset V^{r+s}$, with the convention that $V^r = \{0\}$ if $r > n$.

PROPOSITION 3.11. *Define $L(1) = 1 \otimes 1$, $L(v_i) = \sum_{j=1}^n t_{ij} \otimes v_j$. Then L extends uniquely to $L : V \to \mathscr{F}_R(M(n)) \widetilde{\otimes} V$ such that V is a left $\mathfrak{h}$-comodule algebra for $\mathscr{F}_R(M(n))$.*

For ordered subsets I, J with $\#I = \#J$ we define the elements η_J^I by

$$L(v_I) = \sum_{\#J = \#I} \eta_J^I \otimes v_J \tag{3.9}$$

and $\eta_J^I = 0$ for $\#I \ne \#J$. We denote the corresponding determinant by $\widetilde{\det} = \eta_{\{1,\dots,n\}}^{\{1,\dots,n\}}$.

COROLLARY 3.12. (i) $\Delta(\eta_J^I) = \sum_{\#K=\#I} \eta_K^I \otimes \eta_J^K$ *and* $\varepsilon(\eta_J^I) = \delta_{IJ} T_{-\omega(I)}$ *for I, J with $\#I = \#J$.*
(ii) $L(V^r) \subset \mathscr{F}_R(M(n)) \widetilde{\otimes} V^r$.

We call the matrix elements η_J^I the dynamical quantum minor determinants of $\mathscr{F}_R(M(n))$ with respect to the subsets I and J. In Theorem 3.17 we prove that the dynamical quantum minor determinants related to the right and left corepresentations are equal, so we can speak of *the* dynamical quantum minor determinants of $\mathscr{F}_R(M(n))$, without mentioning right or left. First we compute an explicit expression of the dynamical quantum minor determinants which we use in the proof.

For any permutation $\sigma \in S_r$, $1 \le r \le n$, and any ordered subset $I = \{i_1,\dots,i_r\}$, we define the generalized sign function $S(\sigma, I) \in M_{\mathfrak{h}^*}$ by

$$S(\sigma, I)(\underline{\lambda}) = \prod_{\{k<l:\sigma(k)>\sigma(l)\}} -h(\lambda_{i_{\sigma(k)}} - \lambda_{i_{\sigma(l)}}) = (-q)^{l(\sigma)} \prod_{\{k<l:\sigma(k)>\sigma(l)\}} \frac{q^{-2\lambda_{i_{\sigma(k)}}} - q^{-2} q^{-2\lambda_{i_{\sigma(l)}}}}{q^{-2\lambda_{i_{\sigma(k)}}} - q^{-2\lambda_{i_{\sigma(l)}}}},$$

$$\tag{3.10}$$

where $l(\sigma)$ denotes the length of the permutation, $l(\sigma) = \#\{k < l : \sigma(k) > \sigma(l)\}$.

LEMMA 3.13. *For any permutation $\sigma \in S_r$ the following relation in W holds:*

$$w_{i_{\sigma(1)}} \cdots w_{i_{\sigma(r)}} = \mu_W(S(\sigma, I)) w_I, \tag{3.11}$$

where $I = \{i_1, \ldots, i_r\}$ is ordered.

Proof. We prove by induction on r, for $r = 2$ and $\sigma = \mathrm{Id}$ it is trivial. If $\sigma = (12)$ it is just (3.2) for $j = i_{\sigma(1)}$, $i = i_{\sigma(2)}$. Denote by I' the ordered subset of I defined by $I \setminus \{i_{\sigma(1)}\}$, then

$$w_{i_{\sigma(1)}} \cdots w_{i_{\sigma(r+1)}} = w_{i_{\sigma(1)}} \prod_{\substack{2 \le k < l \le r+1 \\ \sigma(k) > \sigma(l)}} -h(\lambda_{i_{\sigma(k)}} - \lambda_{i_{\sigma(l)}}) w_{I'} = \prod_{\substack{2 \le k < l \le r+1 \\ \sigma(k) > \sigma(l)}} -h(\lambda_{i_{\sigma(k)}} - \lambda_{i_{\sigma(l)}}) \prod_{\substack{2 \le l \le r+1 \\ \sigma(1) > \sigma(l)}}$$

$$- h(\lambda_{i_{\sigma(1)}} - \lambda_{i_{\sigma(l)}}) w_I = \mu_W(S(\sigma, I)) w_I, \tag{3.12}$$

since $w_{i_{\sigma(1)}}$ commutes with all functions in $M_{\mathfrak{h}^*}$ which are independent of $\lambda_{i_{\sigma(1)}}$. $\square$

Using Lemma 3.13 we calculate the action of the corepresentation R on $w_{j_1} \cdots w_{j_r}$ for an arbitrary unordered set $\{j_1, \ldots, j_r\}$. Then

$$R(w_{j_1} \cdots w_{j_r}) = R(w_{j_1}) \cdots R(w_{j_r}) = \sum_{k_1 = 1}^{n} \cdots \sum_{k_r = 1}^{n} w_{k_1} \cdots w_{k_r} \otimes t_{k_1 j_1} \cdots t_{k_r j_r}, \tag{3.13}$$

and there is only a nonzero contribution in the right-hand side of (3.13) if all $k_i \ne k_j$ for $i \ne j$. Let $I = \{i_1, \ldots, i_r\}$ be ordered, then we see that the contribution on the right-hand side of (3.13) containing the basis element w_I in the first leg of the tensor product is given for those terms for which $\{i_1, \ldots, i_r\} = \{k_1, \ldots, k_r\}$ as unordered sets. So there exists for each nonzero term in (3.13) contributing to the term containing w_I in the first leg of the tensor product precisely one permutation $\sigma \in S_r$ such that $k_p = i_{\sigma(p)}$. So the term containing w_I in the first leg of the tensor product equals

$$\sum_{\sigma \in S_r} w_{i_{\sigma(1)}} \cdots w_{i_{\sigma(r)}} \otimes t_{i_{\sigma(1)} j_1} \cdots t_{i_{\sigma(r)} j_r} = \sum_{\sigma \in S_r} \mu_W(S(\sigma, I)) w_I \otimes t_{i_{\sigma(1)} j_1} \cdots t_{i_{\sigma(r)} j_r}$$

$$= \sum_{\sigma \in S_r} w_I \otimes \mu_I(S(\sigma, I)) t_{i_{\sigma(1)} j_1} \cdots t_{i_{\sigma(r)} j_r}, \tag{3.14}$$

by Lemma 3.13 and Remark 3.4.

PROPOSITION 3.14. *Let J be ordered with $r = \#J$, then $R(w_J) = \sum_{\#I = \#J} w_I \otimes \xi_J^I$ with the dynamical quantum minor determinants given by*

$$\xi_J^I = \mu_r(S(\rho, J)^{-1}) \sum_{\sigma \in S_r} \mu_I(S(\sigma, I)) t_{i_{\sigma(1)} j_{\rho(1)}} \cdots t_{i_{\sigma(r)} j_{\rho(r)}}, \tag{3.15}$$

for any $\rho \in S_r$.

Proof. By Lemma 3.13 and the discussion preceding this proposition we obtain

$$\sum_{\#I=\#J} w_I \otimes \xi_J^I = R(w_J) = \left(1 \otimes \mu_r\big(S(\rho,J)^{-1}\big)\right) R(w_{j_{\rho(1)}} \cdots w_{j_{\rho(r)}})$$

$$= \left(1 \otimes \mu_r\big(S(\rho,J)^{-1}\big)\right) \sum_{\#I=\#J} w_I \otimes \mu_l(S(\sigma,I)) t_{i_{\sigma(1)} j_{\rho(1)}} \cdots t_{i_{\sigma(r)} j_{\rho(r)}}. \tag{3.16}$$

So, the proposition follows from Lemma 3.6.   □

COROLLARY 3.15. *Put* $S(\sigma) = S(\sigma, \{1,\ldots,n\})$ *for* $\sigma \in S_n$, *then for any* $\rho \in S_n$,

$$\det = \mu_r\big(S(\rho)^{-1}\big) \sum_{\sigma \in S_n} \mu_l(S(\sigma)) t_{\sigma(1)\rho(1)} \cdots t_{\sigma(n)\rho(n)}. \tag{3.17}$$

Analogously we obtain an explicit formula for the matrix elements η_J^I of L. We need to define another generalized sign function $\widetilde{S}$ depending on an ordered subset I, $\#I = r$, and a permutation $\sigma \in S_r$;

$$\widetilde{S}(\sigma,I)(\underline{\lambda}) := \prod_{\{k<l:\sigma(k)>\sigma(l)\}} -h\big(\lambda_{i_{\sigma(l)}} - \lambda_{i_{\sigma(k)}}\big) = \frac{1}{S(\sigma,I)(\lambda+1)}, \tag{3.18}$$

where we use $h(-\lambda) = 1/h(\lambda+1)$ for the last equality. Analogous to Lemma 3.13, we have for any permutation $\sigma \in S_r$ the following relation in V:

$$v_{i_{\sigma(r)}} \cdots v_{i_{\sigma(1)}} = \mu_V\big(\widetilde{S}(\sigma,I)\big) v_I, \tag{3.19}$$

where $I = \{i_1,\ldots,i_r\}$ is an ordered subset and $v_I = v_{i_r} \cdots v_{i_1}$. We get the analogous statement of Proposition 3.14.

PROPOSITION 3.16. *Let* $I = \{i_1,\ldots,i_r\}$ *be an ordered subset, then* $L(v_I) = \sum_{\#J=\#I} \eta_J^I \otimes v_J$ *with the dynamical quantum minor determinants given by, for any* $\rho \in S_r$,

$$\eta_J^I = \mu_l\big(\widetilde{S}(\rho,I)^{-1}\big) \sum_{\sigma \in S_r} \mu_r\big(\widetilde{S}(\sigma,J)\big) t_{i_{\rho(r)} j_{\sigma(r)}} \cdots t_{i_{\rho(1)} j_{\sigma(1)}}. \tag{3.20}$$

We now relate the two sets of dynamical quantum minor determinants. For this we need the following identity:

$$\sum_{\sigma \in S_r} \prod_{i<j} \frac{x_{\sigma(i)} - t x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}} = \prod_{i=1}^r \frac{1-t^i}{1-t}, \tag{3.21}$$

for r indeterminates $x_1,\ldots,x_r$. This identity can be found in Macdonald [27, Section III.1, equation (1.4)] as the identity expressing that the Hall-Littlewood polynomials for the zero partition gives 1.

THEOREM 3.17. $\xi_J^I = \eta_J^I$ *in* $\mathscr{F}_R(M(n))$.

Proof. The proof is based on the expressions (3.15) and (3.20), which give the possibility to write a suitable multiple of ξ_J^I as a double sum over S_r, which, by interchanging summations, gives a multiple of η_J^I. The multiples turn out to be equal. The details are as follows.

First we rewrite η_J^I. Define the longest element $\sigma_0 \in S_r$ by $\sigma_0 = \left(\begin{smallmatrix} 1 & 2 & \cdots & r \\ r & r-1 & \cdots & 1 \end{smallmatrix}\right)$. By substituting $\rho \mapsto \rho\sigma_0$ and $\sigma \mapsto \sigma\sigma_0$ in (3.20) we get

$$\eta_J^I = \prod_{\substack{m<p \\ \rho(m)<\rho(p)}} -h(\lambda_{i_{\rho(m)}} - \lambda_{i_{\rho(p)}})^{-1} \sum_{\sigma \in S_r} \prod_{\substack{k<l \\ \sigma(k)<\sigma(l)}} -h(\mu_{j_{\sigma(k)}} - \mu_{j_{\sigma(l)}}) t_{i_{\rho(1)}j_{\sigma(1)}} \cdots t_{i_{\rho(r)}j_{\sigma(r)}}, \quad (3.22)$$

for any $\rho \in S_r$. Using this expression for η_J^I and (3.15) we compute

$$\left(\sum_{\rho \in S_r} \prod_{k<l} -h(\mu_{j_{\rho(k)}} - \mu_{j_{\rho(l)}}) \right) \xi_J^I$$

$$= \sum_{\rho \in S_r} \sum_{\sigma \in S_r} \prod_{\substack{k<l \\ \rho(k)<\rho(l)}} -h(\mu_{j_{\rho(k)}} - \mu_{j_{\rho(l)}}) \prod_{\substack{k<l \\ \sigma(k)>\sigma(l)}} -h(\lambda_{i_{\sigma(k)}} - \lambda_{i_{\sigma(l)}}) t_{i_{\sigma(1)}j_{\rho(1)}} \cdots t_{i_{\sigma(r)}j_{\rho(r)}}$$

$$= \sum_{\sigma \in S_r} \sum_{\rho \in S_r} \prod_{k<l} -h(\lambda_{i_{\sigma(k)}} - \lambda_{i_{\sigma(l)}}) \prod_{\substack{k<l \\ \sigma(k)<\sigma(l)}} -h(\lambda_{i_{\sigma(k)}} - \lambda_{i_{\sigma(l)}})^{-1} \times \prod_{\substack{k<l \\ \rho(k)<\rho(l)}}$$

$$\quad (3.23)$$

$$-h(\mu_{j_{\rho(k)}} - \mu_{j_{\rho(l)}}) t_{i_{\sigma(1)}j_{\rho(1)}} \cdots t_{i_{\sigma(r)}j_{\rho(r)}} = \left(\sum_{\sigma \in S_r} \prod_{k<l} -h(\lambda_{i_{\sigma(k)}} - \lambda_{i_{\sigma(l)}}) \right) \eta_J^I.$$

So it suffices to prove that $A(I)(\underline{\lambda}) := \sum_{\rho \in S_r} \prod_{k<l} -h(\lambda_{i_{\rho(k)}} - \lambda_{i_{\rho(l)}})$ is independent of λ and I:

$$A(I)(\underline{\lambda}) = \sum_{\rho \in S_r} \prod_{k<l} (-q) \frac{q^{-2\lambda_{i_{\rho(k)}}} - q^{-2} q^{-2\lambda_{i_{\rho(l)}}}}{q^{-2\lambda_{i_{\rho(k)}}} - q^{-2\lambda_{i_{\rho(l)}}}} = (-q)^{(1/2)r(r-1)} \prod_{k=1}^{r} \frac{1 - q^{-2k}}{1 - q^{-2}} \neq 0, \quad (3.24)$$

using the explicit expression (2.10) for h and (3.21). $\qquad\qquad\qquad\qquad\qquad \square$

COROLLARY 3.18. $\det = \widetilde{\det}$.

Remark 3.19. (i) The dynamical quantum minor determinant ξ_J^I belongs to the weight space $\mathcal{F}_R(M(n))_{\omega(I),\omega(J)}$, where $\omega(I) = \sum_{k=1}^{r} \omega(i_k) = \sum_{k=1}^{r} e_{i_k}$, $I = \{i_1,\ldots,i_r\}$.

(ii) From Theorem 3.17 we obtain relations in $\mathcal{F}_R(M(n))$. For $r = 2$ we get quadratic relations for the generators t_{ij}, for $\rho = \mathrm{Id}$ in the expressions of ξ_J^I and η_J^I in (3.15) and (3.20) we get the third relation of (2.12). Similarly, from Proposition 3.14 for $r = 2$ and taking the expressions for the dynamical quantum minor for $\rho = \mathrm{Id}$ and $\rho = (12)$ we get (3.6).

Remark 3.20. Hadjiivanov et al. [15] introduce a dynamical determinant in the dynamical analogue of $M(n)$ they consider; see also Remarks 2.7, 2.9. In particular, they show that their dynamical quantum determinant is central; (cf. Lemma 5.1). They work with the Hecke property of the matrix $PR(\underline{\lambda})$, and use the q-antisymmetrisers in the Hecke algebra of type A following the ideas of Gurevich for the quantum $SL(n)$-group; see [6, Section 7.2] for more information.

4. Laplace expansions

In this section, we prove some expansion formulas for the dynamical quantum minor determinants, which are used in the following section to introduce the antipode.

For I_1, I_2 disjoint ordered subsets of $\{1,\dots,n\}$, denote by $\operatorname{sign}(I_1;I_2)$ the element of $M_{\mathfrak{h}^*}$ defined by

$$\operatorname{sign}(I_1;I_2)(\underline{\lambda}) = \prod_{\substack{k>m \\ k\in I_1, m\in I_2}} -h(\lambda_k - \lambda_m). \tag{4.1}$$

Then $w_{I_1} w_{I_2} = \mu_W(\operatorname{sign}(I_1;I_2))w_I$ if $I_1 \cap I_2 = \varnothing$ and $I_1 \cup I_2 = I$. If $I_1 \cap I_2 \neq \varnothing$, then $w_{I_1} w_{I_2} = 0$ and in this case we define $\operatorname{sign}(I_1;I_2)(\underline{\lambda}) = 0$. For $I_1 \cap I_2 = \varnothing$ and $I = I_1 \cup I_2$ as ordered subset we have $\operatorname{sign}(I_1;I_2) = S(\sigma,I)$ where σ is the permutation which maps $I_1 \cup I_2$ to the ordered subset I.

PROPOSITION 4.1 (Laplace expansions). *Let I, J_1, J_2 be subsets of $\{1,\dots,n\}$. If $J = J_1 \cup J_2$, $\#J = \#I$,*

$$\mu_r\big(\operatorname{sign}(J_1;J_2)\big)\xi_J^I = \sum_{I_1 \cup I_2 = I} \mu_l\big(\operatorname{sign}(I_1;I_2)\big)\xi_{J_1}^{I_1} \xi_{J_2}^{I_2},$$

$$\mu_l\Big(T_{-\omega(J_1)}\operatorname{sign}(J_2;J_1)^{-1}\Big)\xi_I^J = \sum_{I_1 \cup I_2 = I} \mu_r\Big(T_{-\omega(I_1)}\operatorname{sign}(I_2;I_1)^{-1}\Big)\xi_{I_1}^{J_1} \xi_{I_2}^{J_2}, \tag{4.2}$$

where the summation runs over all partitions $I_1 \cup I_2 = I$ of I such that $\#I_1 = \#J_1$, $\#I_2 = \#J_2$.

Remark 4.2. (i) Note that the left-hand sides of the expressions in (4.2) are zero if J_1 and J_2 are not disjoint.

(ii) The second relation of (4.2) can be rewritten as

$$\xi_I^J = \sum_{I_1 \cup I_2 = I} \xi_{I_1}^{J_1} \frac{\mu_l\big(\operatorname{sign}(J_2;J_1)\big)}{\mu_r\big(\operatorname{sign}(I_2;I_1)\big)} \xi_{I_2}^{J_2}. \tag{4.3}$$

Proof of Proposition 4.1. We have

$$R(w_{J_1})R(w_{J_2}) = \sum_{I_1 \cap I_2 = \varnothing} w_{I_1} w_{I_2} \otimes \xi_{J_1}^{I_1} \xi_{J_2}^{I_2}$$

$$= \sum_{I_1 \cap I_2 = \varnothing} \mu_W\big(\operatorname{sign}(I_1;I_2)\big)w_I \otimes \xi_{J_1}^{I_1} \xi_{J_2}^{I_2} \tag{4.4}$$

$$= \sum_{I_1 \cap I_2 = \varnothing} w_I \otimes \mu_l\big(\operatorname{sign}(I_1;I_2)\big)\xi_{J_1}^{I_1} \xi_{J_2}^{I_2}.$$

Also, if $J_1 \cap J_2 \neq \varnothing$, then $R(w_{J_1} w_{J_2}) = R(0) = 0$ by (3.2) which proves the first relation of (4.2) using Lemma 3.6 in the case that J_1 and J_2 are not disjoint. If $J_1 \cap J_2 = \varnothing$, then we also have

$$R(w_{J_1})R(w_{J_2}) = R(w_{J_1} w_{J_2}) = (1 \otimes \mu_r(\operatorname{sign}(J_1;J_2)))R(w_J) = \sum_{\#I=\#J} w_I \otimes \mu_r(\operatorname{sign}(J_1;J_2))\xi_J^I.$$

(4.5)

The second relation of (4.2) is proved analogously, using L instead of R and Theorem 3.17. $\qquad\square$

In the special case $\#I = \#J = n$ and either J_1 or J_2 contains one element, we get the following expansion formulas for the determinant element. These expansions can be seen as dynamical equivalent of the cofactor expansion across a row or column of the determinant of a matrix.

COROLLARY 4.3. *For all i and j with $1 \le i, j \le n$,*

$$\delta_{ij} \det = \sum_{k=1}^{n} \frac{\operatorname{sign}(\{k\};\widehat{k})(\underline{\lambda})}{\operatorname{sign}(\{i\};\widehat{\imath})(\underline{\mu})} t_{kj}\xi_{\widehat{\imath}}^{\widehat{k}}, \qquad \delta_{ij} \det = \sum_{k=1}^{n} t_{jk} \frac{\operatorname{sign}(\widehat{\imath};\{i\})(\underline{\lambda})}{\operatorname{sign}(\widehat{k};\{k\})(\underline{\mu})} \xi_{\widehat{k}}^{\widehat{\imath}},$$

$$\delta_{ij} \det = \sum_{k=1}^{n} \frac{\operatorname{sign}(\widehat{k};\{k\})(\underline{\lambda})}{\operatorname{sign}(\widehat{\imath};\{i\})(\underline{\mu})} \xi_{\widehat{\imath}}^{\widehat{k}} t_{kj}, \qquad \delta_{ij} \det = \sum_{k=1}^{n} \xi_{\widehat{k}}^{\widehat{\imath}} \frac{\operatorname{sign}(\{i\};\widehat{\imath})(\underline{\lambda})}{\operatorname{sign}(\{k\};\widehat{k})(\underline{\mu})} t_{jk},$$

(4.6)

with the notation $\widehat{\imath} = \{1,\ldots,i-1,i+1,\ldots,n\}$.

5. The dynamical $\operatorname{GL}(n)$ quantum group

In this section we extend $\mathcal{F}_R(M(n))$ by adjoining an inverse of the determinant. The resulting $\mathfrak{h}$-bialgebroid $\mathcal{F}_R(\operatorname{GL}(n))$ is equipped with an antipode, so it is an $\mathfrak{h}$-Hopf algebroid.

LEMMA 5.1. *In $\mathcal{F}_R(M(n))$, the determinant element commutes with all quantum minor determinants ξ_J^I, for I, J subsets of $\{1,\ldots,n\}$. In particular, det commutes with all generators t_{ij}. Moreover, $\Delta(\det) = \det \otimes \det$ and $\varepsilon(\det) = T_{-\underline{1}}$, with $\underline{1} = (1,\ldots,1) \in \mathfrak{h}^*$.*

Proof. Denote by T the $n \times n$-matrix with elements t_{ij}, where i indicates the row index. Using the notation

$$T_j^i = \frac{\mu_l(\operatorname{sign}(\widehat{\imath};\{i\}))}{\mu_r(\operatorname{sign}(\widehat{\jmath};\{j\}))} \xi_{\widehat{\jmath}}^{\widehat{\imath}},$$

(5.1)

denote by $\widetilde{T}$ the $n \times n$-matrix with elements T_j^i where i indicates the column index. Then the third relation of Corollary 4.3 implies $\widetilde{T}T = \det I$ as n^2 identities in $\mathcal{F}_R(M(n))$, where I is the $n \times n$-identity matrix. So $\det T = T\widetilde{T}T = T \det$ which implies that det commutes with all generators t_{ij}. Since $\det \in \mathcal{F}_R(M(n))_{\underline{1},\underline{1}}$, we see that det commutes with all elements in $M_{\mathfrak{h}^*}$ that only depend on differences $\lambda_i - \lambda_j$. By (3.15), det also commutes with ξ_J^I for all subsets I, J. The last statements follow from Corollary 3.8. $\qquad\square$

So the determinant element commutes with all generators t_{ij}, but since $\det \in \mathscr{F}_R(M(n))_{1,1}$, the element det is not central. However, the set $S = \{\det^k\}_{k \geq 1}$ satisfies the Ore condition, and this implies that we can localize at det; see [28]. We adjoin $\mathscr{F}_R(M(n))$ with the formal inverse $\det^{-1}$, adding the relations $\det \det^{-1} = 1 = \det^{-1} \det$, $t_{ij} \det^{-1} = \det^{-1} t_{ij}$ and $f(\underline{\lambda}) \det^{-1} = \det^{-1} f(\underline{\lambda} - \underline{1})$, $f(\underline{\mu}) \det^{-1} = \det^{-1} f(\underline{\mu} - \underline{1})$. We denote the resulting algebra by $\mathscr{F}_R(\mathrm{GL}(n))$ and equip it with a bigrading $\mathscr{F}_R(\mathrm{GL}(n)) = \bigoplus_{m,p \in \mathbb{Z}^n} \mathscr{F}_R(\mathrm{GL}(n))_{mp}$ by $\det^{-1} \in (\mathscr{F}_R(\mathrm{GL}(n)))_{-\underline{1},-\underline{1}}$. Lemma 5.1 implies that $\det^{-1}$ commutes with all dynamical quantum minor determinants ξ_j^I. By extending the comultiplication and counit of Definition 2.8 by $\Delta(\det^{-1}) = \det^{-1} \otimes \det^{-1}$, $\varepsilon(\det^{-1}) = T_{\underline{1}}$, $\mathscr{F}_R(\mathrm{GL}(n))$ it is easily checked that $\mathscr{F}_R(\mathrm{GL}(n))$ is an $\mathfrak{h}$-bialgebroid.

PROPOSITION 5.2. *The $\mathfrak{h}$-bialgebroid $\mathscr{F}_R(\mathrm{GL}(n))$ is an $\mathfrak{h}$-Hopf algebroid with the antipode S defined on the generators by $S(\det^{-1}) = \det$, $S(\mu_r(f)) = \mu_l(f)$, $S(\mu_l(f)) = \mu_r(f)$ for all $f \in M_{\mathfrak{h}^*}$ and*

$$S(t_{ij}) = \det^{-1} \frac{\mu_l(\mathrm{sign}(\hat{j};\{j\}))}{\mu_r(\mathrm{sign}(\hat{i};\{i\}))} \xi_{\hat{i}}^{\hat{j}}, \qquad (5.2)$$

and extended as an algebra antihomomorphism.

Proof. By [20, Proposition 2.2] it suffices to check that S is well defined and that (2.6) holds on the generators. It is straightforward to check that S preserves the relations (2.11). To see that S preserves the RLL-relations, we apply the antipode to the RLL-relations (2.8). Using (5.1) this gives

$$\sum_{x,y} \det^{-2} T_y^d T_x^b R_{ac}^{xy}(\underline{\mu}) = \sum_{x,y} \det^{-2} T_a^x T_c^y R_{xy}^{bd}(\underline{\lambda}), \qquad (5.3)$$

which is equivalent to

$$\sum_{x,y} R_{ac}^{xy}(\underline{\mu} + \omega(x) + \omega(y)) T_y^d T_x^b \det^{-2} = \sum_{x,y} \det^{-2} T_a^x T_c^y R_{xy}^{bd}(\underline{\lambda}). \qquad (5.4)$$

We have to prove that (5.4) holds in $\mathscr{F}_R(\mathrm{GL}(n))$. To show this, we multiply the RLL-relations (2.8) by $T_d^k T_b^l$ from the right and by $T_j^a T_i^c$ from the left and sum over all a, b, c, and d we get, using Corollary 4.3,

$$\sum_{a,c} T_j^a T_i^c R_{ac}^{lk}(\underline{\lambda}) \det^2 = \sum_{b,d} \det^2 R_{ji}^{bd}(\underline{\mu} + \omega(i) + \omega(j)) T_d^k T_b^l. \qquad (5.5)$$

Multiplying this equation from the left and from the right by $\det^{-2}$ gives (5.4) by the $\mathfrak{h}$-invariance of the R-matrix, so S preserves the RLL-relations.

From the proof of Lemma 5.1 it follows that $S(T)T = TS(T) = I$, where T is defined as in the proof of Lemma 5.1, so (2.6) holds for all generators t_{ij}. The proof of [20, Prop. 2.2] shows that if (2.6) holds for a and b, then it holds for ab, so that in particular (2.6)

holds for det. By Lemma 5.1 we find $S(\det)\det = 1 = \det S(\det)$, so that $S(\det) = \det^{-1}$. An independent proof of this statement is given in Proposition 5.3. With this observation it is easily proved that S also preserves the defining relations involving $\det^{-1}$, and that (2.6) holds for $\det^{-1}$.   □

The relation $S(\det) = \det^{-1}$ is the special case $I = J = \{1,\dots,n\}$ of the following proposition.

PROPOSITION 5.3. *For I and J ordered subsets with* $\#I = \#J$,

$$S(\xi_j^I) = \det^{-1}\frac{\mu_l\big(\mathrm{sign}\,(J^c;J)\big)}{\mu_r\big(\mathrm{sign}\,(I^c;I)\big)}\,\xi_{I^c}^{J^c},\tag{5.6}$$

with I^c *the complement of I in* $\{1,\dots,n\}$.

Proof. We prove this formula by induction on the $r := \#I$ using the Laplace expansions of Proposition 4.1. Another proof uses (2.6) combined with the Laplace expansions. We use a similar induction step in the proof of Lemma 6.1.

For $r = 1$ this is just the definition of the antipode on a generator t_{ij}. For the induction step we use the Laplace expansions of Proposition 4.1 several times. Let $j \in J$, then applying the Laplace expansion twice

$$S(\xi_j^I) = S\!\left(\sum_{i\in I}\frac{\mu_l(\mathrm{sign}(\{i\};I'))}{\mu_r(\mathrm{sign}(\{j\};J'))}\xi_{\{j\}}^{\{i\}}\xi_{J'}^{I'}\right) = \sum_{i\in I}S(\xi_{J'}^{I'})S(\xi_j^i)\frac{\mu_r(\mathrm{sign}(\{i\};I'))}{\mu_l(\mathrm{sign}(\{j\};J'))}$$

$$= \sum_{i\in I}\det^{-2}\frac{\mu_l(\mathrm{sign}(J'^c;J'))}{\mu_r(\mathrm{sign}(I'^c;I'))}\xi_{I'^c}^{J'^c}\frac{\mu_l(\mathrm{sign}(\hat{j};\{j\}))}{\mu_r(\mathrm{sign}(\hat{\imath};\{i\}))}\xi_{\hat{\imath}}^{\hat{j}}\frac{\mu_r(\mathrm{sign}(\{i\};I'))}{\mu_l(\mathrm{sign}(\{j\};J'))}$$

$$= \sum_{i\in I}\det^{-2}\frac{\mu_l(\mathrm{sign}\,(J^c;J'))}{\mu_r(\mathrm{sign}\,(I^c;I'))}\xi_{I'^c}^{J'^c}\frac{\mu_l(\mathrm{sign}(\hat{j};\{j\}))}{\mu_r(\mathrm{sign}(\hat{\imath};\{i\}))}\xi_{\hat{\imath}}^{\hat{j}}$$

$$= \sum_{i\in I}\sum_{\substack{k\in J'^c\\K=J'^c\setminus\{k\}}}\det^{-2}\frac{\mu_l(\mathrm{sign}\,(J^c;J'))}{\mu_r(\mathrm{sign}\,(I^c;I'))}\frac{\mu_l(\mathrm{sign}(K;\{k\}))}{\mu_r(\mathrm{sign}\,(I^c;\{i\}))}\xi_{I^c}^{K}\xi_{\{i\}}^{\{k\}}\frac{\mu_l(\mathrm{sign}(\hat{j};\{j\}))}{\mu_r(\mathrm{sign}(\hat{\imath};\{i\}))}\xi_{\hat{\imath}}^{\hat{j}},$$

$$\tag{5.7}$$

where $J' = J \setminus \{j\}$ and $I' = I \setminus \{i\}$ (so I' depends on the summation index) as ordered subsets. In this computation we use

$$\xi_{I'^c}^{J'^c}\xi_{\hat{\imath}}^{\hat{j}}\frac{\mu_r(\mathrm{sign}(\{i\};I'))}{\mu_l(\mathrm{sign}(\{j\};J'))} = \frac{\mu_r(\mathrm{sign}(\{i\};I'))}{\mu_l(\mathrm{sign}(\{j\};J'))}\xi_{I'^c}^{J'^c}\xi_{\hat{\imath}}^{\hat{j}}\tag{5.8}$$

since $\xi_{I'^c}^{J'^c}\xi_{\hat{\imath}}^{\hat{j}}$ has weight $(\underline{1}+\omega(J^c),\underline{1}+\omega(I^c))$ and $\mathrm{sign}(\{a\};B)\,\mathrm{sign}(A;B) = \mathrm{sign}(\{a\}\cup A;B)$ for all subsets A, B and all elements $a \notin A$. Since $\sum_{\substack{k\in J'^c\\K=J'^c\setminus\{k\}}}\mu_l(\mathrm{sign}(K;\{k\}))\xi_{I^c}^{K}\xi_{\{i\}}^{\{k\}} = 0$ for

all $i \in I^c$ and $\text{sign}(\hat{\imath}; \{i\}) = \text{sign}(I'; \{i\})\, \text{sign}(I^c; \{i\})$ we obtain, using the Laplace expansion once more for the summation over i where the only nonzero term is for $k = j$,

$$
S(\xi_j^I) = \sum_{\substack{k \in J'^c \\ K = J'^c \setminus \{k\}}} \det^{-2} \frac{\mu_l\big(\text{sign}\,(J^c; J')\big)}{\mu_r\big(\text{sign}\,(I^c; I)\big)} \mu_l\big(\text{sign}(K; \{k\})\big) \xi_{I^c}^K \sum_{i=1}^n \xi_{\{i\}}^{\{k\}} \frac{\mu_l\big(\text{sign}(\hat{\jmath}; \{j\})\big)}{\mu_r\big(\text{sign}(\hat{\imath}; \{i\})\big)} \xi_{\hat{\imath}}^{\hat{\jmath}}
$$

$$
= \det^{-2} \frac{\mu_l\big(\text{sign}\,(J^c; J')\big)}{\mu_r\big(\text{sign}\,(I^c; I)\big)} \mu_l\big(\text{sign}\,(J^c; \{j\})\big) \xi_{I^c}^{J^c} \det = \det^{-1} \frac{\mu_l\big(\text{sign}\,(J^c; J)\big)}{\mu_r\big(\text{sign}\,(I^c; I)\big)} \xi_{I^c}^{J^c},
$$

$$(5.9)$$

which proves the proposition. $\qquad\square$

COROLLARY 5.4.

$$
S^2(\xi_j^I) = \frac{\prod_{m \in I, k \in I^c} h(\lambda_m - \lambda_k)}{\prod_{m \in J, k \in J^c} h(\mu_m - \mu_k)} \xi_j^I.
$$

$$(5.10)$$

In particular, S is invertible.

6. The dynamical $U(n)$ quantum group

In this section we prove the existence of a $*$-operator on $\mathscr{F}_R(\text{GL}(n))$, such that it becomes an $\mathfrak{h}$-Hopf $*$-algebroid. Equipped with this $*$-structure we denote the $\mathfrak{h}$-Hopf $*$-algebroid by $\mathscr{F}_R(U(n))$.

LEMMA 6.1. *The $*$-operator defined on the generators by*

$$
t_{ij}^* = \xi_j^{\hat{\imath}} \det^{-1}, \qquad \mu_l(f)^* = \mu_l(\overline{f}), \qquad \mu_r(f) = \mu_r(\overline{f}), \quad (\det^{-1})^* = \det, \qquad (6.1)
$$

and extended as $\mathbb{C}$-antilinear algebra anti-homomorphism is well defined on $\mathscr{F}_R(\text{GL}(n))$.

Proof. Let I and J be ordered subsets of $\{1, \ldots, n\}$, such that $\#I = \#J = r$. Denote by I^c the complement of I in $\{1, \ldots, n\}$, then we have

$$
\big(\xi_j^I\big)^* = \xi_{J^c}^{I^c} \det^{-1}. \qquad (6.2)
$$

From this result and Lemma 5.1 it directly follows that $*$ is an involution. The proof of (6.2) is analogous to the corresponding statement (5.6) for the antipode.

We prove that $*$ preserves the RLL-relations by using that the antipode does so. By definition of S and $*$ it follows that

$$
\mu_r\big(\text{sign}(\hat{k}; \{k\})\big) S(t_{kj}) = \mu_l\big(\text{sign}(\hat{\jmath}; \{j\})\big) t_{jk}^*. \qquad (6.3)
$$

Applying $*$ to the RLL-relations (2.8) we get

$$\sum_{x,y=1}^{n} \frac{\mu_r(\text{sign}(\hat{d};\{d\}))}{\mu_l(\text{sign}(\hat{y};\{y\}))} T_d^y \frac{\mu_r(\text{sign}(\hat{b};\{b\}))}{\mu_l(\text{sign}(\hat{x};\{x\}))} T_b^x \mu_l(R_{ac}^{xy})$$

$$= \sum_{x,y=1}^{n} \frac{\mu_r(\text{sign}(\hat{x};\{x\}))}{\mu_l(\text{sign}(\hat{a};\{a\}))} T_x^a \frac{\mu_r(\text{sign}(\hat{y};\{y\}))}{\mu_l(\text{sign}(\hat{c};\{c\}))} T_y^c \mu_r(R_{xy}^{bd}), \tag{6.4}$$

which is equivalent to

$$\sum_{x,y=1}^{n} \frac{\mu_l(\text{sign}(\hat{a};\{a\}))}{\mu_l(\text{sign}(\hat{y};\{y\}))} \frac{\mu_l(T_{\omega(a)} \text{sign}(\hat{c};\{c\}))}{\mu_l(T_{\omega(y)} \text{sign}(\hat{x};\{x\}))} T_d^y T_b^x \mu_l(R_{ac}^{xy})$$

$$= \sum_{x,y=1}^{n} \frac{\mu_r(\text{sign}(\hat{x};\{x\}))}{\mu_r(\text{sign}(\hat{d};\{d\}))} \frac{\mu_r(T_{\omega(x)} \text{sign}(\hat{y};\{y\}))}{\mu_r(T_{\omega(d)} \text{sign}(\hat{b};\{b\}))} T_x^a T_y^c \mu_r(R_{xy}^{bd}). \tag{6.5}$$

Using (5.3), $*$ preserves the RLL-relations if

$$R_{db}^{yx}(\underline{\mu}) = R_{xy}^{bd}(\underline{\mu}) \frac{\text{sign}(\hat{x};\{x\})(\underline{\mu}-\omega(x)-\omega(y))}{\text{sign}(\hat{d};\{d\})(\underline{\mu}-\omega(x)-\omega(y))} \frac{\text{sign}(\hat{y};\{y\})(\underline{\mu}-\omega(y))}{\text{sign}(\hat{b};\{b\})(\underline{\mu}-\omega(b))}. \tag{6.6}$$

This follows by direct calculations using the explicit expression of R and the fact that $\text{sign}(\hat{x};x)$ is independent of μ_y for all $y < x$, where the only nontrivial cases are for $x = y = b = d$, $x = b, y = d$ and $x = d, y = b$. Using $\det^* = \det^{-1}$ which follows from (6.2), it directly follows that $*$ preserves the other commutation relations. $\qquad\square$

PROPOSITION 6.2. *Denote $\mathscr{F}_R(\text{GL}(n))$ equipped with the $*$-operator of Lemma 6.1 by $\mathscr{F}_R(U(n))$, then $\mathscr{F}_R(U(n))$ is an $\mathfrak{h}$-Hopf $*$-algebroid.*

Proof. From the definition of $*$ and Corollary 3.8 it follows that $(* \otimes *)\Delta(t_{ij}) = \Delta(t_{ij}^*)$ and $(\varepsilon \circ *)(t_{ij}) = (*^{D_{\mathfrak{h}}*} \circ \varepsilon)(t_{ij})$,

$$(* \otimes *)\Delta(\det^{-1}) = \det \otimes \det = \Delta((\det^{-1})^*),$$

$$(\varepsilon \circ *)(\det^{-1}) = T_{-\underline{1}} = (T_{\underline{1}})^* = \varepsilon(\det^{-1})^*. \tag{6.7}$$

So the relations $(* \otimes *) \circ \Delta = \Delta \circ *$ and $\varepsilon \circ * = *^{D_{\mathfrak{h}}*} \circ \varepsilon$ hold on the generators of $\mathscr{F}_R(\text{GL}(n))$ and hence on all of $\mathscr{F}_R(\text{GL}(n))$. $\qquad\square$

From (5.6) and (6.2) it directly follows that

$$S(\xi_J^I)^* = \xi_I^J \frac{\mu_l(\text{sign}(J^c;J))}{\mu_r(\text{sign}(I^c;I))}, \qquad S((\xi_J^I)^*) = \frac{\mu_l(\text{sign}(J;J^c))}{\mu_r(\text{sign}(I;I^c))} \xi_I^J, \tag{6.8}$$

which gives an indication for the unitarisability of the corepresentations R and L of $\mathscr{F}_R(\text{GL}(n))$ defined in Propositions 3.7 and 3.11; for the definition of unitarisability see [21, Section 5].

PROPOSITION 6.3. *The corepresentations R and L are unitarisable corepresentations of $\mathscr{F}_R(U(n))$.*

Proof. We have to define a form $\langle \cdot, \cdot \rangle : W \times W \to M_{\mathfrak{h}^*}$ and check that $\langle R(x), R(y) \rangle = \mu_r(\langle x, y \rangle \mathbf{1})$ for all $x, y \in W$; see [21, Section 5]. It is sufficient to do this for basis elements $\{w_I\}$ of W. Define $\langle w_I, w_J \rangle(\underline{\lambda}) = \delta_{IJ} \operatorname{sign}(I^c; I)(\underline{\lambda} - \omega(I)) \in M_{\mathfrak{h}^*}$, so $\langle w_I, w_J \rangle_D = \delta_{IJ} \operatorname{sign}(I^c; I) \in D_{\mathfrak{h}^*}$. Then

$$
\begin{aligned}
\langle R(w_I), R(w_J) \rangle &= \left\langle \sum_{\#K = \#J} w_K \otimes \xi_J^K, \sum_{\#M = \#J} w_M \otimes \xi_J^M \right\rangle = \sum_{K,M} \langle w_K, w_M \rangle_D \otimes (\xi_J^M)^* \xi_J^K \\
&= \sum_K \mu_l(\operatorname{sign}(K^c; K)) \frac{\mu_r(\operatorname{sign}(J^c; J))}{\mu_l(\operatorname{sign}(K^c; K))} S(\xi_K^J) \xi_I^K = \mu_r(\operatorname{sign}(J^c; J)) \delta_{IJ} \\
&= \mu_r(\langle w_I, w_J \rangle_D \mathbf{1}),
\end{aligned}
\tag{6.9}
$$

using (6.8) and (2.6) on ξ_I^J. Define a form on V by $\langle v_I, v_J \rangle = \delta_{IJ} \operatorname{sign}(I; I^c)^{-1} \in M_{\mathfrak{h}^*}$. By a similar computation it follows that $\langle L(v_I), L(v_J) \rangle = \mu_l(\langle v_I, v_J \rangle_D \mathbf{1})$. $\qquad\square$

Remark 6.4. The above discussion strongly suggests that there are analogues of the dynamical $\mathrm{SL}(n)$ and $\mathrm{SU}(n)$ quantum groups. We refer to [37] for details.

7. A pairing on the dynamical $U(n)$ quantum group

In this section we discuss pairings for the dynamical $\mathrm{GL}(n)$ quantum group and we present a cobraiding on $\mathscr{F}_R(\mathrm{GL}(n))$. For a pairing for $\mathscr{F}_R(\mathrm{GL}(n))^{\mathrm{cop}}$ and $\mathscr{F}_R(\mathrm{GL}(n))$ as $\mathfrak{h}$-Hopf $*$-algebroids, we need a second $*$-operator on $\mathscr{F}_R(\mathrm{GL}(n))$.

7.1. Pairing for $\mathfrak{h}$-Hopf $*$-algebroids.
We start by recalling the definition of a pairing for $\mathfrak{h}$-Hopf $*$-algebroids.

Definition 7.1. A pairing for $\mathfrak{h}$-bialgebroids $\mathscr{U}$ and $\mathscr{A}$ is a $\mathbb{C}$-bilinear map $\langle \cdot, \cdot \rangle : \mathscr{U} \times \mathscr{A} \to D_{\mathfrak{h}^*}$ satisfying

$$
\langle \mathscr{U}_{\alpha\beta}, \mathscr{A}_{\gamma\delta} \rangle \subseteq (D_{\mathfrak{h}^*})_{\alpha+\delta, \beta+\gamma}, \tag{7.1a}
$$

$$
\langle \mu_l^{\mathscr{U}}(f)X, a \rangle = \langle X, \mu_l^{\mathscr{A}}(f)a \rangle = f \circ \langle X, a \rangle, \qquad \langle X\mu_r^{\mathscr{U}}(f), a \rangle = \langle X, a\mu_r^{\mathscr{A}}(f) \rangle = \langle X, a \rangle \circ f, \tag{7.1b}
$$

$$
\langle XY, a \rangle = \sum_{(a)} \langle X, a_{(1)} \rangle T_\rho \langle Y, a_{(2)} \rangle, \qquad \Delta^{\mathscr{A}}(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}, a_{(1)} \in \mathscr{A}_{\gamma\rho}, \tag{7.1c}
$$

$$
\langle X, ab \rangle = \sum_{(X)} \langle X_{(1)}, a \rangle T_\rho \langle X_{(2)}, b \rangle, \qquad \Delta^{\mathscr{U}}(X) = \sum_{(X)} X_{(1)} \otimes X_{(2)}, X_{(1)} \in \mathscr{U}_{\alpha\rho}, \tag{7.1d}
$$

$$
\langle X, 1 \rangle = \varepsilon^{\mathscr{U}}(X), \qquad \langle 1, a \rangle = \varepsilon^{\mathscr{A}}(a), \tag{7.1e}
$$

for all $X \in \mathcal{U}$, $a \in \mathcal{A}$. If moreover, $\mathcal{U}$ and $\mathcal{A}$ are $\mathfrak{h}$-Hopf algebroids, then, in addition, the pairing has to satisfy

$$\langle S^{\mathcal{U}}(X), a \rangle = S^{D_{\mathfrak{h}^*}}(\langle X, S^{\mathcal{A}}(a) \rangle), \quad \forall X \in \mathcal{U}, \, a \in \mathcal{A}. \tag{7.2}$$

If in addition a $*$-operator is defined on $\mathcal{U}$ and $\mathcal{A}$ such that

$$\langle X^*, a \rangle = T_{-\gamma} \circ (\langle X, S^{\mathcal{A}}(a)^* \rangle)^* \circ T_{-\delta}, \quad \forall a \in \mathcal{A}_{\gamma\delta}, \, X \in \mathcal{U}, \tag{7.3}$$

then $\mathcal{U}$ and $\mathcal{A}$ are paired as $\mathfrak{h}$-Hopf $*$-algebroids.

Remark 7.2. Note that (7.1a) implies that $\langle X, a \rangle = 0$ whenever $X \in \mathcal{U}_{\alpha\beta}$, $a \in \mathcal{A}_{\gamma\delta}$ with $\alpha + \delta \neq \beta + \gamma$.

A cobraiding on an $\mathfrak{h}$-bialgebroid $\mathcal{A}$ is a pairing $\langle \cdot, \cdot \rangle : \mathcal{A}^{\mathrm{cop}} \times \mathcal{A} \to D_{\mathfrak{h}^*}$ which in addition satisfies

$$\sum_{(a),(b)} \mu_l^{\mathcal{A}}(\langle a_{(1)}, b_{(1)} \rangle \mathbf{1}) a_{(2)} b_{(2)} = \sum_{(a),(b)} \mu_r^{\mathcal{A}}(\langle a_{(2)}, b_{(2)} \rangle \mathbf{1}) b_{(1)} a_{(1)}, \tag{7.4}$$

as an identity in $\mathcal{A}$ and where $\Delta^{\mathcal{A}}(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$, $\Delta^{\mathcal{A}}(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$. In [34], Rosengren proved that for an $\mathfrak{h}$-bialgebroid constructed by the generalized FRST-construction from an R-matrix, denoted by R, that satisfies the quantum dynamical Yang-Baxter (2.1) there exists a natural cobraiding defined on the generators by

$$\langle L_{ij}, L_{kl} \rangle = R_{ik}^{jl}(\lambda) T_{-\omega(i) - \omega(k)}. \tag{7.5}$$

Note that this is the dynamical analogue of the cobraiding for quantum groups; see, for example, [17, Section VIII.6].

In [21] we proved the following proposition, which we now extend to the level of $\mathfrak{h}$-(co)module algebras. By $\mathcal{A}^{\mathrm{lr}}$ we denote the $\mathfrak{h}$-algebra obtained from an $\mathfrak{h}$-algebra $\mathcal{A}$ by interchanging the moment maps and with weight spaces $(\mathcal{A}^{\mathrm{lr}})_{\alpha\beta} = \mathcal{A}_{\beta\alpha}$.

PROPOSITION 7.3. *Let $\mathcal{U}$ be an $\mathfrak{h}$-algebra and let $\mathcal{A}$ be $\mathfrak{h}$-coalgebroid equipped with a pairing $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{A} \to D_{\mathfrak{h}^*}$, and let V be an $\mathfrak{h}$-space. (i) Let $R : V \to V \widetilde{\otimes} \mathcal{A}$ be a right corepresentation of the $\mathfrak{h}$-coalgebroid $\mathcal{A}$, then $\pi(X)v = (\mathrm{Id} \otimes \langle X, \cdot \rangle T_\beta) R(v)$ for $X \in \mathcal{U}_{\alpha\beta}$ defines an $\mathfrak{h}$-algebra homomorphism $\pi : \mathcal{U} \to (D_{\mathfrak{h}^*, V})^{\mathrm{lr}}$, hence $\pi : \mathcal{U}^{\mathrm{lr}} \to D_{\mathfrak{h}^*, V}$ defines a dynamical representation of $\mathcal{U}^{\mathrm{lr}}$ on V.*

(ii) Let $L : V \to \mathcal{A} \widetilde{\otimes} V$ be a left corepresentation of the $\mathfrak{h}$-coalgebroid $\mathcal{A}$, then $\pi(X)v = (T_\alpha \langle X, \cdot \rangle \otimes \mathrm{Id}) L(v)$ for $X \in \mathcal{U}_{\alpha\beta}$ defines an $\mathfrak{h}$-algebra homomorphism $\pi : \mathcal{U}^{\mathrm{opp}} \to (D_{\mathfrak{h}^*, V})^{\mathrm{lr}}$. In particular, $\pi : (\mathcal{U}^{\mathrm{opp}})^{\mathrm{lr}} \to D_{\mathfrak{h}^*, V}$ defines a dynamical representation of $(\mathcal{U}^{\mathrm{opp}})^{\mathrm{lr}}$ on V. Moreover, if $\mathcal{U}$ is $\mathfrak{h}$-Hopf algebroid, then $X \mapsto \pi(S^{\mathcal{U}}(X))$ defines a dynamical representation of $\mathcal{U}$ on V.

We now extend this result to the level of $\mathfrak{h}$-comodule algebras.

Definition 7.4. Let $\mathcal{A}$ be an $\mathfrak{h}$-bialgebroid and V an $\mathfrak{h}$-space. We call V an $\mathfrak{h}$-module algebra for $\mathcal{A}$ if there exists a dynamical representation $\pi : \mathcal{A} \to D_{\mathfrak{h}^*, V}$ such that

(i) V is an associative algebra such that $\mu_V(f)vw = v\mu_V(T_\alpha f)w$ for all $v \in V_\alpha$, $w \in V$, and $V_\alpha V_\beta \subset V_{\alpha+\beta}$,

(ii) $\pi(a)vw = \sum_{(a)}(\pi(a_{(1)})v)(\pi(a_{(2)})w)$, for all $v, w \in V$ and $X \in \mathcal{A}$ with $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$.

Moreover, if V is unital, then $\pi(a)1 = \mu_V(\varepsilon(a)\mathbf{1})$.

PROPOSITION 7.5. *Let $\mathcal{U}$ and $\mathcal{A}$ be paired as $\mathfrak{h}$-bialgebroids. Let V be a right (left) $\mathfrak{h}$-comodule algebra for $\mathcal{A}$, then π as defined in Proposition 7.3 defines an $\mathfrak{h}$-module algebra for $\mathcal{U}^{\mathrm{lr}}$ $((\mathcal{U}^{\mathrm{opp}})^{\mathrm{lr}})$.*

Proof. We prove the proposition in the case that V is a right $\mathfrak{h}$-comodule algebra, the other statement can be proved analogously. Since V is an $\mathfrak{h}$-comodule algebra, Definition 7.4(i) is satisfied. By Proposition 7.3, $\pi(X)v = (\mathrm{Id} \otimes \langle X, \cdot \rangle T_\beta)R(v), X \in \mathcal{U}_{\alpha\beta}$ is an $\mathfrak{h}$-algebra homomorphism of $\mathcal{U}$ to $(D_{\mathfrak{h}^*, V})^{\mathrm{lr}}$. Then, since R is an algebra homomorphism, we have

$$
\begin{aligned}
\pi(X)vw &= \left(\mathrm{Id} \otimes \langle X, \cdot \rangle T_\beta\right)R(vw) = \sum v_{(1)}w_{(1)} \otimes \langle X, a_{(2)}b_{(2)} \rangle T_\beta \\
&= \sum v_{(1)}w_{(1)} \otimes \langle X_{(1)}, a_{(2)} \rangle T_\gamma \langle X_{(2)}, b_{(2)} \rangle T_\beta = \sum (\pi(X_{(1)})v)(\pi(X_{(2)})w),
\end{aligned}
\tag{7.6}
$$

for $X \in \mathcal{U}_{\alpha\beta}$, $\Delta(X) = \sum_{(X)} X_{(1)} \otimes X_{(2)}$, $X_{(1)} \in \mathcal{U}_{\alpha\gamma}$ and with the notation $R(v) = \sum v_{(1)} \otimes a_{(2)}$, $R(w) = \sum w_{(1)} \otimes b_{(2)}$. So π defines an $\mathfrak{h}$-module algebra for $\mathcal{U}^{\mathrm{lr}}$.

If V is unital, then $\pi(X)1 = 1 \otimes \langle X, 1 \rangle T_\beta = \mu_V(\varepsilon(X)\mathbf{1})$ for $X \in \mathcal{U}_{\alpha\beta}$. $\qquad\square$

7.2. A pairing on the dynamical $\mathrm{GL}(n)$ quantum group. A natural cobraiding on the algebra $\mathscr{F}_R(M(n))$ is given by (7.5). For this pairing we have $\langle t_{ij}, \det \rangle = \delta_{ij}qT_{-\underline{1}-\omega(i)}$. For normalization purposes we multiply the pairing of two generators with a factor $q^{-1/n}$. So we use the pairing $\langle \cdot, \cdot \rangle : \mathscr{F}_R(M(n))^{\mathrm{cop}} \times \mathscr{F}_R(M(n)) \to D_{\mathfrak{h}^*}$ defined on the generators t_{ij} by

$$
\langle t_{ij}, t_{kl} \rangle = q^{-1/n}R_{ik}^{jl}(\underline{\lambda})T_{-\omega(i)-\omega(k)}.
\tag{7.7}
$$

Note that switching R to $q^{-1/n}R$ is a gauge transform, which does not affect the RLL-relations. The nontrivial cases for this pairing on the level of the generators are explicitly given by

$$
\begin{aligned}
\langle t_{ii}, t_{ii} \rangle &= q^{1-1/n}T_{-2\omega(i)}, \quad \forall i, \\
\langle t_{ii}, t_{jj} \rangle &= q^{-1/n}T_{-\omega(i)-\omega(j)}, \quad \forall i < j, \\
\langle t_{jj}, t_{ii} \rangle &= q^{-1/n}g(\lambda_i - \lambda_j)T_{-\omega(i)-\omega(j)}, \quad \forall i < j, \\
\langle t_{ji}, t_{ij} \rangle &= q^{-1/n}h_0(\lambda_i - \lambda_j)T_{-\omega(i)-\omega(j)}, \quad \forall i \neq j.
\end{aligned}
\tag{7.8}
$$

In this section we prove that this pairing can be extended to the level of $\mathfrak{h}$-Hopf $*$-algebroids.

In order to extend the pairing to a cobraiding on $\mathscr{F}_R(\mathrm{GL}(n))$ we need to compute the pairing of a generator t_{ij} with the determinant element. Denote by $\underline{1}$ the vector with all 1's.

LEMMA 7.6. *For the pairing* $\langle\cdot,\cdot\rangle : \mathscr{F}_R(M(n))^{\mathrm{cop}} \times \mathscr{F}_R(M(n)) \to D_{\mathfrak{h}^*}$ *defined by (7.7),*

$$\langle t_{ij},\det\rangle = \delta_{ij} T_{-\underline{1}-\omega(i)},$$

$$\langle\det,t_{ij}\rangle = \delta_{ij} T_{-\underline{1}-\omega(i)}, \qquad (7.9)$$

$$\langle\det,\det\rangle = T_{-2\cdot\underline{1}}.$$

Proof. From Remark 7.2 it immediately follows that $\langle\det,t_{ij}\rangle = \langle t_{ij},\det\rangle = 0$ for $i \neq j$. Using the pairing (7.7) on $\mathscr{F}_R(M(n))$, Propositions 3.7 and 7.5 show that $\pi : (\mathscr{F}_R(M_n)^{\mathrm{cop}})^{\mathrm{lr}} \to (D_{\mathfrak{h}^*}, W)$ gives W an $\mathfrak{h}$-module algebra structure for $(\mathscr{F}_R(M(n))^{\mathrm{cop}})^{\mathrm{lr}}$. Then we have

$$\pi(t_{ii})(w_1 \cdots w_n) = w_1 \cdots w_n \otimes \langle t_{ii},\det\rangle T_{\omega(i)}. \qquad (7.10)$$

Also we compute

$$
\begin{aligned}
\pi(t_{ii}) w_1 \cdots w_n &= \pi(t_{ii})\left(\prod_{k<i} -h(\lambda_i - \lambda_k)^{-1} w_i w_{\hat{\imath}}\right) \\
&= \left(T_{-\omega(i)}\prod_{k<i} -h(\lambda_i - \lambda_k)^{-1}\right)\pi(t_{ii})[w_i w_{\hat{\imath}}] \\
&= \prod_{k<i} -h(\lambda_i - 1 - \lambda_k)^{-1} \sum_{k_1,\ldots,k_n}\sum_{j_1,\ldots,j_{n-1}} w_{k_1} w_{k_2}\cdots w_{k_n} \\
&\quad \otimes \langle t_{j_1 i}, t_{k_1 i}\rangle T_{\omega(j_1)}\langle t_{j_2 j_1}, t_{k_2 1}\rangle \cdots T_{\omega(j_{n-1})}\langle t_{ij_{n-1}}, t_{k_n n}\rangle T_{\omega(i)},
\end{aligned} \qquad (7.11)
$$

using the $\mathfrak{h}$-module algebra structure of W in the third equation. From (7.8) it follows that $\langle t_{j_1 i}, t_{k_1 i}\rangle \neq 0$ only if $j_1 = k_1 = i$. So we get

$$
\begin{aligned}
\prod_{k<i} -h(\lambda_i - 1 - \lambda_k)^{-1} &\sum_{k_2,\ldots,k_n}\sum_{j_2,\ldots,j_{n-1}} w_i w_{k_2}\cdots w_{k_n} \\
&\otimes \langle t_{ii}, t_{ii}\rangle T_{\omega(j_1)}\langle t_{j_2 j_1}, t_{k_2 1}\rangle \cdots T_{\omega(j_{n-1})}\langle t_{ij_{n-1}}, t_{k_n n}\rangle T_{\omega(i)}.
\end{aligned} \qquad (7.12)
$$

Now $\langle t_{j_2 i}, t_{k_2 1}\rangle \neq 0$ only if $k_2 = i$, $j_2 = 1$ or $k_2 = 1$, $j_2 = i$. In the first case, the first leg of the tensor product is equal to 0, so $k_2 = 1$, $j_2 = i$. Continuing in this way and recalling that we have pulled the term corresponding to w_i to the left, we obtain that there is only a nonzero contribution for $j_m = i$ for all m and $k_1 = i$, $k_m = m - 1$ for $2 \leq m \leq i$ and $k_m = m$

for $m > i$. So we get

$$\pi(t_{ii})w_1 \cdots w_n = \prod_{k<i} -h(\lambda_i - 1 - \lambda_k)^{-1} w_i w_{\hat{\imath}} \otimes \langle t_{ii}, t_{ii} \rangle T_{\omega(i)} \langle t_{ii}, t_{11} \rangle T_{\omega(i)} \cdots \langle t_{ii}, t_{nn} \rangle T_{\omega(i)}$$

$$= \prod_{k<i} -h(\lambda_i - 1 - \lambda_k)^{-1} \prod_{k<i} -h(\lambda_i - \lambda_k) w_1 \cdots w_n$$

$$\otimes q^{1-1/n} T_{-\omega(i)} \prod_{k<i} q^{-1/n} g(\lambda_k - \lambda_i) T_{-\omega(k)} \prod_{k>i} q^{-1/n} T_{-\omega(k)}$$

$$= \prod_{k<i} \frac{h(\lambda_i - \lambda_k)}{h(\lambda_i - \lambda_k - 1)} \prod_{k<i} g(\lambda_i - \lambda_k - 1) w_1 \cdots w_n = w_1 \cdots w_n,$$

(7.13)

where the last equality follows from (2.15). So $\langle t_{ii}, \det \rangle = T_{-\underline{1} - \omega(i)}$.

Note that $\mathscr{F}_R(M(n))^{\mathrm{cop}}$ can also be seen as an $\mathfrak{h}$-bialgebroid constructed from the R-matrix $\tilde{R}$ with matrix elements $\tilde{R}^{cd}_{ab} = R^{ba}_{dc}$ by the generalized FRST-construction. Following the lines of the proofs of Section 3 we can prove that V is a right $\mathfrak{h}$-comodule algebra for $\mathscr{F}_R(M(n))^{\mathrm{cop}}$. By inspection it follows that the matrix elements τ^I_J of this corepresentation R^{cop}, defined by $R^{\mathrm{cop}}(v_I) = \sum_J v_J \otimes \tau^I_J$, are equal to ξ^I_J. From Proposition 7.5 it follows that $\pi : \mathscr{F}_R(M(n)) \to D_{\mathfrak{h}^*, V}$ defined by $\pi(a)v = (\mathrm{Id} \otimes \langle \cdot, a \rangle T_\beta) R^{\mathrm{cop}}(v)$ for $a \in \mathscr{F}_R(M_n)_{\alpha\beta}$ and $v \in V$ gives V the structure of an $\mathfrak{h}$-module algebra for $\mathscr{F}_R(M(n))$. Now analogously to the proof of the first part of this lemma we get $\langle \det, t_{ii} \rangle = T_{-\underline{1} - \omega(i)}$.

Using Lemma 7.6, $\pi(t_{ij})w_1 \cdots w_n = 0$ if $i \neq j$ and the explicit expression of det we get

$$\pi(\det)w_1 \cdots w_n = \pi(t_{11} t_{22} \cdots t_{nn}) w_1 \cdots w_n = \pi(t_{11}) \cdots \pi(t_{nn}) w_1 \cdots w_n = w_1 \cdots w_n.$$

(7.14)

Also $\pi(\det)w_1 \cdots w_n = w_1 \cdots w_n \otimes \langle \det, \det \rangle T_{-\underline{1}}$, so $\langle \det, \det \rangle = T_{-2 \cdot \underline{1}}$. $\qquad \square$

LEMMA 7.7. *Define the pairing* $\langle \cdot, \cdot \rangle : \mathscr{F}_R(\mathrm{GL}(n))^{\mathrm{cop}} \times \mathscr{F}_R(\mathrm{GL}(n)) \to D_{\mathfrak{h}^*}$ *on the generators of* $\mathscr{F}_R(\mathrm{GL}(n))$ *by (7.8) and*

$$\langle \det^{-1}, t_{ij} \rangle = \delta_{ij} T_{\omega(\hat{\imath})}, \qquad \langle t_{ij}, \det^{-1} \rangle = \delta_{ij} T_{\omega(\hat{\imath})}, \qquad \langle \det^{-1}, \det^{-1} \rangle = T_{2 \cdot \underline{1}}. \quad (7.15)$$

Then $\mathscr{F}_R(\mathrm{GL}(n))^{\mathrm{cop}}$ *and* $\mathscr{F}_R(\mathrm{GL}(n))$ *are paired as* $\mathfrak{h}$-*bialgebroids.*

Proof. For the pairing $\langle \cdot, \cdot \rangle : \mathscr{F}_R(M(n))^{\mathrm{cop}} \times \mathscr{F}_R(M(n)) \to D_{\mathfrak{h}^*}$ the statement follows from [34]. Since

$$\delta_{ij} T_{-\omega(i)} = \varepsilon(t_{ij}) = \langle t_{ij}, 1 \rangle = \langle t_{ij}, \det \det^{-1} \rangle = \sum_k \langle t_{kj}, \det \rangle T_{\omega(k)} \langle t_{ik}, \det^{-1} \rangle = T_{-\underline{1}} \langle t_{ij}, \det^{-1} \rangle,$$

(7.16)

the pairing is also well defined for $\mathscr{F}_R(\mathrm{GL}(n))$. $\qquad \square$

We want to extend Lemma 7.7 and show that the pairing exists on the level of $\mathfrak{h}$-Hopf algebroids. For this we need to calculate pairings with dynamical quantum minor determinants because of Proposition 7.9, the proof of the following lemma follows the same strategy as the proof of Lemma 7.6.

LEMMA 7.8. *For $i \neq j$,*

$$\langle t_{ii}, \xi_{\hat{\imath}}^{\hat{\imath}} \rangle = q^{-1+1/n} \prod_{k<i} g(\lambda_k - \lambda_i) T_{-\underline{1}}, \qquad \langle t_{ii}, \xi_{\hat{\jmath}}^{\hat{\jmath}} \rangle = q^{1/n} T_{-\omega(\hat{\jmath}) - \omega(i)},$$

$$\langle t_{ij}, \xi_{\hat{\jmath}}^{\hat{\imath}} \rangle = q^{-1+1/n} h_0(\lambda_j - \lambda_i) \prod_{k<j, k\neq i} -h(\lambda_j - \lambda_k) \prod_{k<i, k\neq j} -h(\lambda_k - \lambda_i) T_{-\underline{1}},$$

$$\langle \xi_{\hat{\imath}}^{\hat{\imath}}, t_{ii} \rangle = q^{-1+1/n} \prod_{k>i} g(\lambda_i - \lambda_k) T_{-\underline{1}}, \qquad \langle \xi_{\hat{\jmath}}^{\hat{\jmath}}, t_{ii} \rangle = q^{1/n} T_{-\omega(\hat{\jmath}) - \omega(i)},$$

$$\langle \xi_{\hat{\jmath}}^{\hat{\imath}}, t_{ij} \rangle = q^{-1+1/n} h_0(\lambda_i - \lambda_j) \prod_{k>j, k\neq i} -h(\lambda_k - \lambda_j) \prod_{k>i, k\neq j} -h(\lambda_i - \lambda_k) T_{-\underline{1}}.$$

(7.17)

All other pairings between generators t_{ij} and dynamical quantum minor determinants $\xi_{\hat{l}}^{\hat{k}}$ are zero.

PROPOSITION 7.9. *$\mathscr{F}_R(\mathrm{GL}(n))^{\mathrm{cop}}$ and $\mathscr{F}_R(\mathrm{GL}(n))$ are paired as $\mathfrak{h}$-Hopf algebroids.*

Proof. In Lemma 7.7 we proved that $\mathscr{F}_R(\mathrm{GL}(n))^{\mathrm{cop}}$ and $\mathscr{F}_R(\mathrm{GL}(n))$ are paired as $\mathfrak{h}$-bialgebroids. So it remains to check (7.2) on generators. The only nontrivial cases are $(X, a) = (t_{ii}, t_{ii})$, (t_{jj}, t_{ii}), (t_{ji}, t_{ij}), $(t_{ii}, \det^{-1})$ and $(\det^{-1}, t_{ii})$ for $i \neq j$. From Example 2.5(ii) we know $S^{\mathrm{cop}} = S^{-1}$ and $\mu_l^{\mathrm{cop}} = \mu_r$, $\mu_r^{\mathrm{cop}} = \mu_l$, so

$$S^{\mathrm{cop}}(t_{ij}) = \det^{-1} \xi_{\hat{\imath}}^{\hat{\jmath}} \frac{\mu_r^{\mathrm{cop}}(\mathrm{sign}(j; \hat{\jmath}))}{\mu_l^{\mathrm{cop}}(\mathrm{sign}(i; \hat{\imath}))}.$$

(7.18)

Now we can check the relations by direct computations, using Lemma 7.8. We show the third relation in detail; the other cases can be done analogously. Using Lemma 7.8,

$$\langle S^{\mathrm{cop}}(t_{ji}), t_{ij} \rangle = \left\langle \det^{-1} \xi_{\hat{\jmath}}^{\hat{\imath}} \frac{\mu_r^{\mathrm{cop}}(\mathrm{sign}(\{i\}; \hat{\imath}))}{\mu_l^{\mathrm{cop}}(\mathrm{sign}(\{j\}; \hat{\jmath}))}, t_{ij} \right\rangle$$

$$= \mathrm{sign}(\{j\}; \hat{\jmath})(\underline{\lambda} - \omega(\hat{\jmath}))^{-1} \langle \det^{-1}, t_{ii} \rangle T_{\omega(i)} \langle \xi_{\hat{\jmath}}^{\hat{\imath}}, t_{ij} \rangle \, \mathrm{sign}(\{i\}; \hat{\imath})(\underline{\lambda})$$

$$= q^{-1+1/n} h_0(\lambda_i - \lambda_j) h(\lambda_i - \lambda_j) \prod_{m \neq i,j} -h(\lambda_m - \lambda_j) \prod_{m \neq i,j} -h(\lambda_i - \lambda_m),$$

$$S(\langle t_{ji}, S(t_{ij}) \rangle) = S\left(\left\langle t_{ij}, \det^{-1} \frac{\mu_l(\mathrm{sign}(\hat{\jmath}; \{j\}))}{\mu_r(\mathrm{sign}(\hat{\imath}; \{i\}))} \xi_{\hat{\imath}}^{\hat{\jmath}} \right\rangle \right)$$

$$= \mathrm{sign}(\hat{\jmath}; \{j\})(\underline{\lambda}) \langle t_{ii}, \det^{-1} \rangle T_{\omega(i)} \langle t_{ji}, \xi_{\hat{\imath}}^{\hat{\jmath}} \rangle \, \mathrm{sign}(\hat{\imath}; \{i\})(\underline{\lambda} + \omega(\hat{\imath}))^{-1}$$

$$= q^{-1+1/n} h_0(\lambda_i - \lambda_j) h(\lambda_i - \lambda_j) \prod_{m \neq i,j} -h(\lambda_m - \lambda_j) \prod_{m \neq i,j} -h(\lambda_i - \lambda_m),$$

(7.19)

so $\langle S^{\mathrm{cop}}(t_{ji}), t_{ij} \rangle = S(\langle t_{ji}, S(t_{ij}) \rangle)$. $\square$

7.3. Compatible $*$-structures for the pairing.

If we equip $\mathscr{F}_R(\mathrm{GL}(n))^{\mathrm{cop}}$ and $\mathscr{F}_R(\mathrm{GL}(n))$ with the $*$-operator defined in Lemma 6.1, they are not paired as $\mathfrak{h}$-Hopf $*$-algebroids.

But since the $*$-operator is not unique it is possible that there exists another $*$-operator which gives paired $\mathfrak{h}$-Hopf $*$-algebroids.

LEMMA 7.10. *The $\mathfrak{h}$-Hopf algebroid $\mathcal{F}_R(\mathrm{GL}(n))$ has a $*$-operator, denoted by $\dagger$, defined on the generators by $\mu_l(f)^\dagger = \mu_l(\bar f)$, $\mu_r(f)^\dagger = \mu_r(\bar f)$, and*

$$t_{ij}^\dagger = \frac{\mu_l(s_i)}{\mu_r(s_j)}\xi_j^i\det{}^{-1}, \quad (\det{}^{-1})^\dagger = \det, \tag{7.20}$$

where $s_i(\underline\lambda) = q^{2(1/n\sum_{k=1}^n \lambda_k - \lambda_i)}$ and extended as an $\mathbb{C}$-antilinear algebra anti-homomorphism.

Proof. The proof follows the lines of the proof of Lemma 6.1. On dynamical quantum minor determinants we have

$$(\xi_J^I)^\dagger = \frac{\mu_l(s_I)}{\mu_r(s_J)}\xi_{J^c}^{I^c}\det{}^{-1}, \tag{7.21}$$

where $s_I(\lambda) = q^{2(\#I/n\sum_{k=1}^n \lambda_k - \sum_{i\in I}\lambda_i)}$. This follows using $s_{I\setminus\{i\}}s_{\{i\}} = s_I$. From the claim it follows that $\dagger$ is an involution. Indeed, since $\mu_{l/r}(s_I)\det = \det\mu_{l/r}(s_I)$ and $s_{\{1,\dots,n\}} = 1$, we have

$$(t_{ij}^\dagger)^\dagger = \left(\frac{\mu_l(s_i)}{\mu_r(s_j)}\xi_j^i\det{}^{-1}\right)^\dagger = \det\frac{\mu_l(s_{\hat i})}{\mu_r(s_{\hat j})}t_{ij}\det{}^{-1}\frac{\mu_l(s_i)}{\mu_r(s_j)} = t_{ij}. \tag{7.22}$$

Since the $*$-operator $*$ preserves the commutation relations and $(t_{ij})^\dagger = (\mu_l(s_i)/\mu_r(s_j))(t_{ij})^*$ it follows directly from $R_{ab}^{xy} = 0$ if $\omega(x) + \omega(y) \neq \omega(a) + \omega(b)$ that $\dagger$ preserves the RLL-relations. By direct computations we can check $\varepsilon \circ \dagger = *^{D_\mathfrak{h}*} \circ \varepsilon$ and $(\dagger \otimes \dagger) \circ \Delta = \Delta \circ \dagger$ on the generators and so on $\mathcal{F}_R(\mathrm{GL}(n))$. $\square$

THEOREM 7.11. *$(\mathcal{F}_R(\mathrm{GL}(n))^{\mathrm{cop}}, \dagger)$ and $(\mathcal{F}_R(\mathrm{GL}(n)), *)$ are paired as $\mathfrak{h}$-Hopf $*$-algebroids.*

Proof. From Lemma 7.10 it follows that it remains to prove (7.3) for generators. We have to check this relation for five nontrivial cases: $(X, a) = (t_{ii}, t_{ii}), (t_{ii}, t_{jj}), (t_{ij}, t_{ij}), (\det^{-1}, t_{ii})$ and $(t_{ii}, \det^{-1})$ for $i \neq j$. We give the proof of the second case, which is the most involved one, in detail; the others can be proved analogously. Since $T_{\omega(\hat i)}s_i(\underline\lambda) = s_i(\underline\lambda)q^{2(1-1/n)}$, $T_{\omega(\hat j)}s_i(\underline\lambda) = s_i(\underline\lambda)q^{-2/n}$, for $i \neq j$, we have

$$\langle t_{ii}^\dagger, t_{jj}\rangle = s_i(\underline\lambda)^{-1}\langle\xi_i^i, t_{jj}\rangle T_{\omega(j)}\langle\det{}^{-1}, t_{jj}\rangle s_i(\underline\lambda)q^{2(1-1/n)}$$
$$= s_i(\underline\lambda)^{-1}q^{1/n}T_{-\omega(\hat i)+\omega(\hat j)}s_i(\underline\lambda)q^{2(1-1/n)} = q^{-1/n}T_{\omega(i)-\omega(j)}. \tag{7.23}$$

For $i < j$ we also have

$$\langle t_{ii}, S(t_{jj})^*\rangle = \left\langle t_{ii}, t_{jj}\frac{\mu_l(\mathrm{sign}(\hat j; \{j\}))}{\mu_r(\mathrm{sign}(\hat j; \{j\}))}\right\rangle$$
$$= \mathrm{sign}(\hat j; \{j\})(\underline\lambda - \omega(j))q^{-1/n}T_{-\omega(i)-\omega(j)}\,\mathrm{sign}(\hat j; \{j\})(\underline\lambda)^{-1} \tag{7.24}$$
$$= q^{-1/n}\frac{\mathrm{sign}(\hat j; \{j\})(\underline\lambda - \omega(j))}{\mathrm{sign}(\hat j; \{j\})(\underline\lambda - \omega(i) - \omega(j))}T_{-\omega(i)-\omega(j)} = q^{-1/n}T_{-\omega(i)-\omega(j)},$$

since $\mathrm{sign}(\hat{\jmath};\{j\})$ is independent of λ_i if $i < j$. For $i > j$ we get

$$
\begin{aligned}
\langle t_{ii}, S(t_{jj})^* \rangle &= \mathrm{sign}(\hat{\jmath};\{j\})(\underline{\lambda} - \omega(j))q^{-1/n}g(\lambda_j - \lambda_i)\,T_{-\omega(i)-\omega(j)}\,\mathrm{sign}(\hat{\jmath};\{j\})(\underline{\lambda})^{-1} \\
&= q^{-1/n}g(\lambda_j - \lambda_i)\frac{\mathrm{sign}(\hat{\jmath};\{j\})(\underline{\lambda} - \omega(j))}{\mathrm{sign}(\hat{\jmath};\{j\})(\underline{\lambda} - \omega(i) - \omega(j))}T_{-\omega(i)-\omega(j)} \qquad (7.25) \\
&= q^{-1/n}g(\lambda_i - \lambda_j)\frac{h(\lambda_i - \lambda_j + 1)}{h(\lambda_i - \lambda_j)}T_{-\omega(i)-\omega(j)} = q^{-1/n}T_{-\omega(i)-\omega(j)},
\end{aligned}
$$

where we use (2.15) in the last equality. So $T_{-\omega(j)}\langle t_{ii}, S(t_{jj})^*\rangle^* T_{-\omega(j)} = q^{-1/n}T_{\omega(i)-\omega(j)}$ which proves the second case. $\qquad\square$

Remark 7.12. Instead of the relation (7.3) we can also require the pairing and $*$-operator to satisfy a similar relation where $*$ and S are interchanged in the right-hand side; see [21]. Also with that relation, the cobraiding (7.7) on the dynamical GL(n) quantum group is not a pairing on the level of $\mathfrak{h}$-Hopf algebroids with the same $*$-operator $*$ on $\mathscr{F}_R(\mathrm{GL}(n))^{\mathrm{cop}}$ and $\mathscr{F}_R(\mathrm{GL}(n))$.

Acknowledgments

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