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Letter to the Editor

Comment on "On the Carleman Classes of Vectors of a Scalar Type Spectral Operator"

Marat V. Markin

Department of Mathematics, California State University, Fresno, 5245 N. Backer Avenue, M/S PB 108, Fresno, CA 93740-8001, USA

Correspondence should be addressed to Marat V. Markin; mmarkin@csufresno.edu

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The results of three papers, in which the author inadvertently overlooks certain deficiencies in the descriptions of the Carleman classes of vectors, in particular the Gevrey classes, of a scalar type spectral operator in a complex Banach space established in "On the Carleman Classes of Vectors of a Scalar Type Spectral Operator," Int. J. Math. Math. Sci. 2004 (2004), no. 60, 3219–3235, are observed to remain true due to more recent findings.

1. Introduction

Certain deficiencies of the descriptions (established in [1]) of the Carleman classes of vectors, in particular the Gevrey classes, of a scalar type spectral operator in a complex Banach space inadvertently overlooked by the author when proving the results of the three papers [2–4] are observed not to affect the validity of the latter due to more recent findings of [5].

2. Preliminaries

For the reader's convenience, we outline in this section certain preliminaries essential for understanding.

Henceforth, unless specified otherwise, A is supposed to be a *scalar type spectral operator* in a complex Banach space $(X, \|\cdot\|)$ and $E_A(\cdot)$ is supposed to be its strongly σ-additive *spectral measure* (the *resolution of the identity*) assigning to each *Borel set* δ of the *complex plane* $\mathbb{C}$ a *projection operator* $E_A(\delta)$ on X and having the operator's *spectrum* $\sigma(A)$ as its *support* [6, 7].

Observe that, in a complex finite-dimensional space, the scalar type spectral operators are those linear operators on the space, for which there is an *eigenbasis* (see, e.g., [6, 7]) and, in a complex Hilbert space, the scalar type spectral operators are precisely those that are similar to the *normal* ones [8].

Associated with a scalar type spectral operator in a complex Banach space is the *Borel operational calculus* analogous to that for a *normal operator* in a complex Hilbert space [6, 7, 9, 10], which assigns to any Borel measurable function $F : \sigma(A) \to \mathbb{C}$ a scalar type spectral operator

$$F(A) := \int_{\sigma(A)} F(\lambda)\, dE_A(\lambda) \tag{1}$$

defined as follows:

$$\begin{gathered} F(A) f := \lim_{n\to\infty} F_n(A) f, \quad f \in D(F(A)), \\ D(F(A)) := \left\{ f \in X \mid \lim_{n\to\infty} F_n(A) f \text{ exists} \right\}, \end{gathered} \tag{2}$$

where

$$F_n(\cdot) := F(\cdot)\, \chi_{\{\lambda\in\sigma(A) \mid |F(\lambda)|\le n\}}(\cdot), \quad n \in \mathbb{N}; \tag{3}$$

$D(\cdot)$ is the *domain* of an operator, $\chi_\delta(\cdot)$ is the *characteristic function* of a set $\delta \subseteq \mathbb{C}$, and $\mathbb{N} := \{1, 2, 3, \ldots\}$ is the set of *natural numbers* and

$$F_n(A) := \int_{\sigma(A)} F_n(\lambda)\, dE_A(\lambda), \quad n \in \mathbb{N}, \tag{4}$$

are *bounded* scalar type spectral operators on X defined in the same manner as for a *normal operator* (see, e.g., [9, 10]).

In particular,

$$A^n = \int_{\sigma(A)} \lambda^n dE_A(\lambda), \quad n \in \mathbb{Z}_+; \tag{5}$$

$\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ is the set of *nonnegative integers*, $A^0 := I$, and I is the *identity operator* on X and

$$e^{zA} := \int_{\sigma(A)} e^{z\lambda} dE_A(\lambda), \quad z \in \mathbb{C}. \tag{6}$$

The properties of the *spectral measure* and *operational calculus* are exhaustively delineated in [6, 7].

For a densely defined closed linear operator $(A, D(A))$ in a (real or complex) Banach space $(X, \|\cdot\|)$, a sequence of positive numbers $\{m_n\}_{n=0}^{\infty}$, and the subspace

$$C^{\infty}(A) := \bigcap_{n=0}^{\infty} D(A^n) \tag{7}$$

of infinite differentiable vectors of A, the subspaces of $C^{\infty}(A)$,

$$\begin{aligned} C_{\{m_n\}}(A) := \{f \in C^{\infty}(A) \mid \exists \alpha > 0\ \exists c > 0 : \|A^n f\| \\ \le c\alpha^n m_n,\ n \in \mathbb{Z}_+\}, \\ C_{(m_n)}(A) := \{f \in C^{\infty}(A) \mid \forall \alpha > 0\ \exists c > 0 : \|A^n f\| \\ \le c\alpha^n m_n,\ n \in \mathbb{Z}_+\}, \end{aligned} \tag{8}$$

are called the *Carleman classes* of ultradifferentiable vectors of the operator A corresponding to the sequence $\{m_n\}_{n=0}^{\infty}$ of *Roumieu* and *Beurling type*, respectively.

The inclusions

$$C_{(m_n)}(A) \subseteq C_{\{m_n\}}(A) \subseteq C^{\infty}(A) \tag{9}$$

are obvious.

If two sequences of positive numbers $\{m_n\}_{n=0}^{\infty}$ and $\{m'_n\}_{n=0}^{\infty}$ are related as follows:

$$\forall \gamma > 0\ \exists c = c(\gamma) > 0 : m'_n \le c\gamma^n m_n, \quad n \in \mathbb{Z}_+, \tag{10}$$

we also have the inclusion

$$C_{\{m'_n\}}(A) \subseteq C_{(m_n)}(A), \tag{11}$$

with the sequences being subject to the condition

$$\exists \gamma_1, \gamma_2 > 0,\ \exists c_1, c_2 > 0 : c_1\gamma_1^n m_n \le m'_n \le c_2\gamma_2^n m_n, \quad n \in \mathbb{Z}_+; \tag{12}$$

their corresponding Carleman classes coincide:

$$\begin{aligned} C_{\{m_n\}}(A) &= C_{\{m'_n\}}(A), \\ C_{(m_n)}(A) &= C_{(m'_n)}(A). \end{aligned} \tag{13}$$

In view of the latter, by *Stirling's formula*, for $\beta \ge 0$,

$$\begin{aligned} \mathscr{E}^{\{\beta\}}(A) &:= C_{\{[n!]^{\beta}\}}(A) = C_{\{n^{\beta n}\}}(A), \\ \mathscr{E}^{(\beta)}(A) &:= C_{([n!]^{\beta})}(A) = C_{(n^{\beta n})}(A) \end{aligned} \tag{14}$$

are the well-known *Gevrey classes* of ultradifferentiable vectors of A of order β of Roumieu and Beurling type, respectively (see, e.g., [11–13]). In particular, $\mathscr{E}^{\{1\}}(A)$ and $\mathscr{E}^{(1)}(A)$ are the classes of *analytic* and *entire* vectors of A, respectively [14, 15], and $\mathscr{E}^{\{0\}}(A)$ and $\mathscr{E}^{(0)}(A)$ (i.e., the classes $C_{\{1\}}(A)$ and $C_{(1)}(A)$ corresponding to the sequence $m_n \equiv 1$) are the classes of *entire* vectors of A of *exponential* and *minimal exponential type*, respectively (see, e.g., [13, 16]).

3. Remarks

If the sequence of positive numbers $\{m_n\}_{n=0}^{\infty}$ satisfies the condition

$$\text{(WGR)}\ \forall \alpha > 0\ \exists c = c(\alpha) > 0 : c\alpha^n \le m_n, \quad n \in \mathbb{Z}_+, \tag{15}$$

which can be equivalently stated as

$$\lim_{n\to\infty} \sqrt[n]{m_n} = \infty, \tag{16}$$

the scalar function

$$T(\lambda) := m_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{m_n}, \quad \lambda \ge 0,\ \left(0^0 := 1\right), \tag{17}$$

first introduced by Mandelbrojt [17], is well defined (cf. [13]). The function is *continuous*, *strictly increasing*, and $T(0) = 1$.

As is shown in [11] (see also [12, 13]), the sequence $\{m_n\}_{n=0}^{\infty}$ satisfying the condition (WGR), for a *normal operator* A in a complex Hilbert space X, the equalities

$$\begin{aligned} C_{\{m_n\}}(A) &= \bigcup_{t>0} D(T(t|A|)), \\ C_{(m_n)}(A) &= \bigcap_{t>0} D(T(t|A|)) \end{aligned} \tag{18}$$

are true; the normal operators

$$T(t|A|) := \int_{\sigma(A)} T(t|\lambda|)\, dE_A(\lambda), \quad t > 0, \tag{19}$$

are understood in the sense of the Borel operational calculus (see, e.g., [9, 10]) and the function $T(\cdot)$ is replaceable with any *nonnegative*, *continuous*, and *increasing* on $[0, \infty)$ function $F(\cdot)$ satisfying

$$c_1 F(\gamma_1 \lambda) \le T(\lambda) \le c_2 F(\gamma_2 \lambda), \quad \lambda \ge R, \tag{20}$$

with some $\gamma_1, \gamma_2, c_1, c_2 > 0$ and $R \ge 0$, in particular, with

$$\begin{aligned} S(\lambda) &:= m_0 \sup_{n\ge 0} \frac{\lambda^n}{m_n}, \quad \lambda \ge 0, \\ \text{or } P(\lambda) &:= m_0 \left[\sum_{n=0}^{\infty} \frac{\lambda^{2n}}{m_n^2}\right]^{1/2}, \quad \lambda \ge 0. \end{aligned} \tag{21}$$

(cf. [13]).

Notably, when $m_n := [n!]^{\beta}$ ($m_n := n^{\beta n}$) with $\beta > 0$, the corresponding function $T(\cdot)$ is replaceable with

$$F(\lambda) = e^{\lambda^{1/\beta}}, \quad \lambda \ge 0 \tag{22}$$

(see [13] for details, cf. also [1]) and

$$\begin{aligned} \mathscr{E}^{\{\beta\}}(A) &= \bigcup_{t>0} D\left(e^{t|A|^{1/\beta}}\right), \\ \mathscr{E}^{(\beta)}(A) &= \bigcap_{t>0} D\left(e^{t|A|^{1/\beta}}\right). \end{aligned} \tag{23}$$

Observe that equalities (18) can be considered to be operator analogues of the classical *Paley-Wiener Theorems* relating the smoothness of the *Fourier transform* $\hat{f}(\cdot)$ of a square-integrable on $\mathbb{R}$ function $f(\cdot)$ to its decay at $\pm\infty$ [18].

In [1, Theorem 3.1] and [1, Corollary 4.1], equalities (18) and (23) are generalized to the case of a *scalar type spectral operator* A in a *reflexive* complex Banach space X, with the reflexivity requirement dropped, the inclusions

$$\begin{aligned} C_{\{m_n\}}(A) &\supseteq \bigcup_{t>0} D(T(t|A|)), \\ C_{(m_n)}(A) &\supseteq \bigcap_{t>0} D(T(t|A|)), \end{aligned} \tag{24}$$

$$\begin{aligned} \mathscr{E}^{\{\beta\}}(A) &\supseteq \bigcup_{t>0} D\left(e^{t|A|^{1/\beta}}\right), \\ \mathscr{E}^{(\beta)}(A) &\supseteq \bigcap_{t>0} D\left(e^{t|A|^{1/\beta}}\right) \end{aligned} \tag{25}$$

are proven only.

In the recent paper [5], the reflexivity requirement is shown to be superfluous and the following statements are proven.

Theorem 1 (see [5, Theorem 3.1]). *Let $\{m_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying the condition (WGR) given by (15). Then, for a scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$, equalities (18) are true; the scalar type spectral operators $T(t|A|)$, $t > 0$, are understood in the sense of the Borel operational calculus and the function $T(\cdot)$ defined by (17) is replaceable with any nonnegative, continuous, and increasing on $[0,\infty)$ function $F(\cdot)$ satisfying condition (20).*

Corollary 2 (see [5, Corollary 4.1]). *Let $\beta > 0$. Then, for a scalar type spectral operator A in a complex Banach space $(X, \|\cdot\|)$, equalities (23) are true.*

In papers [2–4], written before [5], the deficiency of inclusions (24) and (25) for the general case is inadvertently overlooked by the author and wrong conclusions are drawn from them in the "*only if*" parts of [2, Theorem 5.1] and [4, Theorem 5.1] and the *sufficiency* of [3, Theorem 3.1]. Thus far, this circumstance leaving the statements effectively proven only for reflexive spaces, when, by [1, Theorem 3.1] and [1, Corollary 4.1], inclusions (24) and (25) turn into equalities (18) and (23), respectively, also seems to have escaped the attention of the referees and the authors who have cited [2, 3] (see, e.g., [19–21]).

However, the good news for all is that, due to [5, Theorem 3.1] and [5, Corollary 4.1], inclusions' (24) and (25) being actually equalities (18) and (23), respectively, without the requirement of reflexivity, readily amends the faulty logic in the proofs of all the foregoing statements, making them true for an arbitrary complex Banach space.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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