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Research Article

Description of the Magnetic Field and Divergence of Multisolenoid Aharonov-Bohm Potential

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Explicit formulas for the magnetic field and divergence of multisolenoid Aharonov-Bohm potential are obtained; the mathematical essence of this potential is explained. It is shown that the magnetic field and divergence of this potential are very singular generalized functions concentrated at a finite number of thin solenoids. Deficiency index is found for the minimal operator generated by the Aharonov-Bohm differential expression.

1. Introduction

66 years have passed since the publication of Aharonov and Bohm's "Significance of Electromagnetic Potential in the Quantum Theory" [1], and since then interest in this paper has never faded. According to *Web of Science®-Google Scholar*, it has been cited 5680 times (as of December 2014)! Note that there are plenty of both supporters and opponents of this work (see, e.g., [2, 3]).

The purpose of our work is to find explicit formulas for the magnetic field and divergence of multisolenoid Aharonov-Bohm potential and to explain the mathematical essence of it. The obtained formulas show (see Theorems 1 and 3) that the magnetic field and divergence of this potential are very singular generalized functions concentrated at a finite number of thin solenoids perpendicular to the plane x_1Ox_2.

2. Main Results

Let $\xi_k = (x_1^{(k)}, x_2^{(k)})$, $k = 1, 2, \ldots, n$, be pairwise distinct points in R_2, let $a_k : S_1(0) \to R_1$, $k = 1, 2, \ldots, n$, be real, bounded, and measurable functions on the unit circle $S_1(0) \subset R_2$, and $\Omega' = R_2 \setminus \{\xi_k,\ k = 1, 2, \ldots, n\}$. Define the magnetic Aharonov-Bohm potential as follows:

$$A(x) = \sum_{k=1}^{n} a_k \left(\frac{x - \xi_k}{|x - \xi_k|} \right) \frac{1}{|x - \xi_k|^2} \left(-x_2 + x_2^{(k)}, x_1 - x_1^{(k)} \right), \quad x = (x_1, x_2) \in \Omega', \tag{1}$$

where

$$|x - \xi_k| = \sqrt{\left(x_1 - x_1^{(k)}\right)^2 + \left(x_2 - x_2^{(k)}\right)^2}. \tag{2}$$

As far as we know, in all the earlier works (except for [4]) dedicated to the Aharonov-Bohm effect (for short, AB effect), the functions $a_k((x - \xi_k)/|x - \xi_k|)$, $k = 1, 2, \ldots, n$, are constants.

The following theorems are true (in case $n = 1$ they were proved in [4]).

Theorem 1. *Let the magnetic field $B = \nabla \times A$ be generated by the magnetic Aharonov-Bohm potential (1) in the sense of generalized functions. Then the following equality is true:*

$$B = \nabla \times A = \sum_{k=1}^{n} \left[\int_{-\pi}^{\pi} a_k(\theta)\, d\theta \right] \delta(x - \xi_k), \quad (3)$$

where $\delta(x-\xi_k)$, $k = 1, 2, \ldots, n$, are the Dirac functions and $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$ is the gradient operator.

Proof. Let

$$A(x) = \left(A_{x_1}, A_{x_2}\right), \quad (4)$$

where

$$A_{x_1} = \sum_{k=1}^{n} a_k \left(\frac{x-\xi_k}{|x-\xi_k|} \right) \frac{-x_2 + x_2^{(k)}}{|x-\xi_k|^2},$$
$$A_{x_2} = \sum_{k=1}^{n} a_k \left(\frac{x-\xi_k}{|x-\xi_k|} \right) \frac{x_1 - x_1^{(k)}}{|x-\xi_k|^2}. \quad (5)$$

Then the definition of magnetic field

$$B = \nabla \times A = \frac{\partial A_{x_2}}{\partial x_1} - \frac{\partial A_{x_1}}{\partial x_2} \quad (6)$$

implies that for every function $f(x) \in C_0^{\infty}(R_2)$ we have

$$\int_{R_2} Bf(x)\, dx = \int_{R_2} \left(\frac{\partial A_{x_2}}{\partial x_1} - \frac{\partial A_{x_1}}{\partial x_2} \right) f(x_1, x_2)\, dx_1 dx_2. \quad (7)$$

Taking into account the identity

$$\left(\frac{\partial A_{x_2}}{\partial x_1} - \frac{\partial A_{x_1}}{\partial x_2} \right) f(x_1, x_2) = \frac{\partial}{\partial x_1}\left(A_{x_2} f\right) - \frac{\partial}{\partial x_2}\left(A_{x_1} f\right) - A_{x_2}\frac{\partial f}{\partial x_1} + A_{x_1}\frac{\partial f}{\partial x_2} \quad (8)$$

and the Green formula, we rewrite relation (7) as follows:

$$\int_{R_2} Bf(x)\, dx = \int_{R_2} \left(A_{x_1}\frac{\partial f(x_1,x_2)}{\partial x_2} - A_{x_2}\frac{\partial f(x_1,x_2)}{\partial x_1} \right) dx_1 dx_2. \quad (9)$$

Hence, by virtue of (5), we get

$$\begin{aligned}\int_{R_2} Bf(x)\, dx &= \int_{R_2} \left\{ \left[\sum_{k=1}^{n} a_k \left(\frac{x-\xi_k}{|x-\xi_k|} \right) \frac{-x_2 + x_2^{(k)}}{|x-\xi_k|^2} \right] \frac{\partial f(x_1,x_2)}{\partial x_2} - \left[\sum_{k=1}^{n} a_k \left(\frac{x-\xi_k}{|x-\xi_k|} \right) \frac{x_1 - x_1^{(k)}}{|x-\xi_k|^2} \right] \frac{\partial f(x_1,x_2)}{\partial x_1} \right\} dx_1 dx_2 \\ &= \sum_{k=1}^{n} \left\{ \int_{R_2} a_k \left(\frac{x-\xi_k}{|x-\xi_k|} \right) \left[\frac{-x_2 + x_2^{(k)}}{|x-\xi_k|^2} \frac{\partial f(x_1,x_2)}{\partial x_2} - \frac{x_1 - x_1^{(k)}}{|x-\xi_k|^2} \frac{\partial f(x_1,x_2)}{\partial x_1} \right] \right\} dx_1 dx_2 = -\sum_{k=1}^{n} J_k(f),\end{aligned} \quad (10)$$

where

$$J_k(f) = \int_{R_2} a_k \left(\frac{x-\xi_k}{|x-\xi_k|} \right) \left[\frac{x_1 - x_1^{(k)}}{|x-\xi_k|^2} \frac{\partial f(x_1,x_2)}{\partial x_1} + \frac{x_2 - x_2^{(k)}}{|x-\xi_k|^2} \frac{\partial f(x_1,x_2)}{\partial x_2} \right] dx_1 dx_2, \quad k = 1, 2, \ldots, n. \quad (11)$$

Using the transformation of plane into itself defined by the formulas

$$t_1 = x_1 - x_1^{(k)},$$
$$t_2 = x_2 - x_2^{(k)}, \quad (12)$$
$$(t = x - \xi_k),$$

and considering the equalities

$$\frac{\partial f(x_1,x_2)}{\partial x_1} = \frac{\partial f\left(t_1 + x_1^{(k)}, t_2 + x_2^{(k)}\right)}{\partial t_1},$$
$$\frac{\partial f(x_1,x_2)}{\partial x_2} = \frac{\partial f\left(t_1 + x_1^{(k)}, t_2 + x_2^{(k)}\right)}{\partial t_2} \quad (13)$$

in (11), we arrive at the following formula:

$$J_k(f) = \int_{R_2} a_k \left(\frac{t}{|t|} \right) \left[\frac{t_1}{|t|^2} \frac{\partial f\left(t_1 + x_1^{(k)}, t_2 + x_2^{(k)}\right)}{\partial t_1} + \frac{t_2}{|t|^2} \frac{\partial f\left(t_1 + x_1^{(k)}, t_2 + x_2^{(k)}\right)}{\partial t_2} \right] dt_1 dt_2, \quad k = 1, 2, \ldots, n. \quad (14)$$

After transition to polar coordinates

$$t_1 = r\cos\theta,\quad t_2 = r\sin\theta,\quad r > 0,\ -\pi < \theta \le \pi\ \ (t = r(\cos\theta, \sin\theta)),\tag{15}$$

and using the equalities

$$\begin{aligned}\frac{\partial f\left(t_1 + x_1^{(k)}, t_2 + x_2^{(k)}\right)}{\partial t_1} &= \frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial r}\frac{\partial r}{\partial t_1} + \frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial \theta}\frac{\partial \theta}{\partial t_1}\\ &= \frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial r}\cos\theta - \frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial \theta}\frac{\sin\theta}{r},\\ \frac{\partial f\left(t_1 + x_1^{(k)}, t_2 + x_2^{(k)}\right)}{\partial t_2} &= \frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial r}\frac{\partial r}{\partial t_2} + \frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial \theta}\frac{\partial \theta}{\partial t_2}\\ &= \frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial r}\sin\theta + \frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial \theta}\frac{\cos\theta}{r},\end{aligned}\tag{16}$$

we get

$$\begin{aligned}J_k(f) &= \int_0^{+\infty}\int_{-\pi}^{\pi} a_k(\cos\theta, \sin\theta)\Bigg\{\cos\theta\\ &\cdot\left[\frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial r}\cos\theta - \frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial \theta}\frac{\sin\theta}{r}\right] + \sin\theta\\ &\cdot\left[\frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial r}\sin\theta + \frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial \theta}\frac{\cos\theta}{r}\right]\Bigg\}\,dr\,d\theta\\ &= \int_0^{+\infty}\int_{-\pi}^{\pi} a_k(\cos\theta, \sin\theta)\frac{\partial f\left(r\cos\theta + x_1^{(k)}, r\sin\theta + x_2^{(k)}\right)}{\partial r}\,dr\,d\theta.\end{aligned}\tag{17}$$

Taking into account $f(x) \in C_0^\infty(R_2)$ and denoting $a_k(\theta) \equiv a_k(\cos\theta, \sin\theta)$, from (17) we have

$$\begin{aligned}J_k(f) &= -f\left(x_1^{(k)}, x_2^{(k)}\right)\int_{-\pi}^{\pi} a_k(\theta)\,d\theta\\ &= -f(\xi_k)\int_{-\pi}^{\pi} a_k(\theta)\,d\theta.\end{aligned}\tag{18}$$

The Dirac function $\delta(x - \xi_k)$ acts as follows:

$$(\delta(x - \xi_k), f(x)) = f(\xi_k).\tag{19}$$

Then the functional defined by the right-hand side of (18) is a generalized function. Thus, formula (18) can be rewritten in the following way:

$$\begin{aligned}J_k(f) &= -\left[\int_{-\pi}^{\pi} a_k(\theta)\,d\theta\right](\delta(x - \xi_k), f(x))\\ &= -\left(\left[\int_{-\pi}^{\pi} a_k(\theta)\,d\theta\right]\delta(x - \xi_k), f(x)\right),\\ &\qquad k = 1, 2, \ldots, n.\end{aligned}\tag{20}$$

Due to (20), equality (10) has the following form:

$$\begin{aligned}(B, f(x)) &\equiv \int_{R_2} Bf(x)\,dx\\ &= \sum_{k=1}^{n}\left(\left[\int_{-\pi}^{\pi} a_k(\theta)\,d\theta\right]\delta(x - \xi_k), f(x)\right)\\ &= \left(\sum_{k=1}^{n}\left[\int_{-\pi}^{\pi} a_k(\theta)\,d\theta\right]\delta(x - \xi_k), f(x)\right).\end{aligned}\tag{21}$$

Consequently, we have

$$B = \nabla \times A = \sum_{k=1}^{n}\left[\int_{-\pi}^{\pi} a_k(\theta)\,d\theta\right]\delta(x - \xi_k).\tag{22}$$

The theorem is proved. □

Remark 2. The formula

$$B = \sum_{k=1}^{n}\left[\int_{-\pi}^{\pi} a_k(\theta)\,d\theta\right]\delta(x - \xi_k)\tag{23}$$

implies that if the condition

$$\int_{-\pi}^{\pi} a_k(\theta)\, d\theta = 0 \tag{24}$$

holds for every k from $\{1, 2, \ldots, n\}$, then the AB effect is absent because the total magnetic flux of the magnetic field B passing through the closed contour that covers all the points $\xi_k = (x_1^{(k)}, x_2^{(k)})$, $k = 1, 2, \ldots, n$, is equal to zero.

The conditions for both the presence and absence of the AB effect in multiply connected domains are thoroughly studied in [3, 5].

Theorem 3. *Let the divergence* $\operatorname{div} A = \nabla \cdot A$ *be generated by the magnetic Aharonov-Bohm potential (1) in the sense of generalized functions. Then the following equality is true:*

$$\begin{aligned}\operatorname{div} A = \sum_{k=1}^{n} V.p. \Bigg\{ \frac{1}{|x-\xi_k|^2} \Bigg[&- \frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/|x-\xi_k|, \left(x_2 - x_2^{(k)}\right)/|x-\xi_k|\right)}{\partial\left(\left(x_1 - x_1^{(k)}\right)/|x-\xi_k|\right)} \frac{x_2 - x_2^{(k)}}{|x-\xi_k|} \\ &+ \frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/|x-\xi_k|, \left(x_2 - x_2^{(k)}\right)/|x-\xi_k|\right)}{\partial\left(\left(x_2 - x_2^{(k)}\right)/|x-\xi_k|\right)} \frac{x_1 - x_1^{(k)}}{|x-\xi_k|} \Bigg] \Bigg\},\end{aligned} \tag{25}$$

where

$$\begin{aligned}V.p. \Bigg\{ \frac{1}{|x-\xi_k|^2} \Bigg[&- \frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/|x-\xi_k|, \left(x_2 - x_2^{(k)}\right)/|x-\xi_k|\right)}{\partial\left(\left(x_1 - x_1^{(k)}\right)/|x-\xi_k|\right)} \frac{x_2 - x_2^{(k)}}{|x-\xi_k|} \\ &+ \frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/|x-\xi_k|, \left(x_2 - x_2^{(k)}\right)/|x-\xi_k|\right)}{\partial\left(\left(x_2 - x_2^{(k)}\right)/|x-\xi_k|\right)} \frac{x_1 - x_1^{(k)}}{|x-\xi_k|} \Bigg] \Bigg\}, \quad k = 1, 2, \ldots, n,\end{aligned} \tag{26}$$

are singular generalized functions; the letters V.p. mean "Cauchy principal value."

Proof. Let $f(x) \in C_0^{\infty}(R_2)$. Then, by the definition of the derivative of generalized function, using formulas (5), we have

$$\begin{aligned}(\operatorname{div} A, f(x)) &= (\nabla \cdot A, f(x)) = \int_{R_2} \left(\frac{\partial A_{x_1}}{\partial x_1} + \frac{\partial A_{x_2}}{\partial x_2} \right) f(x)\, dx = -\int_{R_2} \left(A_{x_1} \frac{\partial f(x)}{\partial x_1} + A_{x_2} \frac{\partial f(x)}{\partial x_2} \right) dx \\ &= -\sum_{k=1}^{n} \left\{ \lim_{0<\delta\to 0} \int_{|x-\xi_k|\geq\delta} a_k\left(\frac{x-\xi_k}{|x-\xi_k|}\right) \left[\frac{-x_2 + x_2^{(k)}}{|x-\xi_k|^2} \frac{\partial f(x_1, x_2)}{\partial x_1} + \frac{x_1 - x_1^{(k)}}{|x-\xi_k|^2} \frac{\partial f(x_1, x_2)}{\partial x_2} \right] dx_1 dx_2 \right\} \\ &= -\sum_{k=1}^{n} \left[\lim_{0<\delta\to 0} I_{k,\delta}(f) \right],\end{aligned} \tag{27}$$

where

$$\begin{aligned}I_{k,\delta}(f) = \int_{|x-\xi_k|\geq\delta} a_k\left(\frac{x-\xi_k}{|x-\xi_k|}\right) &\cdot \left[\frac{-x_2 + x_2^{(k)}}{|x-\xi_k|^2} \frac{\partial f(x_1, x_2)}{\partial x_1} \right. \\ &\left. + \frac{x_1 - x_1^{(k)}}{|x-\xi_k|^2} \frac{\partial f(x_1, x_2)}{\partial x_2} \right] dx_1 dx_2, \\ &k = 1, 2, \ldots, n.\end{aligned} \tag{28}$$

Using substitutions (12) and (15) and formulas (13) and (16), we obtain

$$I_{k,\delta}(f) = \int_{|t|\geq\delta} a_k\left(\frac{t}{|t|}\right)\left[\frac{-t_2}{|t|^2}\frac{\partial f\left(t_1+x_1^{(k)}, t_2+x_2^{(k)}\right)}{\partial t_1} + \frac{t_1}{|t|^2}\frac{\partial f\left(t_1+x_1^{(k)}, t_2+x_2^{(k)}\right)}{\partial t_2}\right]dt_1 dt_2 = \int_\delta^{+\infty}\int_{-\pi}^{\pi} a_k(\cos\theta,\sin\theta)$$
$$\cdot\left\{-\sin\theta\left[\frac{\partial f\left(r\cos\theta+x_1^{(k)}, r\sin\theta+x_2^{(k)}\right)}{\partial r}\cos\theta - \frac{\partial f\left(r\cos\theta+x_1^{(k)}, r\sin\theta+x_2^{(k)}\right)}{\partial\theta}\frac{\sin\theta}{r}\right] + \cos\theta\right.$$
$$\left.\cdot\left[\frac{\partial f\left(r\cos\theta+x_1^{(k)}, r\sin\theta+x_2^{(k)}\right)}{\partial r}\sin\theta + \frac{\partial f\left(r\cos\theta+x_1^{(k)}, r\sin\theta+x_2^{(k)}\right)}{\partial\theta}\frac{\cos\theta}{r}\right]\right\}dr\,d\theta \tag{29}$$
$$= \int_\delta^{+\infty}\int_{-\pi}^{\pi} a_k(\theta)\frac{\partial f\left(r\cos\theta+x_1^{(k)}, r\sin\theta+x_2^{(k)}\right)}{\partial\theta}\frac{1}{r}dr\,d\theta = \int_\delta^{+\infty}\frac{1}{r}dr\int_{-\pi}^{\pi} a_k(\theta)\frac{\partial f\left(r\cos\theta+x_1^{(k)}, r\sin\theta+x_2^{(k)}\right)}{\partial\theta}d\theta$$
$$= -\int_\delta^{+\infty}\frac{1}{r}dr\int_{-\pi}^{\pi} a_k'(\theta) f\left(r\cos\theta+x_1^{(k)}, r\sin\theta+x_2^{(k)}\right)d\theta = \int_\delta^{+\infty}\frac{1}{r}dr\int_{-\pi}^{\pi} a_k(\theta)\frac{\partial f\left(r\cos\theta+x_1^{(k)}, r\sin\theta+x_2^{(k)}\right)}{\partial\theta}d\theta,$$
$$k = 1,2,\ldots,n.$$

Now, to express $a_k'(\theta)$ in Cartesian coordinates x_1 and x_2, we put

$$M_k(x_1,x_2) \equiv a_k\left(\frac{x_1-x_1^{(k)}}{|x-\xi_k|}, \frac{x_2-x_2^{(k)}}{|x-\xi_k|}\right) = a_k(\cos\theta,\sin\theta) \equiv a_k(\theta), \quad k = 1,2,\ldots,n. \tag{30}$$

Having solved the system of equations

$$\frac{\partial M_k(x_1,x_2)}{\partial x_1} = \frac{\partial M_k}{\partial r}\cos\theta - \frac{\partial M_k}{\partial\theta}\frac{\sin\theta}{r},$$
$$\frac{\partial M_k(x_1,x_2)}{\partial x_2} = \frac{\partial M_k}{\partial r}\sin\theta + \frac{\partial M_k}{\partial\theta}\frac{\cos\theta}{r}, \tag{31}$$
$$k = 1,2,\ldots,n,$$

we find

$$a_k'(\theta) = \frac{\partial M_k}{\partial\theta} = \frac{\begin{vmatrix}\cos\theta & \partial M_k(x_1,x_2)/\partial x_1\\ \sin\theta & \partial M_k(x_1,x_2)/\partial x_2\end{vmatrix}}{\begin{vmatrix}\cos\theta & -\sin\theta/r\\ \sin\theta & \cos\theta/r\end{vmatrix}}$$
$$= r\left(\frac{\partial M_k(x_1,x_2)}{\partial x_2}\cos\theta - \frac{\partial M_k(x_1,x_2)}{\partial x_1}\sin\theta\right) \tag{32}$$
$$= \frac{\partial M_k(x_1,x_2)}{\partial x_2}\left(x_1-x_1^{(k)}\right) - \frac{\partial M_k(x_1,x_2)}{\partial x_1}\left(x_2-x_2^{(k)}\right), \quad k = 1,2,\ldots,n.$$

Differentiating the composite function $M_k(x_1,x_2)$ in x_1 and x_2 and using formula (32), we find

$$a_k'(\theta) = \left[\frac{\partial a_k\left(\left(x_1-x_1^{(k)}\right)/|x-\xi_k|, \left(x_2-x_2^{(k)}\right)/|x-\xi_k|\right)}{\partial\left(\left(x_1-x_1^{(k)}\right)/|x-\xi_k|\right)}\left(\frac{-\left(x_1-x_1^{(k)}\right)\left(x_2-x_2^{(k)}\right)}{|x-\xi_k|^3}\right)\right.$$
$$\left.+\frac{\partial a_k\left(\left(x_1-x_1^{(k)}\right)/|x-\xi_k|, \left(x_2-x_2^{(k)}\right)/|x-\xi_k|\right)}{\partial\left(\left(x_2-x_2^{(k)}\right)/|x-\xi_k|\right)}\left(\frac{1}{|x-\xi_k|} - \frac{\left(x_2-x_2^{(k)}\right)^2}{|x-\xi_k|^3}\right)\right]\left(x_1-x_1^{(k)}\right)$$
$$-\left[\frac{\partial a_k\left(\left(x_1-x_1^{(k)}\right)/|x-\xi_k|, \left(x_2-x_2^{(k)}\right)/|x-\xi_k|\right)}{\partial\left(\left(x_1-x_1^{(k)}\right)/|x-\xi_k|\right)}\left(\frac{1}{|x-\xi_k|} - \frac{\left(x_1-x_1^{(k)}\right)^2}{|x-\xi_k|^3}\right)\right.$$
$$\left.+\frac{\partial a_k\left(\left(x_1-x_1^{(k)}\right)/|x-\xi_k|, \left(x_2-x_2^{(k)}\right)/|x-\xi_k|\right)}{\partial\left(\left(x_2-x_2^{(k)}\right)/|x-\xi_k|\right)}\left(\frac{-\left(x_1-x_1^{(k)}\right)\left(x_2-x_2^{(k)}\right)}{|x-\xi_k|^3}\right)\right]\left(x_2-x_2^{(k)}\right)$$

$$= -\frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|, \left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)}{\partial\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|\right)} \frac{x_2 - x_2^{(k)}}{\left|x-\xi_k\right|} + \frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|, \left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)}{\partial\left(\left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)} \cdot \frac{x_1 - x_1^{(k)}}{\left|x-\xi_k\right|}, \quad k = 1, 2, \ldots, n. \tag{33}$$

Passing to the limit in (29) as $\delta \to 0$ and taking into account (33), we obtain

$$\lim_{0<\delta\to 0} I_{k,\delta}(f) = -\text{V.p.} \int_{R_2} \left\{ \frac{1}{\left|x-\xi_k\right|^2} \left[-\frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|, \left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)}{\partial\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|\right)} \frac{x_2 - x_2^{(k)}}{\left|x-\xi_k\right|} + \frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|, \left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)}{\partial\left(\left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)} \frac{x_1 - x_1^{(k)}}{\left|x-\xi_k\right|} \right] \right\} f(x)\, dx, \quad k = 1, 2, \ldots, n. \tag{34}$$

It is seen from (27) and (34) that the following equality is true for every $f(x) \in C_0^\infty(R_2)$:

$$\begin{aligned}(\operatorname{div} A, f(x)) &= \sum_{k=1}^{n} \text{V.p.} \int_{R_2} \left\{ \frac{1}{\left|x-\xi_k\right|^2} \left[-\frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|, \left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)}{\partial\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|\right)} \frac{x_2 - x_2^{(k)}}{\left|x-\xi_k\right|} + \frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|, \left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)}{\partial\left(\left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)} \frac{x_1 - x_1^{(k)}}{\left|x-\xi_k\right|} \right] \right\} \cdot f(x)\, dx \\ &= \sum_{k=1}^{n} \left(\text{V.p.} \frac{1}{\left|x-\xi_k\right|^2} \left[-\frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|, \left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)}{\partial\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|\right)} \frac{x_2 - x_2^{(k)}}{\left|x-\xi_k\right|} + \frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|, \left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)}{\partial\left(\left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)} \frac{x_1 - x_1^{(k)}}{\left|x-\xi_k\right|} \right], f(x) \right).\end{aligned} \tag{35}$$

Thus, the following equality is true in the sense of generalized functions:

$$\operatorname{div} A = \sum_{k=1}^{n} \text{V.p.} \left\{ \frac{1}{\left|x-\xi_k\right|^2} \left[-\frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|, \left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)}{\partial\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|\right)} \frac{x_2 - x_2^{(k)}}{\left|x-\xi_k\right|} + \frac{\partial a_k\left(\left(x_1 - x_1^{(k)}\right)/\left|x-\xi_k\right|, \left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)}{\partial\left(\left(x_2 - x_2^{(k)}\right)/\left|x-\xi_k\right|\right)} \frac{x_1 - x_1^{(k)}}{\left|x-\xi_k\right|} \right] \right\}. \tag{36}$$

The theorem is proved. □

Screening every thin solenoid $\tilde{\xi}_k = (x_1^{(k)}, x_2^{(k)}, x_3)$ ($k = 1, 2, \ldots, n$, $x_3 \in R_1$) with the use of Dirac function $\delta(x - \xi_k)$ ($k = 1, 2, \ldots, n$), we obtain a multicenter Schrödinger operator

$$(i\nabla + A(x))^2 - b_1\delta(x - \xi_1) - b_2\delta(x - \xi_2) - \cdots - b_n\delta(x - \xi_n), \tag{37}$$

with the magnetic Aharonov-Bohm potential of type (1), where b_k's ($k = 1, 2, \ldots, n$) are real numbers.

Consider in $L_2(R_2)$ the symmetric operator H_0 with the domain $D(H_0) = C_0^\infty(R_2 \setminus \{\xi_1, \xi_2, \ldots, \xi_n\})$ ($C_0^\infty(\Omega')$ is the totality of all infinitely differentiable finite functions in Ω'), which acts as follows:

$$H_0\psi(x) = (i\nabla + A(x))^2 \psi(x), \quad \psi(x) \in C_0^\infty(R_2 \setminus \{\xi_1, \xi_2, \ldots, \xi_n\}). \tag{38}$$

We denote by H the closure of the operator H_0.

Let

$$\int_{-\pi}^{\pi} a_k(\theta)\,d\theta = \widetilde{a}_k + \alpha_k, \quad k = 1, 2, \ldots, n, \tag{39}$$

where $\widetilde{a}_k$ is the integral part and α_k is the fractional part of the number $\int_{-\pi}^{\pi} a_k(\theta)d\theta$. Obviously, $0 \le \alpha_k < 1$, $k = 1, 2, \ldots, n$. Without loss of generality, we will assume that there exists an integer $l \le n$ such that

$$\begin{gathered} 0 < \alpha_j < 1, \quad \text{if } j = 1, 2, \ldots, l, \\ \alpha_j = 0, \quad \text{if } j = l+1, l+2, \ldots, n. \end{gathered} \tag{40}$$

Theorem 4. *(i) The domain $D(H_0^*)$ of the conjugate operator H_0^* coincides with the set*

$$\begin{aligned} D(H_0^*) = \{&\psi(x) : \psi(x) \in L_2(R_2) \\ &\cap W_{2,\mathrm{loc}}^2(\Omega'),\ (i\nabla + A(x))^2\psi(x) \in L_2(R_2)\}, \end{aligned} \tag{41}$$

where $W_{2,\mathrm{loc}}^2(\Omega')$ is a local second-order Sobolev space.

(ii) Deficiency index of the operator H is $(2l, 2l)$, where l is an integer $(l \le n)$ defined in (40).

Proof. (i) As the domain of the operator H_0 is dense in $L_2(R_2)$, it has a conjugate operator H_0^*. The domain of this conjugate operator $D(H_0^*)$ is the totality of all $\psi(x)$ from $L_2(R_2)$ for which there exist $u(x) \in L_2(R_2)$ such that

$$(H_0\varphi(x), \psi(x)) = (\varphi(x), u(x)), \tag{42}$$

for every $\varphi(x) \in D(H_0)$, and $H_0\psi(x) = u(x)$. From

$$(H_0\varphi(x), \psi(x)) = (\varphi(x), H_0^*\psi(x)), \tag{43}$$

it follows that $u(x) = (i\nabla + A(x))^2\psi(x)$ in the sense of generalized functions in $C_0^\infty(\Omega')$. Hence, in view of the ellipticity of the operator $(i\nabla + A(x))^2$, we have $\psi(x) \in W_{2,\mathrm{loc}}^2(\Omega')$ (see [6]).

(ii) Considering the notations

$$\begin{aligned} A_1(x) &= \sum_{k=1}^{l} a_k\left(\frac{x-\xi_k}{|x-\xi_k|}\right) \\ &\quad \cdot \frac{1}{|x-\xi_k|^2}\left(-x_2 + x_2^{(k)}, x_1 - x_1^{(k)}\right), \\ &\qquad x = (x_1, x_2) \in \Omega', \\ A_2(x) &= \sum_{k=l+1}^{n} a_k\left(\frac{x-\xi_k}{|x-\xi_k|}\right) \\ &\quad \cdot \frac{1}{|x-\xi_k|^2}\left(-x_2 + x_2^{(k)}, x_1 - x_1^{(k)}\right), \\ &\qquad x = (x_1, x_2) \in \Omega', \end{aligned} \tag{44}$$

in (1), we rewrite the potential $A(x)$ in the form of the sum of two summands:

$$A(x) = A_1(x) + A_2(x). \tag{45}$$

Now we introduce the magnetic l-flux potential

$$\begin{aligned} B(x) = \sum_{k=1}^{l} \frac{\alpha_k}{|x-\xi_k|^2}\left(-x_2 + x_2^{(k)}, x_1 - x_1^{(k)}\right), \\ x = (x_1, x_2) \in \Omega', \end{aligned} \tag{46}$$

where α_k is the fractional part of the number $\int_{-\pi}^{\pi} a_k(\theta)d\theta$.

It is proved in [7] that the minimal operator $H_{0,B}$ generated by the differential expression $(i\nabla + B(x))^2$ has the deficiency index $(2l, 2l)$. It follows from the results of [7, 8] that $A_1(x) \sim B(x)$ and $A_2(x) \sim 0$; that is, the pairs of potentials $(A_1(x), B(x))$ and $(A_2(x), 0)$ are gauge equivalent. Consequently, the assertion (ii) of the theorem follows from the gauge equivalence of the potentials $A(x)$ and $B(x)$. The theorem is proved. □

Remark 5. The assertions of Theorem 4 stay true if the Aharonov-Bohm solenoids lie in a homogeneous magnetic field of intensity γ, that is, for potentials of the form

$$A(x) + \left(-\frac{\gamma}{2}x_2, \frac{\gamma}{2}x_1\right). \tag{47}$$

Now let us make a few concluding remarks about the mathematical justification for the AB effect. Proceeding from Berezin and Faddeev's idea (see [9]), we arrive at the conclusion that the rigorous mathematical justification for the Aharonov-Bohm effect is that the pure Aharonov-Bohm operator H_{AB} lies among the self-adjoint extensions of the operator H_0; that is,

$$H_0 \subset H_{\mathrm{AB}} \subset H_0^*. \tag{48}$$

For local and nonlocal δ-interactions without magnetic field this idea was confirmed in many works (see, e.g., [10–13]), while for the Aharonov-Bohm operator it was confirmed in [7, 8, 14]. So the following question remains open for the potential of form (1): which of the self-adjoint extensions of the operator H_0 corresponds to the pure Aharonov-Bohm operator H_{AB}?

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] Y. Aharonov and D. Bohm, "Significance of electromagnetic potentials in the quantum theory," *Physical Review Letters*, vol. 115, pp. 485–491, 1959.

[2] S. N. M. Ruijsenaars, "The Aharonov-Bohm effect and scattering theory," *Annals of Physics*, vol. 146, no. 1, pp. 1–34, 1983.

[3] G. N. Afanasiev, "Old and new problems in the theory of the Aharonov-Bohm effect," *Physics of Elementary Particles and Atomic Nuclei*, vol. 21, no. 1, pp. 172–250, 1990 (Russian).

[4] B. Alziary, J. Fleckinger-Pellé, and P. Takáč, "Eigenfunctions and Hardy inequalities for a magnetic Schrödinger operator in $\mathbb{R}^2$," *Mathematical Methods in the Applied Sciences*, vol. 26, no. 13, pp. 1093–1136, 2003.

[5] G. N. Afanasiev, "On the single-valuedness of wave functions in multiply connected spaces," *JINR Rapid Communications*, vol. 6, pp. 17–26, 1985 (Russian).

[6] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, vol. 2 of *Van Nostrand Mathematical Studies*, Van Nostrand, Princeton, NJ, USA, Prepared for Publication by B. F. Jones Jr. with the Assistance of G. W. Batten Jr., 1965.

[7] T. Mine, "The Aharonov-Bohm solenoids in a constant magnetic field," *Annales Henri Poincaré*, vol. 6, no. 1, pp. 125–154, 2005.

[8] Y. Nambu, "The Aharonov-Bohm problem revisited," *Nuclear Physics B*, vol. 579, no. 3, pp. 590–616, 2000.

[9] F. A. Berezin and L. D. Faddeev, "Remark on the Schrödinger equation with singular potential," *Doklady Akademii Nauk SSSR*, vol. 137, pp. 1011–1014, 1961 (Russian).

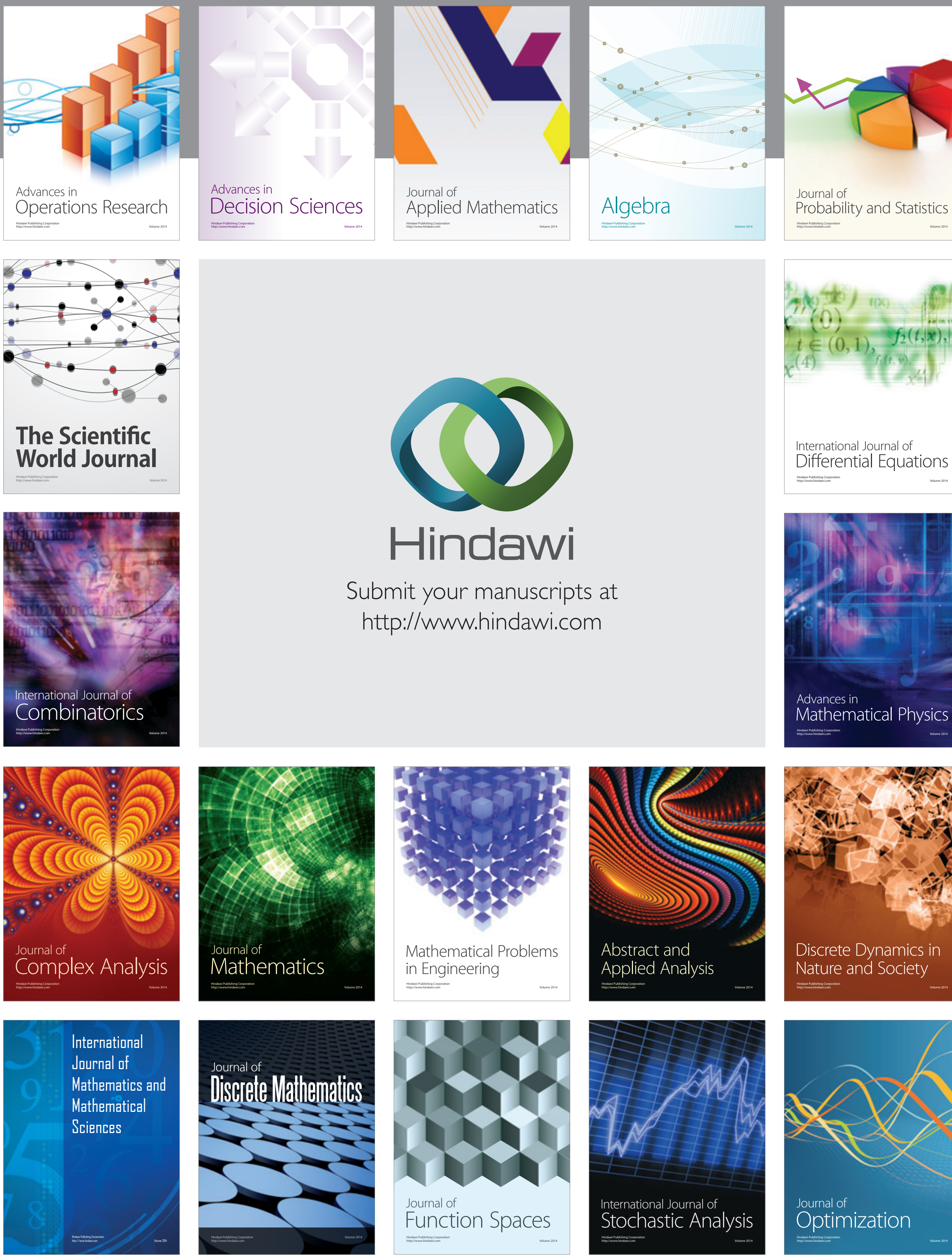
[10] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, Springer, Berlin, Germany; Mir Publishers, Moscow, Russia, 1988.

[11] E. H. Eyvazov, "Expansion in decisions scattering problem for the operator $H_\alpha + q(x)$," *Doklady Akademii Nauk Azerbaĭdzhana*, vol. 33, no. 10, pp. 8–11, 1977 (Russian).

[12] E. H. Eyvazov, "Higher order differential operators with finite number of δ-interactions in multivariate case," *Journal of Mathematical Physics, Analysis, Geometry*, vol. 6, no. 3, pp. 266–276, 2010.

[13] L. Dabrowski and H. Grosse, "On nonlocal point interactions in one, two, and three dimensions," *Journal of Mathematical Physics*, vol. 26, no. 11, pp. 2777–2780, 1985.

[14] L. Dabrowski and P. Stovcek, "Aharonov-Bohm effect with delta type interaction," *Journal of Mathematical Physics*, vol. 39, pp. 47–62, 1998.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