Internat. J. Math. & Math. Sci.
VOL. 14 NO. 2 (1991) 227-232

AN EXTENSION OF HELSON-EDWARDS THEOREM TO BANACH MODULES

SIN-EI TAKAHASI

Yamagata University
Department of Basic Technology
Faculty of Engineering
Yonezawa 992
JAPAN

(Received September 23, 1988)

ABSTRACT. An extension of the Helson-Edwards theorem for the group algebras to Banach modules over commutative Banach algebras is given. This extension can be viewed as a generalization of Liu-Rooij-Wang's result for Banach modules over the group algebras.

KEY WORDS AND PHRASES. Multiplier, Banach modules, bounded approximate identity, compact abelian group, completely regular.

1. INTRODUCTION.

Let A be a commutative complex Banach algebra with a bounded approximate identity $\{u_\lambda\}$ of norm β and denote by Φ_A the class of all nonzero homomorphisms of A into the field of complex numbers. The space Φ_A, with the Gelfand topology, is called the carrier space of A. Let X be a Banach left A-module. A continuous module homomorphism of A into X is called a multiplier of X. We introduce a family $\{X_\phi: \phi \in \Phi_A\}$ of Banach A-modules such that any multiplier T of X can be represented as a function T on Φ_A with $T(\phi) \in X_\phi$ for each $\phi \in \Phi_A$. In this setting we give an extension of the Helson-Edwards theorem for the group algebras to Banach modules. We also observe that this extension can be viewed as a generalization of Liu-Rooij-Wang's result for Banach modules over the group algebras. We further consider a local property of multipliers when A is completely regular.

2. REPRESENTATION THEOREM OF MULTIPLIERS.

For each $\phi \in \Phi_A$, let M_ϕ denote the maximal modular ideal of A corresponding to ϕ and define

$$X^\phi = \overline{sp}\{M_\phi X + (1 - e_\phi)X\},$$

where $\overline{sp}$ denotes the closed linear span and e_ϕ is an element of A with $\phi(e_\phi) = 1$. Note that X^ϕ does not depend on the choise of e_ϕ.

Throughout the remainder of this note we will assume

$$\bigcap_{\phi \in \Phi_A} \overline{sp}(M_\phi) = \{0\}. \tag{2.1}$$

In the case of X = A, the condition (2.1) is equivalent to the semisimplicity of A. The space $\overline{sp}(AX)$ is called the essential part of X and is denoted by X_e. Since A has a bounded approximate identity, it follows that X_e = AX from the Cohen-Hewitt factorization theorem (see Doran-Wichman [1]). We also have

$$X^\phi \quad X_e = \overline{sp}(M_\phi X) \tag{2.2}$$

for all $\phi \in \Phi_A$. In fact, let $\phi \in \Phi_A$, $x \in X^\phi$ X_e and $\varepsilon > 0$. Since $x \in X^\phi$, there exist $a_1,\dots,a_n$ M_ϕ and $x_1,\dots,x_n$, y X such that

$$||x - \sum_{i=1}^{n} a_i x_i - (1 - e_\phi)y|| < \varepsilon/\beta.$$

Therefore for each λ, we have

$$||u_\lambda x - (\sum_{i=1}^{n} u_\lambda a_i x_i + (u_\lambda - u_\lambda e_\phi)y)|| < \varepsilon.$$

Letting $\varepsilon \downarrow 0$, we obtain that $u_\lambda x \in \overline{sp}(M_\phi X)$ for all λ. Since $x \in X_e$, $\lim u_\lambda x = x$. Consequently, we have that $x \in \overline{sp}(M_\phi X)$ and hence $X^\phi \cap X_e \subset \overline{sp}(M_\phi X)$. The reverse inclusion is immediate.

We denote by M(A, X), or simply M(X), the class of all multipliers of X. Then M(X) also becomes a Banach A-module under the module multiplication defined by (aT)b = a(Tb). For each $x \in X$, the mapping τ_x of A into X defined by $\tau_x(a) = ax$ is a multiplier of X, so that τ becomes a module homomorphism of X into M(X). Also it can be easily observed that

$$TA \subset X_e \text{ and } TM_\phi \subset \overline{M_\phi X} \tag{2.3}$$

for all $T \in M(X)$ and $\phi \in \Phi_A$, where the bar denotes the norm closure.

Now, for each $\phi \in \Phi_A$, let $X_\phi = X/X^\phi$ be the quotient of X by X^ϕ. So X_ϕ becomes a Banach A-module under the natural module structure and the quotient norm. For each $x \in X$, let $x(\phi) = x + X^\phi$ be the natural image of x in X^ϕ. A vector field on Φ_A is a function σ defined on Φ_A with $\sigma(\phi) \in X_\phi$ for each $\phi \in \Phi_A$. Of course, $\hat{x}(x \in X)$ is a vector field on Φ_A. Denote by ΠX_ϕ the class of all vector fields on Φ_A and so it becomes an A-module under the module multiplication defined by $(a\sigma)(\phi) = \hat{a}(\phi)\sigma(\phi)$, where a denotes the Gelfand transform of a A. Define

$$\Pi^b X_\phi = \{\sigma \quad \Pi X_\phi : ||\sigma||_\infty = \sup_{\phi \in \Phi_A} ||\sigma(\phi)|| < +\infty\}.$$

Then $\Pi^b X_\phi$ beomes a Banach A-module under the norm $|| \ ||_\infty$ and $X = \{\hat{x} : x \in X\} \subset \Pi^b X_\phi$.

With the above notations, we have the following representation theorem of multipliers.

THEOREM 2.1. (i) If $T \in M(X)$, then there exists a unique vector field $\hat{T}$ on Φ_A such that $\widehat{Ta} = a\hat{T}$ for all $a \in A$. (ii) The mapping $T \to \hat{T}$ is a continuous module isomorphism of $M(X)$ into $\Pi^b X_\phi$.

PROOF. Let $T \in M(X)$, $a \in A$ and $\phi \in \Phi_A$. Since $e_\phi au_\lambda - \hat{a}(\phi)e_\phi u_\lambda \in M_\phi$ for all λ, it follows from (2.3) that $T(e_\phi au_\lambda) - \hat{a}(\phi)T(e_\phi u_\lambda) \in \overline{M_\phi X}$ for all λ. Hence, after taking the limit with respect to λ, we obtain $T(e_\phi a) - \hat{a}(\phi)Te_\phi \in \overline{M_\phi X}$. Note also that $Ta \in X_e$ from (2.3). Then there exist $c \in A$ and $y \in X$ such that $Ta = cy$, so that

$$Ta - T(e_\phi a) = Ta - e_\phi Ta = (c - e_\phi c)y \in M_\phi X.$$ We therefore have

$$Ta - \hat{a}(\phi)Te_\phi = (Ta - Te_\phi a) + (Te_\phi a - \hat{a}(\phi)Te_\phi) \in M_\phi X + \overline{M_\phi X} \subset \overline{sp}(M_\phi X) \subset X^\phi.$$

Setting $\hat{T}(\phi) = \widehat{Te_\phi}(\phi)$, we obtain that $\widehat{Ta}(\phi) = \hat{a}(\phi)\hat{T}(\phi) = (\widehat{aT})(\phi)$. In other words, $\widehat{Ta} = a\hat{T}$ for all $a \in A$. If $\sigma \in \Pi X_\phi$ such that $\widehat{Ta} = a\sigma$ for all $a \in A$, then $\hat{T}(\phi) = \widehat{Te_\phi}(\phi) = \hat{e}_\phi(\phi)\sigma(\phi) = \sigma(\phi)$ for all $\phi \in \Phi_A$, so that $\hat{T} = \sigma$. This proves (i). It is immediate from (i) that $T \to \hat{T}$ is a continuous module homomorphism of $M(X)$ into $\Pi^b X_\phi$. To show that this mapping is injective, let $T \in M(X)$ with $\hat{T} = 0$. Then $\widehat{TA} = A\hat{T} = \{0\}$ from (i), so $TA \subset \bigcap_{\phi \in \Phi_A} X^\phi$. Also $TA \subset X_e$ from (2.3). Therefore, by (2.2) and our assumption (2.1),

$$TA \subset \bigcap_{\phi \in \Phi_A} X_e \cap X^\phi = \bigcap_{\phi \in \Phi_A} \overline{sp}(M_\phi X) = \{0\}.$$

We thus obtain $T = 0$, and (ii) is proved.

A Banach left A-module X is said to be order-free if for every $x \in X$ with $x \neq 0$ there exists $a \in A$ with $ax \neq 0$.

COROLLARY 2.2. Let $x \in X$. If either $x \in X_e$ or X is order-free, then $\hat{x} = 0$ implies $x = 0$.

PROOF. Note first that

$$\widehat{ax} = a\hat{x}, \ a \in A, \ x \in X. \tag{2.4}$$

In fact, for each $\phi \in \Phi_A$,

$$ax - \hat{a}(\phi)x = (a - \hat{a}(\phi)e_\phi)x - \hat{a}(\phi)(1 - e_\phi)\, x \in M_\phi X - (1-e_\phi)\, X \subset X^\phi.$$

This implies (2.4). Now let $x \in X$ with $\hat{x} = 0$. By the above theorem and (2.4), we have

$$\hat{\tau}_x(\phi) = \hat{e}_\phi(\phi)\hat{\tau}_x(\phi) = (e_\phi\hat{\tau}_x)(\phi) = \hat{\tau}_x e_\phi(\phi) = \widehat{e_\phi x}(\phi) = \hat{e}_\phi(\phi)\hat{x}(\phi) = \hat{x}(\phi)$$

for all $\phi \in \Phi_A$, so that $\hat{\tau}_x = \hat{x}$. Then $\hat{\tau}_x = 0$ and hence $Ax = \{0\}$. Accordingly, if either $x \in X_e$ or X is order-free, then $x = 0$.

3. EXTENSION OF HELSON-EDWARDS THEOREM.

We give a characterization of multipliers of an order-free Banach A-module which is similar to [2, Theorem 1.2.4] and Liu, van Rooij, and Wang [3, Lemma 1.3].

COROLLARY 3.1. Let X be order-free and T a mapping of A into X. Then the following conditions are equivalent.

(i) $T \in M(X)$.

(ii) T is linear and continuous; $TM_\phi \subset X^\phi$ for every $\phi \in \Phi_A$.

(iii) $T(ab) = aTb$ for all $a,b \in A$.

PROOF. (i) ==> (ii) follows immediately from (2.3). (ii) ==> (iii). Let $a,b \in A$ and $\phi \in \Phi_A$. Since $abu_\lambda - \hat{a}(\phi)bu_\lambda \in M_\phi$ for all λ, it follows from (ii) that $T(abu_\lambda) - \hat{a}(\phi)T(bu_\lambda) \in TM_\phi \subset X^\phi$ for all λ. Hence, after taking the limit with respect to λ, we obtain $T(ab) - \hat{a}(\phi)Tb \in X^\phi$. Then, by (2.4), $\widehat{T(ab)} = \widehat{aTb} = \hat{a}\widehat{Tb}$, so that $T(ab) = aTb$ by Corollary 2.2.

(iii) ==> (i). To show that T is linear, let $a,b \in A$ and α,β scalars. Then

$$cT(\alpha a + \beta b) = T(\alpha ac - \beta bc) = (\alpha a + \beta b)Tc = \alpha aTc + \beta bTc$$

$$= \alpha cTa + \beta cTb = c(\alpha Ta + \beta Tb)$$

for all $c \in A$. Since X is order-free, $T(\alpha a + \beta b) = \alpha Ta + \beta Tb$.

To show the continuity of T, let $\lim a_n = a \in A$ and $\lim Ta_n = x \in X$. Then

$$bTa = aTb = \lim a_nTb = \lim bTa_n = bx$$

for all $b \in A$. So $Ta = x$ and hence T is continuous by the closed graph theorem.

Let $\widehat{M(A)} = \{\hat{T}: T \in M(X)\}$. The following result is an extension of the Helson-Edwards theorem for the group algebra of a locally compact Abelian group (see Rudin [4, Theorem 3.8.1]).

THEOREM 3.2. Let $\sigma \in \Pi X_\phi$. Then, $A\sigma \subset \widehat{M(X)}$ if and only if $\sigma \in \widehat{M(X)}$.

PROOF. Note first that $\hat{\tau}_x = \hat{x}$ for all $x \in X$ as observed in the proof of Corollary 2.2. If $T \in M(X)$ with $\hat{T} = \sigma$, then, by Theorem 2.1, $a\sigma = \widehat{aT} = \widehat{Ta} = \hat{\tau}_{Ta} \in \widehat{M(X)}$ for all a A.

Suppose conversely that $A\sigma \subset \widehat{M(X)}$. Let $a \in A$. By the Cohen-Hewitt factorization theorem, a can be written as $a = bc$ for some $b, c \in A$. Choose $S \in M(X)$ with $c\sigma = \hat{S}$. Then, $a\sigma = bc\sigma = b\hat{S} = \widehat{Sb} \in \hat{X}_e$ from (2.3). Hence, by Corollary 2.2, there is a unique element of X_e, say Ta, such that $a\sigma = \widehat{Ta}$. If a,b are arbitrary elements of A, then $\widehat{T(ab)} = ab\sigma = a(b\sigma) = a\widehat{Tb} = \hat{a}\widehat{Tb}$ by (2.4). Since $TA \quad X_e$, $T(ab) = aTb$ by Corollary 3.1. Note that X_e is an order-free Banach A-module. Then, by Corollary 3.1, $T \in M(A, X_e) \subset M(A, X) = M(X)$. Consequently, $\sigma = \hat{T} \in \widehat{M(X)}$ and the theorem is proved.

We will observe that Theorem 3.2 can be viewed as a generalization of Liu-Rooij-Wang's result [3, Theorem 2.3].

Let G be a compact Abelian group and X a Banach L^1 (G)-module. Let $X_\gamma = \gamma X$ for each $\gamma\hat{G}$ the dual group of G. Also denote by πX_γ the class of all mappings ρ of $\hat{G}$ into X such that $\rho(\gamma) \in X_\gamma$ for every $\gamma \in G$. Set $\phi_\gamma(f) = \hat{f}(\gamma)$ $(\gamma \in \hat{G}, f \in L^1(G))$, where $\hat{f}$ is the Fourier transform of f. Note that for each $\gamma \in \hat{G}, X^{\phi\gamma} = (1-\gamma)X$ and $X^{\phi\gamma}$ is isometrically module-isomorphic to X_γ. Also since $\overline{\text{sp}}\ \hat{G} = L^1(G)$, it follows that

$$\bigcap_{\gamma \in \hat{G}} X^{\phi_\gamma} = \{0\}$$

and hence X satisfies (2.1). For each $x \in X$, denote by $\tilde{x}$ the restriction of τ_x to $\hat{G}$ and set $\tilde{X} = \{\tilde{x}: x \in X\}$.

COROLLARY (Liu-Rooij-Wang). $\rho \in \Pi X_\gamma$ can be extended to a multiplier of X if and only if $\hat{f}\rho \in \tilde{X}$ for every $f \in L^1(G)$.

PROOF. Clearly $\hat{f}(\gamma)\gamma = \gamma * f$ ($\gamma \in \hat{G}$, $f \in L^1(G)$). So if $\rho = T|\hat{G}$ for some $T \in M(X)$, then $\hat{f}\rho = \widetilde{Tf} \in \tilde{X}$ for every $f \in L^1(G)$. Suppose conversely that $\rho \in \Pi X_\gamma$ and $\hat{f}\rho \in \tilde{X}$ for every $f \in L^1(G)$. Then for each $f \in L^1(G)$, choose $x_f \in X$ with $\hat{f}\rho = \tilde{x}_f$. set $\sigma(\phi_\gamma) = \widehat{\rho(\gamma)}(\phi_\gamma)$ for each $\gamma \in \hat{G}$. We then have

$$(f\sigma)(\phi_\gamma) = \hat{f}(\gamma)\widehat{\rho(\gamma)}(\phi_\gamma) = (\tilde{x}_f(\gamma))\ (\phi_\gamma)$$

$$= \widehat{\gamma x_f}(\phi_\gamma) = \hat{x}_f(\phi_\gamma)$$

for all $\gamma \in \hat{G}$ and $f \in L^1(G)$. Thus $f\sigma = \hat{x}_f \in \hat{X}$ for all $f \in L^1(G)$ and hence $\sigma = \hat{T}$ for some $T \in M(X)$ from Theorem 3.2. Therefore,

$$\widehat{\rho(\gamma)}(\phi_\gamma) = \hat{T}(\phi_\gamma) = (\widehat{\gamma T})(\phi_\gamma) = \widehat{T\gamma}(\phi_\gamma),$$

so that $\rho(\gamma) - T\gamma \in X^{\phi_\gamma} = (1 - \gamma)X$ for all $\gamma \in \hat{G}$. But $\rho(\gamma)$, $T\gamma \in \gamma X$ and so $\rho(\gamma) - T\gamma \in \gamma X$ for all $\gamma \in \hat{G}$. Consequently, $\rho = T|\hat{G}$.

4. LOCAL PROPERTIES OF MULTIPLIERS.

We will consider local properties of multipliers. To do this, we introduce the following notation which is exactly similar to one given in Rickart [5, 2.7.13].

DEFINITION. Let $\sigma \in \Pi X_\phi$ and $\Sigma \subset \Pi X_\phi$. Then σ is said to belong to Σ near a point $\phi \in \Phi_A$ (or at infinity) provided there exists a neighborhood V of ϕ (or infinity) and an element σ' Σ such that $\sigma|V = \sigma'|V$. If σ belongs to near every point of Φ_A and at infinity, then σ is said to belong locally to Σ.

The following result is similar to one given in [5, 2.7.16] and we refer to the proof of one.

THEOREM 4.1. Assume A to be completely regular and let Σ be a submodule of ΠX_ϕ. If $\sigma \in \Pi X_\phi$ belongs locally to Σ, then $\sigma \in \Sigma$.

PROOF. Since σ belongs to Σ at infinity, there exists an open set U_0 of Φ_A with compact complement K and $\sigma_0 \in \Sigma$ with $\sigma_0|U_0 = \sigma|U_0$. Also since σ belongs to Σ near every point of K, there exists a finite open covering $\{U_1,\dots,U_n\}$ of K and a finite subset $\{\sigma_1,\dots,\sigma_n\}$ of Σ with $\sigma_i|U_i = \sigma|U_i$ $(i = 1,\dots,n)$. Note that A admits a partion of the identity (cf. [5, Theorem 2.7.12]). Then there exists $e_1,\dots,e_n$ A such that $e = e_1 + \dots + e_n$ is an identity for A modulo ker K and $e_i \in \ker(\Phi_A - U_i)$ $(i = 1,\dots, n)$, where ker K denotes the kernel of K. Set

$$\sigma' = (1 - e)\sigma_0 + e_1\sigma_1 + \dots + e_n\sigma_n .$$

Then σ' is obviously in Σ. We further assert $\sigma' = \sigma$. In fact, if $\phi \in U_0$, then we have

$$\sigma'(\phi) = (1 - \hat{e}(\phi))\sigma(\phi) + \sum_{\phi \in U_i} \hat{e}_i(\phi)\sigma(\phi)$$

$$= (1 - \hat{e}(\phi) + \sum_{\phi \in U_i} \hat{e}_i(\phi))\sigma(\phi)$$

$$= \sigma(\phi).$$

If $\phi \in K$, then $\hat{e}(\phi) = 1$ and $\{i: 1 \leq i \leq n, \phi \in U_i\} \neq \emptyset$, so that

$$\sigma'(\phi) = \sum_{\phi \in U_i} \hat{e}_i(\phi)\sigma_i(\phi) = \sum_{\phi \in U_i} \hat{e}_i(\phi)\sigma(\phi)$$

$$= \hat{e}(\phi)\sigma(\phi) = \sigma(\phi).$$

Consequently, $\sigma' = \sigma$ and the theorem is proved.

Because M(X) is a submodule of ΠX_ϕ, we obtain the following local property of multipliers from the preceding theorem.

COROLLARY 4.2. Assume A to be completely regular. If $\sigma \in \Pi X_\phi$ belongs locally to M(X), then $\sigma \in M(X)$.

Let A contain local identities (cf. [5, 3.6.11]) and $T \in M(X)$. The closure of $\{\phi \in \Phi_A: \hat{T}(\phi) \neq 0\}$ is called the support of T and is denoted by supp T. If supp T is compact, then there exists a unique $x \in X_e$ with $T = \tau_x$. In fact, by [5, Theorem 3.6.13], A has an identity for A modulo ker(supp $\hat{T}$), say e. Set x = Te. So the desired result follows from Theorem 2.1 and Corollary 2.2.

Similarly, we obtain that for each compact set K of Φ_A, there exists $x \in X_e$ with $\hat{T}|K = \hat{x}|K$.

REFERENCES

1. DORAN, R.S. and WICHMANN, J., Approximate Identities and Factorization in Banach Modules, Lecture Notes in Math. 768, Springer-Verlag, New York-Heidelberg, 1979.

2. LARSEN, R., An Introduction to the Theorey of Multipliers, Springer-Verlag, New York-Heidelberg, 1971.

3. LIU, T.S., VAN ROOIJ, A.C.M. and WANG, J.K., A Generalized Fourier Transformation for $L_1(G)$-modules, J. Austral. Math. Soc. (Series A) 36 (1984), 365-377.

4. RUDIN, W., Fourier Analysis on Groups, New York, N.Y.: Interscience Publishers, Inc., 1962.

5. RICKART, C.E., General Theorey of Banach Algebras, Van Nostrand, Princeton, N.J.. 1960.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization