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ISOMETRIES OF A FUNCTION SPACE

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ABSTRACT. It is proved here that an isometry on the subset of all positive functions of $L^1 \cap L^p(\mathbb{R})$ can be characterized by means of a function h together with a Borel measurable mapping ϕ of $\mathbb{R}$, thus generalizing the Banach-Lamparti theorem of L^p spaces.

KEY WORDS AND PHRASES. Borel measure, function space, Banach-Lamparti Theorem.
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1. INTRODUCTION.

Edwards [1] proves that all bipositive isomorphisms of L^p $(1 \leq p < \infty)$ convolution algebras of a compact group are induced by bicontinuous isomorphisms of the group. By changing the algebra isomorphism from bipositive to an isometry Strichartz [2] establishes the same type of result with the exception of $p = 2$. Here we consider isometries of $L^1 \cap L^p(\mathbb{R})$, $p \neq 2$, and give a general form to the Banach-Lamparti theorem proving the isometry equivalent to a combination of a function h and a Borel measurable mapping ϕ of $\mathbb{R}$.

2. THE ELEMENTARY LEMMAS.

The norm of a function in $L^1 \cap L^p(\mathbb{R})$, denoted by $\|f\|_\cap$, is defined by

$$\|f\|_\cap = \|f\|_p + \|f\|_1 .$$

A condition equivalent to the equality of norms of f+g and f-g for positive functions of $L^1 \cap L^p(\mathbb{R})$ is given in the following lemma.

LEMMA 1. Let $f, g \in L^1 \cap L^p(\mathbb{R})$ and $f, g \geq 0$. Then

$$\|f + g\|_\cap = \|f - g\|_\cap \iff f.g. = 0 \text{ a.e.}$$

PROOF. From Royden [3] we have

$$\|f + g\|_p^p + \|f - g\|_p^p = 2(\|f\|_p^p + \|g\|_p^p) \iff fg = 0 \text{ a.e.} \tag{2.1}$$

Now, $\|f + g\|_p^p = \int (f+g)^p = \int f^p + \int g^p \iff fg = 0.$

Thus, $$\|f + g\|_p^p = \|f\|_p^p + \|g\|_p^p \iff fg = 0. \tag{2.2}$$

From (2.1) and (2.2), we get

$$\|f + g\|_p^p = \|f - g\|_p^p \iff fg = 0$$

or
$$\|f + g\|_p = \|f - g\|_p \iff fg = 0 \tag{2.3}$$

In particular for $p = 1$ (2.3) becomes

$$\|f + g\|_1 = \|f - g\|_1 \iff fg = 0 \tag{2.4}$$

Addition of (2.3) and (2.4) yields

$$fg = 0 \implies \|f + g\|_{\cap} = \|f - g\|_{\cap} .$$

Conversely, if $\|f + g\|_{\cap} = \|f - g\|_{\cap}$, then

$$(\|f + g\|_p - \|f - g\|_p) + (\|f + g\|_1 - \|f - g\|_1) = 0.$$

Since both of the terms in parentheses are positive, we obtain $fg = 0$ by (2.3) and (2.4).

In the next lemma we show that for positive functions lf $L^1 \cap L^p$ ($\mathbb{R}$) on isometry preserves the disjointness of supports.

LEMMA 2. Let $f, g \in L^1 \cap L^p$ and $f, g \neq 0$. Let U be an isometry on $L^1 \cap L^p$. Then

$$(\text{Supp } f) \cap (\text{Supp } g) = \phi \iff (\text{Supp } Uf) \cap (\text{Supp } Ug) = \phi$$

PROOF. $(\text{Supp } f) \cap (\text{Supp } g) = \phi \iff fg = 0$

$$\iff \|f + g\|_{\cap} = \|f - g\|_{\cap} \qquad \text{(Lemma 1)}$$

$$\iff \|Uf + Ug\|_{\cap} = \|Uf - Ug\|_{\cap}$$

$$\iff (Uf)\,(Ug) = 0$$

$$\iff (\text{Supp } Uf) \cap (\text{Supp } Ug) = \phi.$$

□

3. THE THEOREM.

Let $1 \le p < \infty$, $p \neq 2$ and U be a ono-one onto linear transformation on positive functions of $L^1 \cap L^p$ ($\mathbb{R}$) such that $\|Uf\|_{\cap} = \|f\|_{\cap}$. Then there is a one-one Borel measurable mapping of $\mathbb{R}$ onto itself and a function h such that

$$Uf = h\,(f\,(\phi)) \text{ for all positive } f \in L^1 \cap L^p .$$

PROOF. Let M_0 denote the family of sets of measure zero. Clearly M_0 is a σ-ideal of $\mathcal{B}$, where $\mathcal{B}$ is the family of Borel sets.

For the σ-algebra

$$\mathcal{B}/M_0 = \{ A \mid A = \text{Supp} f, \ f \text{ positive}, \ f \in L^1 \cap L^p\}.$$

Define a map

$$\Phi : \mathcal{B}/M_0 \to \mathcal{B}/M_0 \quad \text{by}$$

$$\Phi(A) = \text{Supp } Ue^{-x^2}\chi_A , \quad \text{where } A = \text{Supp} f.$$

We shall prove that Φ is an σ-isomorphism.
For this we must prove the following:

(i) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$

(ii) $\Phi(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} \Phi(A_i)$

(iii) $\Phi(\mathbb{R}) = \mathbb{R}$

(iv) $\Phi(\bar{A}) = (\overline{\Phi(A)})$ ($\bar{A}$ = the complement of A)

(v) Φ is a bijection

(i) Let $A, B \in \mathcal{B}/M_o$ and $A \cap B = \phi$. Then there are $f, g \in L^1 \cap L^p$ such that $A = \text{supp} f$ and $B = \text{supp} g$. So $(\text{supp} f) \cap (\text{Supp } g) = \phi$. Therefore by lemma 2 we get

$$(\text{Supp } Uf) \cap (\text{Supp } Ug) = \phi \tag{3.1}$$

Since $A \cap B = \phi$, we have $\chi_{A \cup B} = \chi_A + \chi_B$, and therefore

$$Ue^{-x^2}\chi_{A \cup B} = Ue^{-x^2}\chi_A + Ue^{-x^2}\chi_B, \qquad \text{(by (3.1))}$$

This gives,

$$\text{Supp } Ue^{-x^2}\chi_{A \cup B} = \text{Supp } Ue^{-x^2}\chi_A + \text{supp } Ue^{-x^2}\chi_B$$

Thus,

$$\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$$

(ii) Let $(A_i)_{i \in \mathbb{N}}$ be a disjoint family of members of $\mathcal{B}/M_o$ and let $A = \bigcup_{i=1}^{\infty} A_i$, then $\chi_A = \lim \sum_{i=1}^{n} \chi_{A_i}$. So by linearity of U we obtain

$$Ue^{-x^2}\chi_A = \lim \sum_{i=1}^{n} U \chi_{A_i}.$$

Therefore we get

$$\text{Supp } Ue^{-x^2}\chi_A = \lim \sum_{i=1}^{n} \text{supp } U\chi_{A_i}$$

and hence $\Phi(A) = \bigcup_{i=1}^{\infty} \Phi(A_i)$.

(iii) First we show that $\Phi(A) \subset \Phi(B)$ whenever $A \subset B$. For, if $A \subset B$ then, B is the disjoint union of A and B-A and $\chi_B = \chi_A + \chi_{B-A}$. An application of lemma (2) gives

$$\text{Supp } Ue^{-x^2}\chi_B = (\text{Supp } Ue^{-x^2}\chi_A) \cup (\text{Supp } Ue^{-x^2}\chi_{B-A})$$

proving $\Phi(B) = \Phi(A) \cup \Phi(B-A)$ which in turn gives

$$\Phi(A) \subset \Phi(B) \tag{3.2}$$

Now in order to prove $\Phi(\mathbb{R}) = \mathbb{R}$, suppose that $\mathbb{R} - \Phi(\mathbb{R}) = E \neq \phi$, and consider $e^{-x^2}\chi_E \in L^1 \cap L^p$. Since U is onto there exists $f \in L^1 \cap L^p$ such that $Uf = e^{-x^2}\chi_E$. Therefore supp Uf = supp $e^{-x^2}\chi_E = E$, giving $\Phi(A) = E$, where A = supp f. Thus we obtained $\mathbb{R} - \Phi(\mathbb{R}) = \Phi(A)$ which implies $\Phi(A) \not\subset \Phi(\mathbb{R})$, contradicting (3.2). Hence $\Phi(\mathbb{R}) = \mathbb{R}$.

(iv) Since the sum of the characteristic functions on A and its complement is unity we easily obtain (supp $Ue^{-x^2}\chi_A$) $\cup$ (Supp $Ue^{-x^2}\chi_{\bar{A}}$) = supp Ue^{-x^2}. This implies $\Phi(A) \cup \Phi(\bar{A}) = \Phi(\mathbb{R})$ and so using (iii) we get $\Phi(A) \cup \Phi(\bar{A}) = \mathbb{R}$. Further from $\Phi(A) \cap \Phi(\bar{A}) = \phi$ we obtain $\overline{\Phi(A)} = \Phi(\bar{A})$ as required.

(v) If we take $\Phi(A) = \Phi(B)$ then supp $Ue^{-x^2}\chi_A$ = supp $Ue^{-x^2}\chi_B$ and this implies supp $e^{-x^2}\chi_{\bar{A}} \cap$ supp $e^{-x^2}\chi_B = \phi$ by lemma (2). Thus $\bar{A} \cap B = \phi$ which implies $B \subset A$. Interchanging the roles of A and B gives $A \subset B$ so that $A=B$ and thus Φ is one-one.

Now since U is onto, corresponding to $e^{-x^2}\chi_A$, there exists $g \in L^1 \cap L^p$ such that $Ue^{-x^2}\chi_A = g$. Therefore supp $Ue^{-x^2}\chi_A$ = supp$g$ and $\Phi(A)$ = supp$g \in \mathcal{B}/M_o$ proving Φ is onto.

Now it follows from a theorem of Royden [3] that Φ is a σ-isomorphism of $\mathcal{B}/M_o$ onto itself. Thus there is a one-one mapping ϕ of $\mathcal{B}/M_o$ onto itself such that ϕ and ϕ^{-1} are Borel measurable and

$$\Phi(A) = \phi^{-1}(A) \text{ modulo } M_o.$$

Now, consider $\chi_{[0,1]} \in L^1 \cap L^p$ and take $h_1 = U(\chi_{[0,1]})$. If A_1 is any Borel set of $\mathbb{R}$ contained in $[0,1]$ then $\chi_{[0,1]} = \chi_{A_1} + \chi_{[0,1]-A_1}$. So $h_1 = U\chi_{A_1} + U\chi_{[0,1]-A_1}$. But supp χ_{A_1} is disjoint from (supp $\chi_{[0,1]-A_1}$) therefore from lemma (2) we get

$$(\text{supp } U\chi_{A_1}) \cap (\text{supp } U\chi_{[0,1]-A_1}) = \phi$$

That is $U\chi_{A_1}$ equals h_1 on the support $U\chi_{A_1}$.

Therefore
$$U\chi_{A_1} = h_1 \chi_{\text{supp } U\chi_{A_1}}$$

$$= h_1 \chi_{\text{supp } U e^{-x^2}\chi_{A_1}}$$

$$= h_1 \chi_{\Phi(A_1)}$$

$$= h_1(\chi_{A_1}\phi)$$

In general if A_n is a Borel set contained in $[n,n+1]$ where $n \in Z$, then we can have $U\chi_{A_n} = h_n(\chi_{A_n}\phi)$. Further if A is any Borel set of $\mathbb{R}$ then there exists a Borel set of A_n of $\mathbb{R}$ for all n such that $A = \bigcup_{n=-\infty}^{\infty} A_n$, $A_m \cap A_n = \phi$ whenever $m \neq n$.

Now $U\chi_A = U(\chi_{\cup A_n})$

$$= \lim_{n\to\infty} h_n(\chi_A \phi)$$

$$= h(\chi_A \phi), \text{ where } h = \lim_{n\to\infty} h_n.$$

If ψ is any simple function we have

$$U\psi = h(\psi(\phi)).$$

Further, functions in $L^1 \cap L^P(\mathbb{R})$ can be approximated in norm by a simple function, and U is norm preserving, we get

$$U f = h(f(\phi)).$$

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