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ANALYTIC APPROACH TO $q\bar{q}$ SYSTEMS IN POTENTIAL MODELS

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ABSTRACT. Analytic solutions for $q\bar{q}$ systems obtained from a cut-off type approximation to the funnel potential are applied to $b\bar{b}$ and $c\bar{c}$ systems. Perturbative corrections to oscillator energy levels due to inclusion of short range a/r effect are also obtained.

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1. INTRODUCTION.

Bound state characteristics of heavy quark-antiquark pairs are generally well described in the framework of non-relativistic potential models. The funnel potential

$$V_1(r) = - a/r + gr + C \tag{1.1}$$

which takes both confinement and asymptotic freedom into account by the linear and inverse r terms respectively, is generally considered the most appropriate [1]. It has been shown y inverse scattering methods [2], [3], using charmonium and upsilon parameters that quarkonium potentials are flavour-independent for the range 0.1 fm $\leq r \leq 1$ fm. It has also been found that solutions obtained from $V_1(r)$ are generally in qualitative, and often in quantitative agreement with those obtained from the oscillator potential $V_2(r) = Kr^2/2$.

However, $V_2(r)$ completely ignores the short-range Coulombic type potential. As radial wave functions evaluated at the origin are required in expressions for the decay width, it does not seem a good approximation to extrapolate $V_2(r)$ to $r = 0$ when it is $(1/r)$ part of $V_1(r)$ that dominates close to r=0.

In this communication we approximate $V_1(r)$ by a cut-off type potential

$$\begin{aligned} V(r) &= - a/r & 0 \leq r \leq r_1 \\ &= Kr^2/2 & r_1 \leq r \leq r_2 \\ &= 0 & r_2 \leq r < \infty \end{aligned} \tag{1.2}$$

with $K = m\omega_0^2$. m = reduced mass of the system.

We investigate the solution obtained for Quarkonium systems in such a potential, and compare the results with those using $V_1(r)$.

The use of $V(r)$ permits us to obtain exact analytic wave functions which are not possible with $V_1(r)$, and avoids the necessity of neglecting the Coulombic part of $V_1(r)$ that occurs if $V_2(r)$ alone is used. We take a = .27 from funnel potential parameters [3].

We calculate also the perturbation to the oscillator energy due to the inclusion of a $(-a/r)$ potential for $0 \leq r \leq r_1$, relying heavily on the assumption that the colour interaction has a small coupling constant for small r, thereby justifying use of perturbation techniques.

2. THE WAVE FUNCTIONS FOR EACH REGION.

The radial Schrödinger equation which is operative for a potential $(-a/r)$ can be reduced to the confluent form

$$x \frac{\partial^2 y}{\partial x^2} + (C - \chi) \frac{\partial y}{\partial x} - ay = 0 \tag{2.1}$$

with

$$x = \alpha r, \; c = 2\ell + 2, \; a = \ell + 1 - \lambda$$

This yields general solutions $F(a, c, x)$ and $\Psi(a,c,x)$ for integer c which coincides with our case. As $\Psi(a,c,x)$ is not regular at $x = 0$ (or $r = 0$) we reject this, and retain $F(a,c,x)$ as the solution. So,

$$R_1 = A\, e^{-\alpha r/2} \, {}_1F_1(\ell + 1 - \lambda, \, 2\ell + 2, \, \alpha r) \tag{2.2}$$

where

$$\alpha = 2\sqrt{2mE'}, \quad \lambda = a\sqrt{2m}/(2\sqrt{E'}), \quad E' = |E|$$

We assume m = mass of the constituent quarks of the heavy bound quarkonium system.

The radial Schrödinger equation with non-zero ℓ for potential $V_2(r) = Kr^2/2$ can be reduced by suitable substitutions to the form

$$\frac{\partial^2 y}{\partial z^2} + \{(\ell + \tfrac{3}{2}) - z\} \frac{\partial y}{\partial z} - \left(\frac{\lambda' + 2 + 3}{4}\right) y = 0 \tag{2.3}$$

where $z = (\alpha r)^2$ for the range

$$r_1 \leq r \leq r_2.$$

This yields solutions for R as

$$R_2 = B\, e^{-(\alpha' r)^2/2} (\alpha' r)^{\ell} \, {}_1F_1\left(\frac{\lambda' + 2\ell + 3}{4}, \, \ell + 3/2, \, (\alpha' r)^2\right)$$
$$+ C\, e^{-(\alpha' r)^2/2} (\alpha' r)^{-2\ell-2} \, {}_1F_1\left(\frac{\lambda' - 2\ell + 1}{4}, \, \frac{1}{2} - \ell, \, (\alpha' r)^2\right) \tag{2.4}$$

where

$$\alpha' = 4\sqrt{2mK}, \quad \lambda' = E'\sqrt{\frac{2m}{K}}, \quad \hbar = c = 1.$$

3. ESTIMATE OF HEAVY QUARK MASSES.

At the outer boundary, which we expect to be the hadronic radius or confinement radius r_2,

$$R_2 = 0 \tag{3.1}$$

and at the inner boundary

$$R_1 = R_2 \tag{3.2}$$

For values of $E = 3100$ MeV [4], $\omega_o = 300$ MeV [3] we have evaluated m, using the boundary conditions. We note, for m = 1 GeV and r_1 and r_2 are of order of inverse pion mass $(M_\pi)\cdot(\alpha' r_1)^2, > 23$ and $(\alpha\ r_1) > 36$.

So that asymptotic expansion of the confluent hypergeometric functions at the boundary may be considered valid. This yields the ratio of the wave function and their derivatives at the boundary, as the equation

$$H\ [\frac{\alpha}{2} - (1 + \lambda/\alpha\ r)] = \frac{B}{\Gamma(\frac{\lambda' + 3}{4})}[H_1 + H_2 + H_3] + H_4 \tag{3.3}$$

where

$$H_1 = [-\alpha'/2(\alpha' r_1)^2],\ H_2 = [1/2\ (\alpha' r_1)^2],$$

$$H_3 = [\alpha'^2 r_1 + (\lambda'/2 - 1/2)/(\alpha' r_1)]/[2(\alpha' r_1)]$$

$$H_4 = C\ [(\alpha'^2 r_1) + (\lambda'/\ 2 - 1/2)/(\alpha' r_1)]/\Gamma(\frac{\lambda' + 1}{4})$$

$$((\lambda + 1)/\alpha r) << \alpha/2 \text{ and } [(\lambda'/2 - 1/2)/(\alpha' r_1)] << \alpha'^2 r_1$$

$$H = [B/\{2\alpha' r_1\ \Gamma\ (\frac{\lambda' + 3}{4})\}] + [C/\Gamma\ [(\frac{\lambda + 1}{4})]$$

From (3.2) we have B = pc, where

$$p = -\frac{1}{\alpha'^2 r_2^2}\ \frac{(\frac{\lambda' + 3}{4})\ \Gamma\ (1/2)(\alpha' r_2)^{1/2}}{(1/2)\ \Gamma\ (\frac{\lambda' + 1}{4})\ \Gamma(1/2)} \tag{3.4}$$

$$p = -\ .06 \quad \text{for} \quad m \simeq 1 \text{ GeV}.$$

Equation (3.3) can be reduced to

$$\sqrt{2mE'} = \frac{p\ 4\sqrt{2}\ \sqrt{m\omega_0}}{2}\ \frac{\Gamma(\frac{\lambda' + 1}{4})}{\Gamma(\frac{\lambda' + 3}{4})} + r_1\sqrt{2}\ m\omega_0 \tag{3.5}$$

Neglecting the 1st term on the r.h.s., since it is of order .02m, compared to $\sim 1.5m^3$ and $87m^2\sqrt{m}$, ($m \sim 1$ GeV), we estimate

$$\sqrt{m} = \sqrt{E'} / r_1 \omega_0$$

Thus m is sensitive to the value of r_1 and ω_0, and is not dependent much on r_2 and on the magnitude of a, when a is small.

We find the funnel potential prediction of mass m = 1650 MeV is obtained for $r_1 = .0046\ \mathrm{MeV}^{-1}$ when $\omega_0 = 300$ MeV.

We can then choose $r_2 = 1/M_\pi = .0073\ \mathrm{MeV}^{-1}$. Alternatively, if we wish to fix r_1 at 1/2 fermi = $.0025\ \mathrm{MeV}^{-1}$, which is assumed generally to be the region where inverse r behaviour falls off and the linear confining effect sets in, we get a much larger value for m = 5.5116 GeV.

Further, from $|R_{1s}(r=0)|^2 = 0.5\ (\mathrm{GeV})^2$ we can get straight away $A = \sqrt{5 \times 10^6} = 707.107$ from Eq. (2.2). Similarly B and C may be found.

If we assume the Upsilon Υ potential to be of the same form with the same value of ω_0 as obtained by level fitting from charmonium a value of ~ 5 GeV for the mass of the beauty quark is obtained.

A value 372 MeV for ω_0 obtained by level fitting of 1S, 2S levels of Υ yields a lower value of m_b.

For the case of Upsilon, using E = 9460 [4], we obtain by similar analysis m = 4967.416 for the beauty quark with Coulombic radius $= 0.0046\ \mathrm{MeV}^{-1}$ while a value 372 MeV for ω_0 from ν level fitting yields $m_q = 3230.647$ MeV for $r_1 = 0.0046\ \mathrm{MeV}^{-1}$.

The analytic solutions obtained can be used in problems involving the non-relativistic potential models for heavy quarkonium systems assuming the potential to be flavour independent.

4. CORRECTIONS TO ENERGY LEVELS.

We now estimate the correction to the first oscillator energy level due to the perturbation by the Coulombic potential for r between zero and r_1.

The perturbation energy for the L = 0, J = 0 level may be written

$$\Delta E = \int (V-V_0)\, \Psi_0\, dr \tag{4.1}$$

where Ψ_0 = the unperturbed wave function and the perturbation potential $(V-V_0)$ is

$$\begin{aligned}(V-V_0) &= a/r - V_0 && \text{for} \quad 0 < r < r_1 \\ &= 0 && \text{for} \quad r_1 < r < r_2 \\ &= -V_0 && \text{for} \quad r_2 < r < \alpha\end{aligned}$$

where V_0 = unperturbed potential.

Due to confinement $\Psi_0 = 0$ for $r > r_2$, and we have eventually

$$\Delta E = 4\pi(\alpha'/2\pi V)\,[-a\, I_1 - K\, I_2/2] \tag{4.2}$$

where $I_1 = \int_0^{r_1} r\, e^{-br^2} dr$ and $I_2 = \int_0^{r_1} r^4\, e^{-br^2}\, dr$ We have used for the unperturbed wavefunction

$$\Psi_n = \left(\frac{\alpha}{\sqrt{2}\; 2^n n!}\right)^{1/2} e^{-(\alpha\, r/2)^2}\, H_n\,(\alpha r) \tag{4.3}$$

as given by [6].

I_1 and I_2 are evaluated using the error function [6]. We get, for r_1 = - 0026 ~ .5 fm, ω_0 = 300 MeV, m_c = 1.17 GeV.

$$\Delta E = -\ 0.002 \text{ MeV}$$

Alternatively if we take r_1 = .0020, m_q = 1.65 MeV, retaining ω_0 = 300 MeV, we obtain

$$\Delta E = -\ 0.0035 \text{ MeV}$$

The small magnitude of the perturbation energy shows that perturbation corrections are justified. However, if the short range coupling was greater than a = .27, the perturbation energy would increase correspondingly through (aI_1) term.

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