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Research Article

Dynamics of a Computer Virus Propagation Model with Delays and Graded Infection Rate

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A four-compartment computer virus propagation model with two delays and graded infection rate is investigated in this paper. The critical values where a Hopf bifurcation occurs are obtained by analyzing the distribution of eigenvalues of the corresponding characteristic equation. In succession, direction and stability of the Hopf bifurcation when the two delays are not equal are determined by using normal form theory and center manifold theorem. Finally, some numerical simulations are also carried out to justify the obtained theoretical results.

1. Introduction

In recent years, with the fast development and popularization of computer technologies and network, Internet has offered numerous functionalities and facilities to the world. Meanwhile, Internet has also become a powerful mechanism for propagating computer viruses. Computer viruses are computer programs which have serious effects on individual and corporate computer systems in the network, such as modifying data and formatting disks [1, 2].

In order to analyze the propagation laws of computer viruses in the network, many epidemiological models have been borrowed to depict the spread of computer viruses because of the high similarity between the computer viruses and the biological viruses [3–5]. In [6–11], Mishra et al. proposed SIRS computer virus models in different forms. Yuan and Chen presented the SEIR computer virus propagation model in [12] and they studied the stability of the model. Based on the the work in [12], Dong et al. proposed the SEIR computer virus model with time delay in [13] and they investigated the Hopf bifurcation of the model. There are also some other different computer virus models which have been proposed by other scholars in recent years and one can refer to [14–18]. However, all the computer virus models above which incorporate the latent status of the viruses assume that the latent computers have no infection ability. This is not consistent with the reality, because an infected computer which is in latency can also infect other computers through file copying or file downloading. Based on this fact, Yang et al. established a computer virus propagation model with graded infection rate in [19]:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \mu - \beta_1 S(t) L(t) - \beta_2 S(t) A(t) + \alpha R(t) \\
&\quad - \mu S(t), \\
\frac{dL(t)}{dt} &= \beta_1 S(t) L(t) + \beta_2 S(t) A(t) - \varepsilon L(t) - \mu L(t), \\
\frac{dA(t)}{dt} &= \varepsilon L(t) - \gamma A(t) - \mu A(t), \\
\frac{dR(t)}{dt} &= \gamma A(t) - \alpha R(t) - \mu R(t),
\end{aligned} \tag{1}$$

where $S(t)$, $L(t)$, $A(t)$, and $R(t)$ are the percentages of susceptible computers, latent computers, active computers, and recovered computers on the Internet, at time t, respectively. μ is the rate at which external computers are connected to the Internet and it is also the rate at which internal computers are disconnected from the Internet; β_1 is the infected rate of the susceptible computers by the latent computers; β_2 is the infected rate of the susceptible computers by the active computers; α is the rate at which the recovered computers become susceptibly virus-free again; ε is the rate at which the latent computers break out; and γ is the rate at which the active computers are cured by the antivirus software.

As pointed out in [9], one of the typical features of computer viruses is their latent characteristic. Therefore, they need a period to become active computers for the latent ones. Likewise, the antivirus software needs a period to clean the viruses in the active computers. Based on this and motivated by the work about the dynamical system with delay in [20–24], we incorporate two delays into system (1) and obtain the following delayed computer virus model:

$$\begin{aligned}\frac{dS(t)}{dt} &= \mu - \beta_1 S(t) L(t) - \beta_2 S(t) A(t) + \alpha R(t) - \mu S(t),\\ \frac{dL(t)}{dt} &= \beta_1 S(t) L(t) + \beta_2 S(t) A(t) - \varepsilon L(t-\tau_1) - \mu L(t),\\ \frac{dA(t)}{dt} &= \varepsilon L(t-\tau_1) - \gamma A(t-\tau_2) - \mu A(t),\\ \frac{dR(t)}{dt} &= \gamma A(t-\tau_2) - \alpha R(t) - \mu R(t),\end{aligned} \tag{2}$$

where τ_1 is the latent period of the computer viruses and τ_2 is the period that the antivirus software needs to clean the viruses in the active computers.

The rest of this paper is organized as follows. In Section 2, we present the existence of the viral equilibrium and conditions for the local stability of the viral equilibrium and existence of the Hopf bifurcation are derived. Direction and stability of the Hopf bifurcation are studied in Section 3 and some numerical simulations are performed in Section 4 to justify the obtained theoretical findings by taking some relevant values of the parameters in system (2) and using the Matlab software package. Finally, we end this paper with concluding remarks in Section 5.

2. Existence of Local Hopf Bifurcation

By a simple computation, we know that if $(\varepsilon+\mu)(\gamma+\mu)/(\beta_1(\gamma+\mu)+\beta_2\varepsilon) < 1$ and $(\varepsilon+\mu)(\gamma+\mu)/\varepsilon > \alpha\gamma/(\alpha+\mu)$, then system (2) has a unique viral equilibrium $E_*(S_*, L_*, A_*, R_*)$, where

$$\begin{aligned}S_* &= \frac{(\gamma+\mu)(\varepsilon+\mu)}{\beta_1(\gamma+\mu)+\beta_2\varepsilon},\\ L_* &= \frac{\gamma+\mu}{\varepsilon} A_*,\\ R_* &= \frac{\gamma}{\alpha+\mu} A_*,\\ A_* &= \frac{A_{1*}}{A_{2*}},\\ A_{1*} &= \mu - \frac{\mu(\varepsilon+\mu)(\gamma+\mu)}{\beta_1(\gamma+\mu)+\beta_2\varepsilon},\\ A_{2*} &= \frac{(\varepsilon+\mu)(\gamma+\mu)}{\varepsilon} - \frac{\alpha\gamma}{\alpha+\mu}.\end{aligned} \tag{3}$$

Let $\overline{S}(t) = S(t) - S_*$, $\overline{L}(t) = L(t) - L_*$, $\overline{A}(t) = A(t) - A_*$, $\overline{R}(t) = R(t) - R_*$. Dropping the bars, system (2) becomes

$$\begin{aligned}\frac{dS(t)}{dt} &= a_1 S(t) + a_2 L(t) + a_3 A(t) + a_4 R(t) - \beta_1 S(t) L(t) - \beta_2 S(t) A(t),\\ \frac{dL(t)}{dt} &= a_5 S(t) + a_6 L(t) + a_7 A(t) + b_1 L(t-\tau_1) + \beta_1 S(t) L(t) + \beta_2 S(t) A(t),\\ \frac{dA(t)}{dt} &= a_8 A(t) + b_2 L(t-\tau_1) + c_1 A(t-\tau_2),\\ \frac{dR(t)}{dt} &= a_9 R(t) + c_2 A(t-\tau_2),\end{aligned} \tag{4}$$

where

$$\begin{aligned}a_1 &= -(\beta_1 L_* + \beta_2 A_* + \mu),\\ a_2 &= -\beta_1 S_*,\\ a_3 &= -\beta_2 S_*,\\ a_4 &= \alpha,\\ a_5 &= \beta_1 L_* + \beta_2 A_*,\\ b_1 &= -\varepsilon,\\ a_6 &= \beta_1 S_* - \mu,\\ a_7 &= \beta_2 S_*,\\ a_8 &= -\mu,\\ b_2 &= \varepsilon,\\ c_1 &= -\gamma,\\ a_9 &= -(\alpha+\mu) R_*,\\ c_2 &= \gamma.\end{aligned} \tag{5}$$

The linear system of system (4) is

$$\begin{aligned}\frac{dS(t)}{dt} &= a_1 S(t) + a_2 L(t) + a_3 A(t) + a_4 R(t),\\ \frac{dL(t)}{dt} &= a_5 S(t) + a_6 L(t) + a_7 A(t) + b_1 L(t-\tau_1),\\ \frac{dA(t)}{dt} &= a_8 A(t) + b_2 L(t-\tau_1) + c_1 A(t-\tau_2),\\ \frac{dR(t)}{dt} &= a_9 R(t) + c_2 A(t-\tau_2).\end{aligned} \tag{6}$$

The corresponding characteristic equation is

$$\begin{aligned}\lambda^4 &+ a_{03}\lambda^3 + a_{02}\lambda^2 + a_{01}\lambda + a_{00}\\ &+ \left(b_{03}\lambda^3 + b_{02}\lambda^2 + b_{01}\lambda + b_{00}\right)e^{-\lambda\tau_1}\\ &+ \left(c_{03}\lambda^3 + c_{02}\lambda^2 + c_{01}\lambda + c_{00}\right)e^{-\lambda\tau_2}\\ &+ \left(d_{02}\lambda^2 + d_{01}\lambda + d_{00}\right)e^{-\lambda(\tau_1+\tau_2)} = 0,\end{aligned} \quad (7)$$

where

$$\begin{aligned}a_{00} &= (a_1a_6 - a_2a_5)\,a_8a_9,\\ a_{01} &= (a_2a_5 - a_1a_6)(a_8 + a_9),\\ a_{02} &= a_1a_6 - a_2a_5 + a_8a_9 + (a_1 + a_6)(a_8 + a_9),\\ a_{03} &= -(a_1 + a_6 + a_8 + a_9),\\ b_{00} &= a_1a_8a_9b_1 + (a_3a_5 - a_1a_7)\,a_9b_2,\\ b_{01} &= a_7b_2(a_1 + a_9) - b_1(a_1a_8 + a_1a_9 + a_8a_9),\\ b_{02} &= b_1(a_1 + a_8 + a_9) - a_7b_2,\\ b_{03} &= -b_1,\\ c_{00} &= (a_1a_6 - a_2a_5)\,a_9c_1,\\ c_{01} &= c_1(a_2a_5 - a_1a_6 - a_1a_9 - a_6a_9),\\ c_{02} &= c_1(a_1 + a_6 + a_9),\\ c_{03} &= -c_1,\\ d_{00} &= a_1a_9b_1c_1 + a_4a_5b_2c_2,\\ d_{01} &= -b_1c_1(a_1 + a_9),\\ d_{02} &= b_1c_1.\end{aligned} \quad (8)$$

Case 1 ($\tau_1 = \tau_2 = 0$). For $\tau_1 = \tau_2 = 0$, (7) becomes

$$\lambda^4 + a_{13}\lambda^3 + a_{12}\lambda^2 + a_{11}\lambda + a_{10} = 0, \quad (9)$$

where

$$\begin{aligned}a_{10} &= a_{00} + b_{00} + c_{00} + d_{00},\\ a_{11} &= a_{01} + b_{01} + c_{01} + d_{01},\\ a_{12} &= a_{02} + b_{02} + c_{02} + d_{02},\\ a_{13} &= a_{03} + b_{03} + c_{03}.\end{aligned} \quad (10)$$

Thus, according to the Routh-Hurwithz theorem, we know that if conditions (H_{11}) $a_{10} > 0$, $a_{13} > 0$, and $a_{13}a_{12} > a_{11}$ hold, then viral equilibrium $E_*(S_*, L_*, A_*, R_*)$ of system (2) without delay is locally asymptotically stable.

Case 2 ($\tau_1 > 0, \tau_2 = 0$). For $\tau_1 > 0$ and $\tau_2 = 0$, we can get the following from (7):

$$\begin{aligned}\lambda^4 &+ a_{23}\lambda^3 + a_{22}\lambda^2 + a_{21}\lambda + a_{20}\\ &+ \left(b_{23}\lambda^3 + b_{22}\lambda^2 + b_{21}\lambda + b_{20}\right)e^{-\lambda\tau_1} = 0,\end{aligned} \quad (11)$$

where

$$\begin{aligned}a_{20} &= a_{00} + c_{00},\\ a_{21} &= a_{01} + c_{01},\\ a_{22} &= a_{02} + c_{02},\\ a_{23} &= a_{03} + c_{03},\\ b_{20} &= b_{00} + d_{00},\\ b_{21} &= b_{01} + d_{01},\\ b_{22} &= b_{02} + d_{02},\\ b_{23} &= b_{03}.\end{aligned} \quad (12)$$

We assume that $\lambda = i\omega_1$ $(\omega_1 > 0)$ is a root of (11). Then,

$$\begin{aligned}&\left(b_{21}\omega_1 - b_{23}\omega_1^3\right)\sin\tau_1\omega_1 + \left(b_{20} - b_{22}\omega_1^2\right)\cos\tau_1\omega_1\\ &\quad = a_{22}\omega_1^2 - \omega_1^4 - a_{20},\\ &\left(b_{21}\omega_1 - b_{23}\omega_1^3\right)\cos\tau_1\omega_1 - \left(b_{20} - b_{22}\omega_1^2\right)\sin\tau_1\omega_1\\ &\quad = a_{23}\omega_1^3 - a_{21}\omega_1,\end{aligned} \quad (13)$$

which implies that

$$\omega_1^8 + g_{23}\omega_1^6 + g_{22}\omega_1^4 + g_{21}\omega_1^2 + g_{20} = 0, \quad (14)$$

with

$$\begin{aligned}g_{20} &= a_{20}^2 - b_{20}^2,\\ g_{21} &= a_{21}^2 - b_{21}^2 - 2a_{20}a_{22} + 2b_{20}b_{22},\\ g_{22} &= a_{22}^2 - b_{22}^2 + 2b_{21}b_{23} - 2a_{21}a_{23} + 2a_{20},\\ g_{23} &= a_{23}^2 - b_{23}^2 - 2a_{22}.\end{aligned} \quad (15)$$

Let $\omega_1^2 = v_1$; then (14) becomes

$$v_1^4 + g_{23}v_1^3 + g_{22}v_1^2 + g_{21}v_1 + g_{20} = 0. \quad (16)$$

Discussion about distribution of roots for (16) is similar to that in [25]. Therefore, we directly assume that (H_{21}) (16) has at least one positive equilibrium v_{10}.

If (H_{21}) holds, we know that (14) has at least one positive root $\omega_{10} = \sqrt{v_{10}}$ such that (11) has a pair of purely imaginary roots $\pm i\omega_{10}$. For ω_{10},

$$\tau_{10} = \frac{1}{\omega_{10}}\arccos\frac{h_{21}(\omega_{10})}{h_{22}(\omega_{10})}, \quad (17)$$

where

$$h_{21}(\omega_{10}) = (b_{22} - a_{23}b_{23})\,\omega_{10}^6 + (a_{21}b_{23} + a_{23}b_{21} - a_{22}b_{22} - b_{20})\,\omega_{10}^4 + (a_{20}b_{22} + a_{22}b_{20} - a_{21}b_{21})\,\omega_{10}^2 - a_{20}b_{20},$$
$$h_{22}(\omega_{10}) = b_{23}^2\omega_{10}^6 + \left(b_{22}^2 - 2b_{21}b_{23}\right)\omega_{10}^4 + \left(b_{21}^2 - 2b_{20}b_{22}\right)\omega_{10}^2 + b_{20}^2. \tag{18}$$

Differentiating both sides of (11) with respect to τ_1, one can obtain

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = -\frac{4\lambda^3 + 3a_{23}\lambda^2 + 2a_{22}\lambda + a_{21}}{\lambda\left(\lambda^4 + a_{23}\lambda^3 + a_{22}\lambda^2 + a_{21}\lambda + a_{20}\right)} + \frac{3b_{23}\lambda^2 + 2b_{22}\lambda + b_{21}}{\lambda\left(b_{23}\lambda^3 + b_{22}\lambda^2 + b_{21}\lambda + b_{20}\right)} - \frac{\tau_1}{\lambda}. \tag{19}$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{10}} = \frac{f_1'(v_1^*)}{h_{22}(\omega_{10})}, \tag{20}$$

where $v_1^* = \omega_{10}^2$ and $f_1(v_1) = v_1^4 + g_{23}v_1^3 + g_{22}v_1^2 + g_{21}v_1 + g_{20}$. Therefore, if condition (H_{22}) $f_1'(v_1^*) \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_1]^{-1}_{\tau_1=\tau_{10}} \neq 0$. Based on the discussion above and according to the Hopf bifurcation theorem in [26], we obtain the following.

Theorem 1. *If conditions (H_{21})-(H_{22}) hold, then*

(i) *viral equilibrium $E_*(S_*, L_*, A_*, R_*)$ of system (2) is locally asymptotically stable for $\tau_1 \in [0, \tau_{10})$;*

(ii) *system (2) undergoes a Hopf bifurcation at viral equilibrium $E_*(S_*, L_*, A_*, R_*)$ when $\tau_1 = \tau_{10}$ and a family of periodic solutions bifurcate from $E_*(S_*, L_*, A_*, R_*)$.*

Case 3 $(\tau_1 = 0, \tau_2 > 0)$. For $\tau_1 = 0$ and $\tau_2 > 0$, (7) becomes

$$\lambda^4 + a_{33}\lambda^3 + a_{32}\lambda^2 + a_{31}\lambda + a_{30} + \left(c_{33}\lambda^3 + c_{32}\lambda^2 + c_{31}\lambda + c_{30}\right)e^{-\lambda\tau_2} = 0, \tag{21}$$

where

$$a_{30} = a_{00} + b_{00},\quad a_{31} = a_{01} + b_{01},\quad a_{32} = a_{02} + b_{02},\quad a_{33} = a_{03} + b_{03},$$
$$c_{30} = c_{00} + d_{00},\quad c_{31} = c_{01} + d_{01},\quad c_{32} = c_{02} + d_{02},\quad c_{33} = c_{03}. \tag{22}$$

Let $\lambda = i\omega_2$ $(\omega_2 > 0)$ be a root of (21). Then,

$$\left(c_{31}\omega_2 - c_{33}\omega_2^3\right)\sin\tau_2\omega_2 + \left(c_{30} - c_{32}\omega_2^2\right)\cos\tau_2\omega_2 = a_{32}\omega_2^2 - \omega_2^4 - a_{30},$$
$$\left(c_{31}\omega_2 - c_{33}\omega_2^3\right)\cos\tau_2\omega_2 - \left(c_{30} - c_{32}\omega_2^2\right)\sin\tau_2\omega_2 = a_{33}\omega_2^3 - a_{31}\omega_2. \tag{23}$$

It follows that

$$\omega_2^8 + g_{33}\omega_2^6 + g_{32}\omega_2^4 + g_{31}\omega_2^2 + g_{30} = 0, \tag{24}$$

with

$$g_{30} = a_{30}^2 - c_{30}^2,$$
$$g_{31} = a_{31}^2 - c_{31}^2 - 2a_{30}a_{32} + 2c_{30}b_{32},$$
$$g_{32} = a_{32}^2 - c_{32}^2 + 2c_{31}c_{33} - 2a_{31}a_{33} + 2a_{30},$$
$$g_{33} = a_{33}^2 - b_{33}^2 - 2a_{32}. \tag{25}$$

Let $\omega_2^2 = v_2$; then, we have

$$v_2^4 + g_{33}v_2^3 + g_{32}v_2^2 + g_{31}v_2 + g_{30} = 0. \tag{26}$$

Similar to Case 2, we make the following assumption. (H_{31}) (26) has at least one positive root v_{20}. If condition (H_{31}) holds, then there exists $\omega_{20} = \sqrt{v_{20}}$ such that (21) has a pair of purely imaginary roots $\pm i\omega_{20}$. For ω_{20},

$$\tau_{20} = \frac{1}{\omega_{20}}\arccos\frac{h_{31}(\omega_{20})}{h_{32}(\omega_{20})}, \tag{27}$$

where

$$h_{31}(\omega_{20}) = (c_{32} - a_{33}c_{33})\,\omega_{20}^6 + (a_{31}c_{33} - a_{32}c_{32} + a_{33}c_{31} - c_{30})\,\omega_{20}^4 + (a_{30}c_{32} - a_{31}c_{31} + a_{32}c_{30})\,\omega_{20}^2 - a_{30}c_{30},$$
$$h_{32}(\omega_{20}) = c_{33}^2\omega_{20}^6 + \left(b_{32}^2 - 2c_{31}c_{33}\right)\omega_{20}^4 + \left(c_{31}^2 - 2c_{30}c_{32}\right)\omega_{20}^2 + c_{30}^2. \tag{28}$$

In addition, we have

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = -\frac{4\lambda^3 + 3a_{33}\lambda^2 + 2a_{32}\lambda + a_{31}}{\lambda\left(\lambda^4 + a_{33}\lambda^3 + a_{32}\lambda^2 + a_{31}\lambda + a_{30}\right)} + \frac{3c_{33}\lambda^2 + 2c_{32}\lambda + c_{31}}{\lambda\left(c_{33}\lambda^3 + c_{32}\lambda^2 + c_{31}\lambda + c_{30}\right)} - \frac{\tau_2}{\lambda}. \tag{29}$$

Further,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\tau_2=\tau_{20}} = \frac{f_2'(v_2^*)}{h_{32}(\omega_{20})}, \tag{30}$$

where $v_2^* = \omega_{20}^2$ and $f_2(v_2) = v_2^4 + g_{33}v_2^3 + g_{32}v_2^2 + g_{31}v_2 + g_{30}$. Therefore, if condition (H_{32}) $f_2'(v_2^*) \neq 0$ holds, then $\mathrm{Re}[d\lambda/d\tau_2]^{-1}_{\tau_2=\tau_{20}} \neq 0$. Based on the discussion above and according to the Hopf bifurcation theorem in [26], we obtain the following.

Theorem 2. *If conditions (H_{31})-(H_{32}) hold, then*

(i) *viral equilibrium $E_*(S_*, L_*, A_*, R_*)$ of system (2) is locally asymptotically stable for $\tau_2 \in [0, \tau_{20})$;*

(ii) *system (2) undergoes a Hopf bifurcation at viral equilibrium $E_*(S_*, L_*, A_*, R_*)$ when $\tau_2 = \tau_{20}$ and a family of periodic solutions bifurcate from $E_*(S_*, L_*, A_*, R_*)$.*

Case 4 ($\tau_1 = \tau_2 = \tau > 0$). For $\tau_1 = \tau_2 = \tau > 0$, we have

$$\begin{aligned}\lambda^4 &+ a_{43}\lambda^3 + a_{42}\lambda^2 + a_{41}\lambda + a_{40} \\ &+ \left(b_{43}\lambda^3 + b_{42}\lambda^2 + b_{41}\lambda + b_{40}\right)e^{-\lambda\tau} \\ &+ \left(d_{42}\lambda^2 + d_{41}\lambda + d_{40}\right)e^{-2\lambda\tau} = 0,\end{aligned} \tag{31}$$

with

$$\begin{aligned} a_{40} &= a_{00}, \\ a_{41} &= a_{01}, \\ a_{42} &= a_{02}, \\ a_{43} &= a_{03}, \\ b_{40} &= b_{00} + c_{00}, \\ b_{41} &= b_{01} + c_{01}, \\ b_{42} &= b_{02} + c_{02}, \\ b_{43} &= b_{03} + c_{03}, \\ d_{40} &= d_{00}, \\ d_{41} &= d_{01}, \\ d_{42} &= d_{02}. \end{aligned} \tag{32}$$

Multiplying by $e^{\lambda\tau}$, (31) becomes the following:

$$\begin{aligned} b_{43}\lambda^3 &+ b_{42}\lambda^2 + b_{41}\lambda + b_{40} \\ &+ \left(\lambda^4 + a_{43}\lambda^3 + a_{42}\lambda^2 + a_{41}\lambda + a_{40}\right)e^{\lambda\tau} \\ &+ \left(d_{42}\lambda^2 + d_{41}\lambda + d_{40}\right)e^{-\lambda\tau} = 0. \end{aligned} \tag{33}$$

Let $\lambda = i\omega$ $(\omega > 0)$ be a root of (37); it is easy to get

$$\begin{aligned} g_{41}(\omega)\cos\tau\omega - g_{42}(\omega)\sin\tau\omega &= g_{43}(\omega), \\ g_{44}(\omega)\sin\tau\omega + g_{45}(\omega)\cos\tau\omega &= g_{46}(\omega), \end{aligned} \tag{34}$$

where

$$\begin{aligned} g_{41}(\omega) &= \omega^4 - (a_{42} + d_{42})\omega^2 + a_{40} + d_{40}, \\ g_{42}(\omega) &= (a_{41} - d_{41})\omega - a_{43}\omega^3, \\ g_{43}(\omega) &= b_{42}\omega^2 - b_{40}, \\ g_{44}(\omega) &= \omega^4 - (a_{42} - d_{42})\omega^2 + a_{40} - d_{40}, \\ g_{45}(\omega) &= (a_{41} + d_{41})\omega - a_{43}\omega^3, \\ g_{46}(\omega) &= b_{43}\omega^3 - b_{41}\omega. \end{aligned} \tag{35}$$

It leads to

$$\begin{aligned} \cos\tau\omega &= \frac{g_{42}(\omega)\times g_{46}(\omega) + g_{43}(\omega)\times g_{44}(\omega)}{g_{41}(\omega)\times g_{44}(\omega) + g_{42}(\omega)\times g_{45}(\omega)}, \\ \sin\tau\omega &= \frac{g_{41}(\omega)\times g_{46}(\omega) - g_{43}(\omega)\times g_{45}(\omega)}{g_{41}(\omega)\times g_{44}(\omega) + g_{42}(\omega)\times g_{45}(\omega)}. \end{aligned} \tag{36}$$

Thus, we can get the following equation with respect to ω:

$$\cos^2\tau\omega + \sin^2\tau\omega = 1. \tag{37}$$

Next, we make the following assumption. (H_{41}) (37) has at least one positive root ω_0. Then, for ω_0, we have

$$\begin{aligned} \tau_0 &= \frac{1}{\omega_0} \\ &\cdot \arccos\frac{g_{42}(\omega_0)\times g_{46}(\omega_0) + g_{43}(\omega_0)\times g_{44}(\omega_0)}{g_{41}(\omega_0)\times g_{44}(\omega_0) + g_{42}(\omega_0)\times g_{45}(\omega_0)}. \end{aligned} \tag{38}$$

Taking the derivative of λ with respect to τ, we obtain

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{g_{47}(\lambda)}{g_{48}(\lambda)} - \frac{\tau}{\lambda}, \tag{39}$$

where

$$\begin{aligned} g_{47}(\lambda) &= 3b_{43}\lambda^2 + 2b_{42}\lambda + b_{41} \\ &\quad + \left(4\lambda^3 + 3a_{43}\lambda^2 + 2a_{42}\lambda + a_{41}\right)e^{\lambda\tau} \\ &\quad + (2d_{42}\lambda + d_{41})e^{-\lambda\tau}, \\ g_{48}(\lambda) &= \left(\lambda^5 + a_{43}\lambda^4 + a_{42}\lambda^3 + a_{41}\lambda^2 + a_{40}\lambda\right)e^{\lambda\tau} \\ &\quad - \left(d_{42}\lambda^3 + d_{41}\lambda^2 + d_{40}\lambda\right)e^{-\lambda\tau}. \end{aligned} \tag{40}$$

Then, we can get that

$$\begin{aligned} \mathrm{Re}&\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} \\ &= -\frac{h_{41}(\omega_0)\times h_{43}(\omega_0) + h_{42}(\omega_0)\times h_{44}(\omega_0)}{h_{43}^2(\omega_0) + h_{44}^2(\omega_0)}, \end{aligned} \tag{41}$$

where

$$\begin{aligned}
&h_{41}(\omega_0)\\
&= \left(a_{41} + d_{41} - 3a_{43}\omega_0^2\right)\cos\tau_0\omega_0\\
&\quad - 2\left(\left(a_{42} - d_{42}\right)\omega_0 - 2\omega_0^3\right)\sin\tau_0\omega_0 + b_{41}\\
&\quad - 3b_{43}\omega_0^2,\\
&h_{42}(\omega_0)\\
&= \left(a_{41} - d_{41} - 3a_{43}\omega_0^2\right)\sin\tau_0\omega_0\\
&\quad + 2\left(\left(a_{42} + d_{42}\right)\omega_0 - 2\omega_0^3\right)\cos\tau_0\omega_0 + 2b_{42}\omega_0,\\
&h_{43}(\omega_0)\\
&= \left(a_{43}\omega_0^4 - \left(a_{41} + d_{41}\right)\omega_0^2\right)\cos\tau_0\omega_0\\
&\quad - \left(\omega_0^5 - \left(a_{42} + d_{42}\right)\omega_0^3 + \left(a_{40} - d_{40}\right)\omega_0\right)\sin\tau_0\omega_0,\\
&h_{44}(\omega_0)\\
&= \left(a_{43}\omega_0^4 - \left(a_{41} - d_{41}\right)\omega_0^2\right)\sin\tau_0\omega_0\\
&\quad + \left(\omega_0^5 - \left(a_{42} + d_{42}\right)\omega_0^3 + \left(a_{40} + d_{40}\right)\omega_0\right)\cos\tau_0\omega_0.
\end{aligned} \tag{42}$$

We assume that (H_{42}) $h_{41}(\omega_0) \times h_{43}(\omega_0) + h_{42}(\omega_0) \times h_{44}(\omega_0) \neq 0$. Clearly, if condition (H_{42}) holds, then we can conclude that $\mathrm{Re}[d\lambda/d\tau]^{-1}_{\tau=\tau_0} \neq 0$. Therefore, according to the Hopf bifurcation theorem in [26], we obtain the following.

Theorem 3. *If conditions (H_{41})-(H_{42}) hold, then*

(i) *viral equilibrium $E_*(S_*, L_*, A_*, R_*)$ of system (2) is locally asymptotically stable for $\tau \in [0, \tau_0)$;*

(ii) *system (2) undergoes a Hopf bifurcation at viral equilibrium $E_*(S_*, L_*, A_*, R_*)$ when $\tau = \tau_0$ and a family of periodic solutions bifurcate from $E_*(S_*, L_*, A_*, R_*)$.*

Case 5 ($\tau_1 > 0$, $\tau_2 \in (0, \tau_{20})$). Let $\lambda = i\omega_1'$ be the root of (7). Then,

$$\begin{aligned}
g_{51}(\omega_1')\sin\tau_1\omega_1' + g_{52}(\omega_1')\cos\tau_1\omega_1' &= g_{53}(\omega_1'),\\
g_{51}(\omega_1')\cos\tau_1\omega_1' - g_{52}(\omega_1')\sin\tau_1\omega_1' &= g_{54}(\omega_1'),
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
g_{51}(\omega_1') &= b_{01}\omega_1' - b_{03}(\omega_1')^3 + d_{01}\omega_1'\cos\tau_2\omega_1'\\
&\quad - \left(d_{00} - d_{02}(\omega_1')^2\right)\sin\tau_2\omega_1',\\
g_{52}(\omega_1') &= b_{00} - b_{02}(\omega_1')^2 + d_{01}\omega_1'\sin\tau_2\omega_1'\\
&\quad + \left(d_{00} - d_{02}(\omega_1')^2\right)\cos\tau_2\omega_1',\\
g_{53}(\omega_1') &= a_{02}(\omega_1')^2 - (\omega_1')^4 - a_{00}\\
&\quad - \left(c_{01}\omega_1' - c_{03}(\omega_1')^3\right)\sin\tau_2\omega_1'\\
&\quad - \left(c_{00} - c_{02}(\omega_1')^2\right)\cos\tau_2\omega_1',\\
g_{54}(\omega_1') &= a_{03}(\omega_1')^3 - a_{01}\omega_1'\\
&\quad - \left(c_{01}\omega_1' - c_{03}(\omega_1')^3\right)\cos\tau_2\omega_1'\\
&\quad + \left(c_{00} - c_{02}(\omega_1')^2\right)\sin\tau_2\omega_1'.
\end{aligned} \tag{44}$$

Thus, one can get the following equation with respect to ω_1':

$$g_{51}^2(\omega_1') + g_{52}^2(\omega_1') = g_{53}^2(\omega_1') + g_{54}^2(\omega_1'). \tag{45}$$

Similar to Case 4, we assume that (H_{51}) (45) has at least one positive root ω_{10}'. Then, for ω_{10}', we have

$$\begin{aligned}
\tau_{10}' &= \frac{1}{\omega_{10}'}\\
&\quad \cdot \arccos\frac{g_{51}(\omega_{10}') \times g_{54}(\omega_{10}') + g_{52}(\omega_{10}') \times g_{53}(\omega_{10}')}{g_{51}^2(\omega_{10}') + g_{52}^2(\omega_{10}')}.
\end{aligned} \tag{46}$$

Differentiating (7) with respect to τ_1, we get

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{g_{55}(\lambda)}{g_{56}(\lambda)} - \frac{\tau_1}{\lambda}, \tag{47}$$

with

$$\begin{aligned}
g_{55}(\lambda) &= 4\lambda^3 + 3a_{03}\lambda^2 + 2a_{02}\lambda + a_{01} + \left(3b_{03}\lambda^2\right.\\
&\quad \left.+ 2b_{02}\lambda + b_{01}\right)e^{-\lambda\tau_1} + \left[\left(3c_{03} - \tau_2 c_{02}\right)\lambda^2 - \tau_2 c_{03}\lambda^3\right.\\
&\quad \left.+ \left(2c_{02} - \tau_2 c_{01}\right)\lambda + c_{01} - \tau_2 c_{00}\right]e^{-\lambda\tau_2}\\
&\quad + \left[\left(2d_{02} - \tau_2 d_{01}\right)\lambda - \tau_2 d_{02}\lambda^2 + d_{01} - \tau_2 d_{00}\right]\\
&\quad \cdot e^{-\lambda(\tau_1+\tau_2)},\\
g_{56}(\lambda) &= \left(c_{03}\lambda^4 + c_{02}\lambda^3 + c_{01}\lambda^2 + c_{00}\lambda\right)e^{-\lambda\tau_2}\\
&\quad + \left(d_{02}\lambda^3 + d_{01}\lambda^2 + d_{00}\lambda\right)e^{-\lambda(\tau_1+\tau_2)}.
\end{aligned} \tag{48}$$

Then, we obtain

$$\begin{aligned}
&\mathrm{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{10}'}\\
&\quad = \frac{h_{51}(\omega_{10}') \times h_{53}(\omega_{10}') + h_{52}(\omega_{10}') \times h_{54}(\omega_{10}')}{h_{53}^2(\omega_{10}') + h_{54}^2(\omega_{10}')},
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
h_{51}(\omega_{10}') &= \Big[2b_{02}\omega_{10}' + (2d_{02} - \tau_2 d_{01})\cos\tau_2\omega_{10}' \\
&\quad - \left(\tau_2(\omega_{10}')^2 + d_{01} - \tau_2 d_{00}\right)\sin\tau_2\omega_{10}'\Big]\sin\tau_{10}'\omega_{10}' \\
&\quad + \Big[b_{01} - 3b_{03}(\omega_{10}')^2 + (2d_{02} - \tau_2 d_{01})\sin\tau_2\omega_{10}' \\
&\quad + \left(\tau_2(\omega_{10}')^2 + d_{01} - \tau_2 d_{00}\right)\cos\tau_2\omega_{10}'\Big]\cos\tau_{10}'\omega_{10}' \\
&\quad + \left[(2c_{02} - \tau_2 c_{01})\omega_{10}' + \tau_2 c_{03}(\omega_{10}')^3\right]\sin\tau_2\omega_{10}' \\
&\quad + \left[c_{01} - \tau_2 c_{00} - (3c_{03} - \tau_2 c_{02})(\omega_{10}')^2\right]\cos\tau_2\omega_{10}' \\
&\quad + a_{01} - 3a_{03}(\omega_{10}')^2, \\
h_{52}(\omega_{10}') &= \Big[2b_{02}\omega_{10}' + (2d_{02} - \tau_2 d_{01})\cos\tau_2\omega_{10}' \\
&\quad - \left(\tau_2(\omega_{10}')^2 + d_{01} - \tau_2 d_{00}\right)\sin\tau_2\omega_{10}'\Big]\cos\tau_{10}'\omega_{10}' \\
&\quad - \Big[b_{01} - 3b_{03}(\omega_{10}')^2 + (2d_{02} - \tau_2 d_{01})\sin\tau_2\omega_{10}' \\
&\quad + \left(\tau_2(\omega_{10}')^2 + d_{01} - \tau_2 d_{00}\right)\cos\tau_2\omega_{10}'\Big]\sin\tau_{10}'\omega_{10}' \\
&\quad + \left[(2c_{02} - \tau_2 c_{01})\omega_{10}' + \tau_2 c_{03}(\omega_{10}')^3\right]\cos\tau_2\omega_{10}' \\
&\quad - \left[c_{01} - \tau_2 c_{00} - (3c_{03} - \tau_2 c_{02})(\omega_{10}')^2\right]\sin\tau_2\omega_{10}' \\
&\quad + a_{02}\omega_{10}' - 4(\omega_{10}')^3, \\
h_{53}(\omega_{10}') &= \Big[c_{00}\omega_{10}' - c_{02}(\omega_{10}')^3 \\
&\quad + \left(d_{00}\omega_{10}' - d_{02}(\omega_{10}')^3\right)\cos\tau_{10}'\omega_{10}' \\
&\quad + d_{01}(\omega_{10}')^2\sin\tau_{10}'\omega_{10}'\Big]\sin\tau_2\omega_{10}' + \Big[c_{03}(\omega_{10}')^4 \\
&\quad - c_{01}(\omega_{10}')^2 + \left(d_{00}\omega_{10}' - d_{02}(\omega_{10}')^3\right)\sin\tau_{10}'\omega_{10}' \\
&\quad - d_{01}(\omega_{10}')^2\cos\tau_{10}'\omega_{10}'\Big]\cos\tau_2\omega_{10}', \\
h_{54}(\omega_{10}') &= \Big[c_{00}\omega_{10}' - c_{02}(\omega_{10}')^3 \\
&\quad + \left(d_{00}\omega_{10}' - d_{02}(\omega_{10}')^3\right)\cos\tau_{10}'\omega_{10}' \\
&\quad + d_{01}(\omega_{10}')^2\sin\tau_{10}'\omega_{10}'\Big]\cos\tau_2\omega_{10}' - \Big[c_{03}(\omega_{10}')^4 \\
&\quad - c_{01}(\omega_{10}')^2 + \left(d_{00}\omega_{10}' - d_{02}(\omega_{10}')^3\right)\sin\tau_{10}'\omega_{10}' \\
&\quad - d_{01}(\omega_{10}')^2\cos\tau_{10}'\omega_{10}'\Big]\sin\tau_2'\omega_{10}'.
\end{aligned} \tag{50}$$

We assume that (H_{52}) $h_{51}(\omega_{10}') \times h_{53}(\omega_{10}') + h_{52}(\omega_{10}') \times h_{54}(\omega_{10}') \neq 0$. Thus, we know that $\mathrm{Re}[d\lambda/d\tau_1]^{-1}_{\tau_1=\tau_{10}'} \neq 0$, if condition (H_{52}) holds. Therefore, according to the Hopf bifurcation theorem in [26], we obtain the following.

Theorem 4. *If conditions (H_{51})-(H_{52}) hold and $\tau_2 \in (0, \tau_{20})$, then*

(i) *viral equilibrium $E_*(S_*, L_*, A_*, R_*)$ of system (2) is locally asymptotically stable for $\tau_1 \in [0, \tau_{10}')$;*

(ii) *system (2) undergoes a Hopf bifurcation at viral equilibrium $E_*(S_*, L_*, A_*, R_*)$ when $\tau_1 = \tau_{10}'$ and a family of periodic solutions bifurcate from $E_*(S_*, L_*, A_*, R_*)$.*

3. Properties of the Hopf Bifurcation

In this section, we shall investigate direction of the Hopf bifurcation and stability of the bifurcating periodic solution of system (2) when $\tau_1 = \tau_{10}'$ and $\tau_2 = \tau_{20}' \in (0, \tau_{20})$ by using the center manifold theorem and the normal form theory which has been developed by Hassard et al. [26].

Let $\tau_1 = \tau_{10}' + \mu$, $u_1 = S(\tau_1 t)$, $u_2 = L(\tau_1 t)$, $u_3 = A(\tau_1 t)$, $u_4 = R(\tau_1 t)$, $\mu \in R$. Then, $\mu = 0$ is the Hopf bifurcation value of system (2) and system (2) can be rewritten as

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{51}$$

where $u(t) = (u_1, u_2, u_3, u_4)^T \in C = C([-1, 0], R^4)$ and L_μ: $C \to R^4$ and $F : R \times C \to R^4$ are given, respectively, by

$$L_\mu\phi = (\tau_{10}' + \mu)\left(A^*\phi(0) + C^*\phi\left(-\frac{\tau_{20}'}{\tau_{10}'}\right) + B^*\phi(-1)\right), \tag{52}$$

with

$$\begin{aligned}
A_* &= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & 0 \\ 0 & 0 & a_8 & 0 \\ 0 & 0 & 0 & a_9 \end{bmatrix}, \\
B_* &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
C_* &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & 0 \\ 0 & 0 & c_2 & 0 \end{bmatrix}, \\
F_1 &= -\beta_1\phi_1(0)\phi_2(0) - \beta_2\phi_1(0)\phi_3(0), \\
F_2 &= \beta_1\phi_1(0)\phi_2(0) + \beta_2\phi_1(0)\phi_3(0), \\
\phi &= (\phi_1, \phi_2, \phi_3, \phi_4)^T \in C([-1, 0], R^4).
\end{aligned} \tag{53}$$

Using Riesz representation theorem, there exists matrix $\eta(\theta,\mu):[-1,0]\to R^4$ such that

$$L_\mu\phi=\int_{-1}^{0}d\eta(\theta,\mu)\,\phi(\theta),\quad \phi\in C\left([-1,0],R^4\right). \tag{54}$$

In fact, choose

$$\eta(\theta,\mu)=\begin{cases}\left(\tau'_{10}+\mu\right)\left(A_*+B_*+C_*\right), & \theta=0,\\ \left(\tau'_{10}+\mu\right)\left(B'+C_*\right), & \theta\in\left[-\dfrac{\tau'_{20}}{\tau'_{10}},0\right),\\ \left(\tau'_{10}+\mu\right)B_*, & \theta\in\left(-1,-\dfrac{\tau'_{20}}{\tau'_{10}}\right),\\ 0, & \theta=-1.\end{cases} \tag{55}$$

For $\phi\in C([-1,0],R^4)$, define

$$A(\mu)\phi=\begin{cases}\dfrac{d\phi(\theta)}{d\theta}, & -1\le\theta<0,\\ \displaystyle\int_{-1}^{0}d\eta(\theta,\mu)\,\phi(\theta), & \theta=0,\end{cases}$$
$$R(\mu)\phi=\begin{cases}0, & -1\le\theta<0,\\ F(\mu,\phi), & \theta=0.\end{cases} \tag{56}$$

Then, system (51) can be rewritten in the following form:

$$\dot u(t)=A(\mu)u_t+R(\mu)u_t. \tag{57}$$

For $\phi\in C([-1,0],R^4)$, $\varphi\in C^1([-1,0],(R^4)^*)$,

$$A^*(\varphi)=\begin{cases}-\dfrac{d\varphi(s)}{ds}, & 0<s\le 1,\\ \displaystyle\int_{-1}^{0}d\eta^T(s,0)\,\varphi(-s), & s=0,\end{cases} \tag{58}$$

and bilinear form

$$\langle\varphi,\phi\rangle=\overline{\varphi}(0)\phi(0)-\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{\varphi}(\xi-\theta)\,d\eta(\theta)\,\phi(\xi)\,d\xi, \tag{59}$$

are defined, where $\eta(\theta)=\eta(\theta,0)$, $A=A(0)$, and A^* are adjoint operators.

Based on the discussion above, one can conclude that $\pm i\omega'_{10}\tau'_{10}$ are common eigenvalues of $A(0)$ and A^*. The eigenvectors of $A(0)$ and A^* can be calculated corresponding to $+i\omega'_{10}\tau'_{10}$ and $-i\omega'_{10}\tau'_{10}$, respectively. Let $q(\theta)=(1,q_2,q_3,q_4)^T e^{i\omega'_{10}\tau'_{10}\theta}$ be the eigenvector of $A(0)$ corresponding to $+i\omega'_{10}\tau'_{10}$ and $q^*(\theta)=K(1,q_2^*,q_3^*,q_4^*)e^{i\omega'_{10}\tau'_{10}s}$ be the eigenvector of A^* corresponding to $-i\omega'_{10}\tau'_{10}$. By some complex computations, we obtain

$$\begin{aligned}q_2&=\frac{a_5}{q_{20}},\\ q_3&=\frac{b_2e^{-i\tau'_{10}}\omega'_{10}q_2}{i\omega'_{10}-a_8-c_1e^{-i\tau'_{20}\omega'_{10}}},\\ q_4&=\frac{c_2e^{-i\tau'_{20}\omega'_{10}}q_3}{i\omega'_{10}-a_9},\\ q_2^*&=-\frac{i\omega'_{10}+a_1}{a_5},\\ q_3^*&=\frac{a_2+\left(i\omega'_{10}+a_6+b_1e^{i\tau'_{10}\omega'_{10}}\right)q_2^*}{b_2e^{i\tau'_{10}\omega'_{10}}},\\ q_4^*&=-\frac{a_3+a_7q_2^*+\left(i\omega'_{10}+a_8+c_1e^{i\tau'_{20}\omega'_{10}}\right)q_3^*}{c_2e^{i\tau'_{20}\omega'_{10}}},\end{aligned} \tag{60}$$

with

$$q_{20}=i\omega'_{10}-a_6-b_1e^{-i\tau'_{10}\omega'_{10}}-\frac{a_7b_2e^{-i\tau'_{10}\omega'_{10}}}{i\omega'_{10}-a_8-c_1e^{-i\tau'_{20}\omega'_{10}}}. \tag{61}$$

From (59), we get

$$\begin{aligned}\overline{K}=\Big[&1+q_2\overline{q}_2^*+q_3\overline{q}_3^*+q_4\overline{q}_4^*\\ &+\tau'_{10}e^{-i\omega'_{10}\tau'_{10}}q_2\left(b_1\overline{q}_2^*+b_2\overline{q}_3^*\right)\\ &+\tau'_{20}e^{-i\omega'_{10}\tau'_{10}}q_3\left(c_1\overline{q}_3^*+c_2\overline{q}_4^*\right)\Big]^{-1},\end{aligned} \tag{62}$$

such that $\langle q^*,q\rangle=1$. In what follows, we can obtain the coefficients by using the method introduced in [26]:

$$\begin{aligned}g_{20}&=2\tau'_{10}\overline{K}\left(\overline{q}_2^*-1\right)\left(\beta_1q_2+\beta_2q_3\right),\\ g_{11}&=\tau'_{10}\overline{K}\left(\overline{q}_2^*-1\right)\left(\beta_1\operatorname{Re}\{q_2\}+\beta_2\operatorname{Re}\{q_3\}\right),\\ g_{02}&=2\tau'_{10}\overline{K}\left(\overline{q}_2^*-1\right)\left(\beta_1\overline{q}_2+\beta_2\overline{q}_3\right),\\ g_{21}&=2\tau'_{10}\overline{K}\left(\overline{q}_2^*-1\right)\Big(\beta_1\Big(W_{11}^{(1)}(0)q_2+\frac{1}{2}W_{20}^{(1)}(0)\overline{q}_2\\ &\quad+W_{11}^{(2)}(0)+\frac{1}{2}W_{20}^{(2)}(0)\Big)+\beta_2\Big(W_{11}^{(1)}(0)q_3\\ &\quad+\frac{1}{2}W_{20}^{(1)}(0)\overline{q}_3+W_{11}^{(3)}(0)+\frac{1}{2}W_{20}^{(3)}(0)\Big)\Big),\end{aligned} \tag{63}$$

with

$$W_{20}(\theta) = \frac{ig_{20}q(0)}{\tau'_{10}\omega'_{10}} e^{i\tau'_{10}\omega'_{10}\theta} + \frac{i\overline{g}_{02}\overline{q}(0)}{3\tau'_{10}\omega'_{10}} e^{-i\tau'_{10}\omega'_{10}\theta} + E_1 e^{2i\tau'_{10}\omega'_{10}\theta},$$
$$W_{11}(\theta) = -\frac{ig_{11}q(0)}{\tau'_{10}\omega'_{10}} e^{i\tau'_{10}\omega'_{10}\theta} + \frac{i\overline{g}_{11}\overline{q}(0)}{\tau'_{10}\omega'_{10}} e^{-i\tau'_{10}\omega'_{10}\theta} + E_2, \quad (64)$$

where

$$E_1 = 2\begin{pmatrix} a_{1*} & -a_2 & -a_3 & -a_4 \\ -a_5 & a_{6*} & a_7 & 0 \\ 0 & -b_2 e^{-i\tau'_{10}\omega'_{10}} & a_{8*} & 0 \\ 0 & 0 & -c_2 e^{-i\tau'_{20}\omega'_{10}} & a_{9*} \end{pmatrix}^{-1} \cdot \begin{pmatrix} -\beta_1 q_2 - \beta_2 q_3 \\ \beta_1 q_2 + \beta_2 q_3 \\ 0 \\ 0 \end{pmatrix},$$
$$E_2 = -\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 + b_1 & a_7 & 0 \\ 0 & b_2 & a_8 + c_1 & 0 \\ 0 & 0 & c_2 & a_9 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -\beta_1 \operatorname{Re}\{q_2\} - \beta_2 \operatorname{Re}\{q_3\} \\ \beta_1 \operatorname{Re}\{q_2\} + \beta_2 \operatorname{Re}\{q_3\} \\ 0 \\ 0 \end{pmatrix}, \quad (65)$$

with

$$a_{1*} = 2i\omega'_{10} - a_1,$$
$$a_{6*} = 2i\omega'_{10} - a_6 - b_1 e^{-i\tau'_{10}\omega'_{10}},$$
$$a_{8*} = 2i\omega'_{10} - a_8 - c_1 e^{-i\tau'_{20}\omega'_{10}},$$
$$a_{9*} = 2i\omega'_{10} - a_9. \quad (66)$$

Thus, we can compute the following coefficients:

$$C_1(0) = \frac{i}{2\omega'_{10}\tau'_{10}}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$
$$\mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau'_{10})\}},$$
$$\beta_2 = 2\operatorname{Re}\{C_1(0)\},$$
$$T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau'_{10})\}}{\omega'_{10}\tau'_{10}}. \quad (67)$$

In conclusion, we have the following results in this section.

Theorem 5. *For system (2),*

(i) *the direction of the Hopf bifurcation is determined by μ_2: if $\mu_2 > 0$, the Hopf bifurcation is supercritical; if $\mu_2 < 0$, the Hopf bifurcation is subcritical;*

(ii) *the stability of the bifurcating periodic solutions is determined by β_2: if $\beta_2 < 0$, the bifurcating periodic solutions are stable; if $\beta_2 > 0$, the bifurcating periodic solutions are unstable;*

(iii) *the period of the bifurcating periodic solution is determined by T_2: if $T_2 > 0$, the period of the bifurcating periodic solutions increases; if $T_2 < 0$, the period of the bifurcating periodic solutions decreases.*

4. Numerical Simulations

In this section, we present some numerical simulation results of system (2) to illustrate our theoretical results. We choose a set of parameters as follows: $\mu = 0.001$, $\beta_1 = 0.1$, $\beta_2 = 0.15$, $\alpha = 0.05$, $\varepsilon = 0.05$, and $\gamma = 0.02$. Then, we get the following system:

$$\frac{dS(t)}{dt} = 0.001 - 0.1S(t)L(t) - 0.15S(t)A(t) + 0.05R(t) - 0.001S(t),$$
$$\frac{dL(t)}{dt} = 0.1S(t)L(t) + 0.15S(t)A(t) - 0.05L(t-\tau_1) - 0.001L(t),$$
$$\frac{dA(t)}{dt} = 0.05L(t-\tau_1) - 0.02A(t-\tau_2) - 0.001A(t),$$
$$\frac{dR(t)}{dt} = 0.02A(t-\tau_2) - 0.05R(t) - 0.001R(t). \quad (68)$$

It is easy to verify that $(\varepsilon + \mu)(\gamma + \mu)/(\beta_1(\gamma + \mu) + \beta_2\varepsilon) < 1$ and $(\varepsilon + \mu)(\gamma + \mu)/\varepsilon > \alpha\gamma/(\alpha + \mu)$, which ensures the fact that system (68) has a unique viral equilibrium $E_*(0.1116, 0.2073, 0.4936, 0.1936)$.

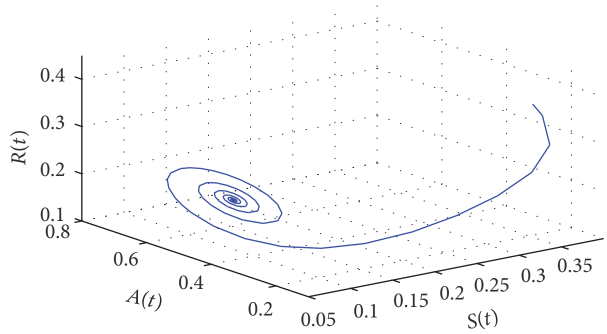

FIGURE 1: The phase plot of states S_*, A_*, and R_* with $\tau_1 = 22.36 < 28.4522 = \tau_{10}$.

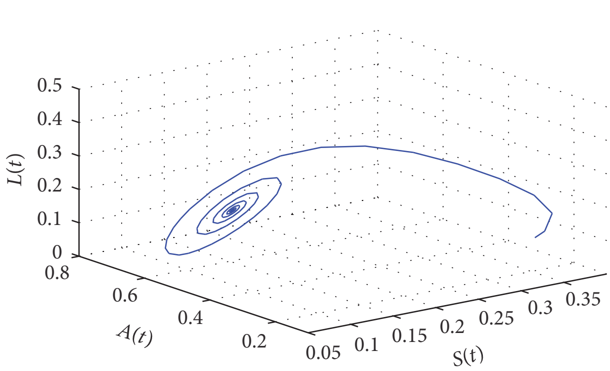

FIGURE 2: The phase plot of states S_*, L_*, and A_* with $\tau_1 = 22.36 < 28.4522 = \tau_{10}$.

For $\tau_1 = \tau_2 = 0$, by direct computation by Matlab 7.0, we can get $a_{10} > 0$, $a_{13} > 0$, and $a_{13}a_{12} > a_{11}$, which means that viral equilibrium $E_*(0.1116, 0.2073, 0.4936, 0.1936)$ is locally asymptotically stable.

For $\tau_1 > 0, \tau_2 = 0$, we obtain that (16) has a unique positive root $v_{10} = 0.0013$ and (14) has a unique positive root $\omega_{10} = 0.0363$. Further, we get the critical value of delay $\tau_{10} = 28.4522$ and $f_1'(v_1^*) = 0.0612 > 0$. Thus, we can see that the conditions in Theorem 1 hold true. It follows that viral equilibrium $E_*(0.1116, 0.2073, 0.4936, 0.1936)$ is locally asymptotically stable for $\tau_1 \in [0, 28.4522)$ and system (68) undergoes a Hopf bifurcation at viral equilibrium $E_*(0.1116, 0.2073, 0.4936, 0.1936)$ when $\tau_1 = 28.4522$. This property can be depicted in Figures 1–4. Similarly, we can also obtain $\omega_{20} = 0.2756$, $\tau_{20} = 95.9606$ for $\tau_1 = 0, \tau_2 > 0$ and $\omega_0 = 0.7114$, $\tau_0 = 24.2508$ for $\tau_1 = \tau_2 = \tau > 0$, respectively. The corresponding phase plots are depicted in Figures 5–8 and Figures 9–12, respectively.

For $\tau_1 > 0$ and $\tau_2 = 65.25 \in (0, \tau_{20})$, we obtain $\omega_{10}' = 0.0089$, $\tau_{10}' = 18.6255$. The simulation results can be seen in Figures 13–16. In addition, we obtain $C_1(0) = -0.2107 + 0.0062i$, $\lambda'(\tau_{10}') = 0.0031 - 0.1086i$ by some complex computations. Further, we have $\mu_2 = 67.9677 > 0$, $\beta_2 = -0.4214 < 0$, $T_2 = 44.4907 > 0$. Therefore, we can conclude that the Hopf bifurcation is supercritical; the bifurcating periodic solutions are stable; and the period of the bifurcating periodic solutions increases.

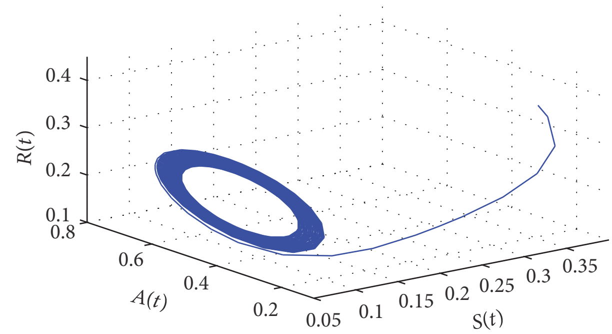

FIGURE 3: The phase plot of states S_*, A_*, and R_* with $\tau_1 = 29.49 > 28.4522 = \tau_{10}$.

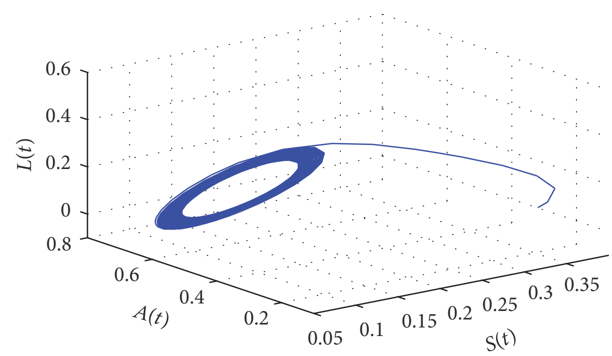

FIGURE 4: The phase plot of states S_*, L_*, and A_* with $\tau_1 = 29.49 > 28.4522 = \tau_{10}$.

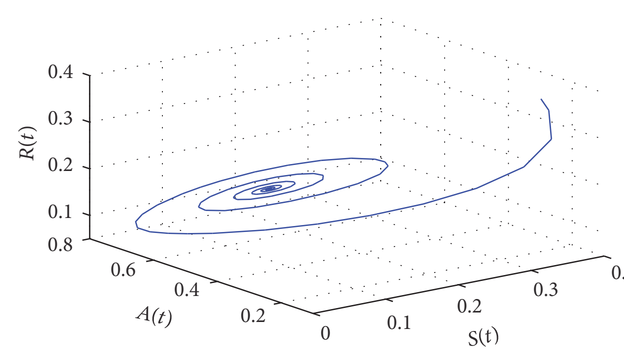

FIGURE 5: The phase plot of states S_*, A_*, and R_* with $\tau_2 = 58.48 < 95.9606 = \tau_{20}$.

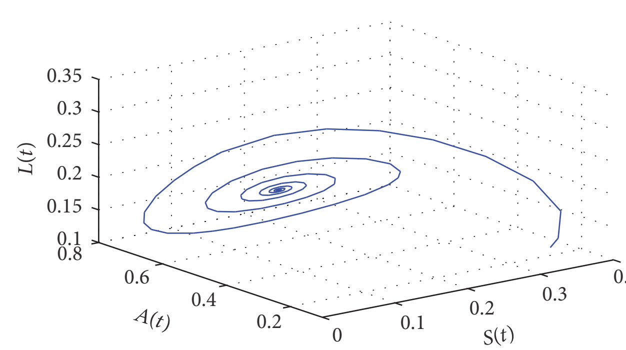

FIGURE 6: The phase plot of states S_*, L_*, and A_* with $\tau_2 = 58.48 < 95.9606 = \tau_{20}$.

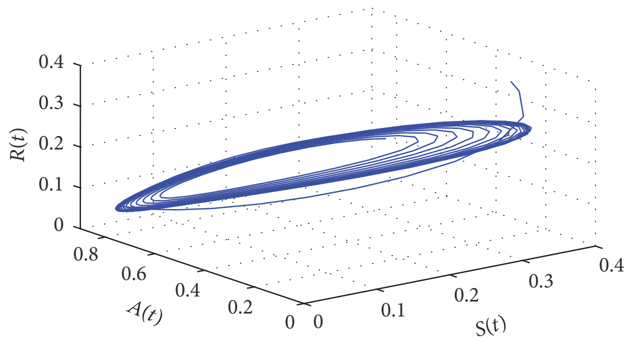

FIGURE 7: The phase plot of states S_*, A_*, and R_* with $\tau_2 = 110.96 > 95.9606 = \tau_{20}$.

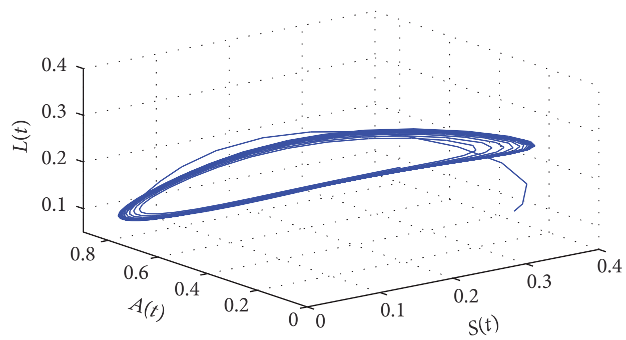

FIGURE 8: The phase plot of states S_*, L_*, and A_* with $\tau_2 = 110.96 > 95.9606 = \tau_{20}$.

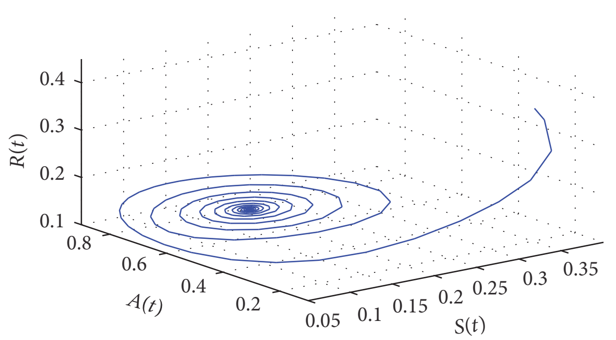

FIGURE 9: The phase plot of states S_*, A_*, and R_* with $\tau = 21.35 < 24.2508 = \tau_0$.

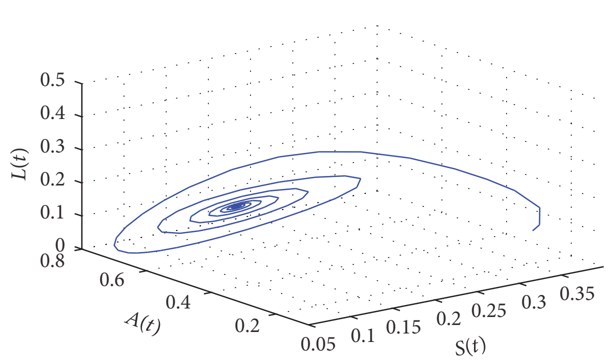

FIGURE 10: The phase plot of states S_*, L_*, and A_* with $\tau = 21.35 < 24.2508 = \tau_0$.

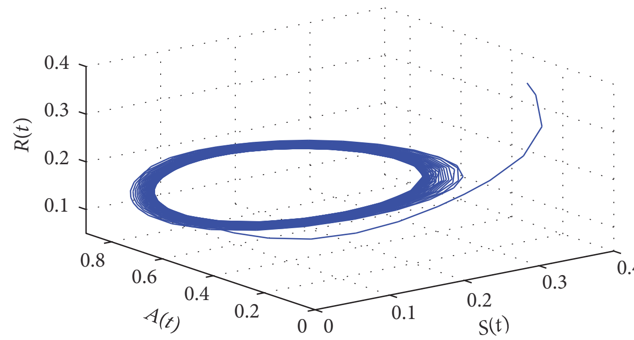

FIGURE 11: The phase plot of states S_*, A_*, and R_* with $\tau = 25.47 > 24.2508 = \tau_0$.

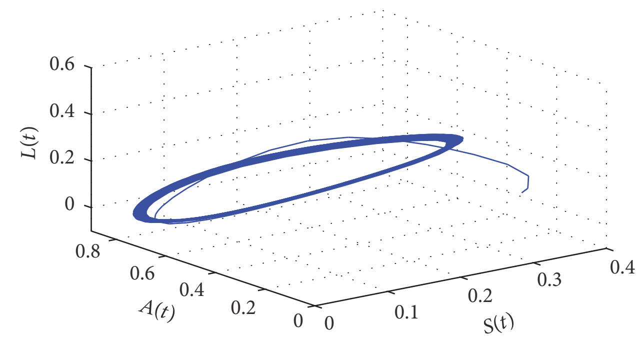

FIGURE 12: The phase plot of the states S_*, L_*, and A_* with $\tau = 25.47 > 24.2508 = \tau_0$.

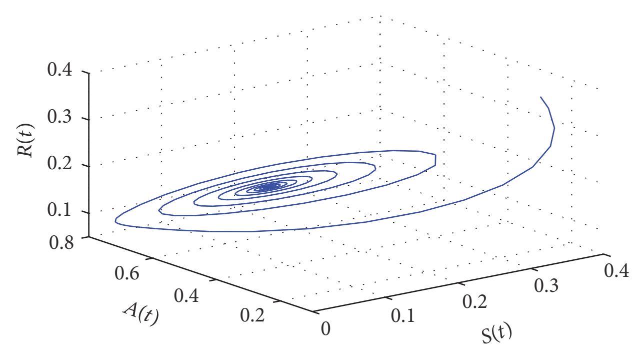

FIGURE 13: The phase plot of states S_*, A_*, and R_* with $\tau_1 = 8.65 < 18.6255 = \tau_{10}'$ and $\tau_2 = 65.25$.

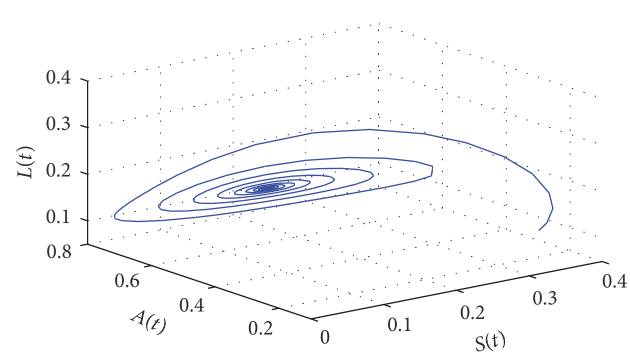

FIGURE 14: The phase plot of states S_*, L_*, and A_* with $\tau_1 = 8.65 < 18.6255 = \tau_{10}'$ and $\tau_2 = 65.25$.

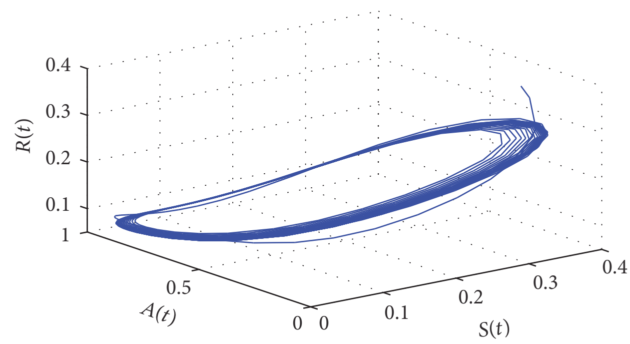

Figure 15: The phase plot of states S_*, A_*, and R_* with $\tau_1 = 22.4057 > 18.6255 = \tau'_{10}$ and $\tau_2 = 65.25$.

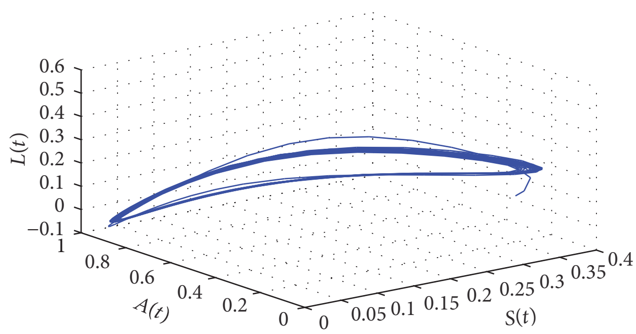

Figure 16: The phase plot of states S_*, L_*, and A_* with $\tau_1 = 22.4057 > 18.6255 = \tau'_{10}$ and $\tau_2 = 65.25$.

5. Conclusions

In the present paper, an improved model for propagation of computer virus propagation model in the network is introduced and studied by incorporating the delay due to the latent period of the computer viruses and the delay due to the period that the antivirus software needs to clean the viruses in the active computers into the model proposed in [19]. We mainly investigate effect of the two delays on the model.

By choosing different combination of the two delays as a bifurcation parameter, it has been found that both the two delays can change the stability of the viral equilibrium of the model under some conditions. When the value of the delay is below corresponding critical value, the model is locally asymptotically stable which indicates that the law of propagation of the computer viruses in system (2) can be predicted. However, when the value of the delay is above the corresponding critical value, a Hopf bifurcation occurs and a family of periodic solutions bifurcate from the viral equilibrium, which suggests that the percentages of susceptible, latent, active, and recovered computers in system (2) will fluctuate periodically in a range. This is not helpful in predicting the law of propagation of the computer viruses. Therefore, we should control the occurrence of the Hopf bifurcation by using some bifurcation control strategies and we leave this as our near future work. Furthermore, the properties of the Hopf bifurcation when $\tau_1 > 0$ and $\tau_2 \in (0, \tau_{20})$ have been investigated in detail. Finally, some numerical simulations are also included to support the theoretical results obtained in the paper.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

Acknowledgments

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