Hindawi
Mathematical Problems in Engineering
Volume 2017, Article ID 3919438, 7 pages
https://doi.org/10.1155/2017/3919438

Research Article

A Good Earthquake Concave Behaviour of a Seismic Isolator Which Supports a Metallic Roof

Iancu Durău,[1] Ionel Rovenţa,[2] and Nicolae Daniel Stoica[1]

[1] *Technical University of Civil Engineering, Bucureşti, Romania*
[2] *Department of Mathematics, University of Craiova, 200585 Craiova, Romania*

Correspondence should be addressed to Ionel Rovenţa; ionelroventa@yahoo.com

Received 12 July 2017; Accepted 18 October 2017; Published 6 November 2017

Academic Editor: Zhen-Lai Han

We combine some important theoretical mathematical results with practical aspects from engineering in a nice framework of K-spiders, which forms a space with global nonpositive curvature. Some concavity properties into such framework are derived. As an application, we consider a mathematical model for an engineering problem. More precisely, we are trying to model the behaviour of tension forces appearing in a metallic roof supported on seismic isolators. In fact, we are looking to find the properties of good seismic isolators in order to reduce the destructive power of an earthquake. The answer consists of a concave behaviour in terms of the displacements on the 3D axes of the Cartesian system.

1. Introduction and Convexity Properties in Global NPC Spaces

The aim of this paper is to use some nice mathematical results in the framework of modelling some practical engineering problems. More precisely, since the convex/concave functions defined on trees can model the tension forces in an arborescent network and the flows from communication networks, it is interesting to study which classical inequalities hold true in this framework. Our aim is to study some properties of a concave function defined on a K-spider, which is a tree endowed with a special metric.

The spaces with global nonpositive curvature (abbreviated, global NPC spaces) are defined as follows.

Definition 1. A global NPC space is a complete metric space $E = (E, d)$ for which the following inequality holds true: for each pair of points $x_0, x_1 \in E$ there exists a point $y \in E$ such that, for all points $z \in E$,

$$d^2(z, y) \leq \frac{1}{2}d^2(z, x_0) + \frac{1}{2}d^2(z, x_1) - \frac{1}{4}d^2(x_0, x_1). \quad (1)$$

These spaces are also named as the Cat [0] spaces. See [1]. It is well-known that in a global NPC space, each pair of points $x_0, x_1 \in E$ can be connected by a unique geodesic (a rectifiable curve γ: $[0,1] \to E$ such that the length of $\gamma|_{[s,t]}$ is $d(\gamma(s), \gamma(t))$ for all $0 \leq s \leq t \leq 1$). Moreover, this geodesic is unique. It can be easily proved that the point y from above is the *midpoint* of x_0 and x_1 and has the property

$$d(x_0, y) = d(y, x_1) = \frac{1}{2}d(x_0, x_1). \quad (2)$$

There are many examples of global NPC spaces: Hilbert spaces (its geodesics are the line segments), the upper half-plane $\mathbf{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, endowed with the Poincaré metric (the geodesics are the semicircles in $\mathbf{H}$ perpendicular to the real axis and the straight vertical lines ending on the real axis), and Riemannian manifolds with complete, simply connected, and of nonpositive sectional curvature.

Besides manifolds, other important examples of global NPC spaces are the Bruhat-Tits buildings (in particular, the trees). See [1]. More details on global NPC spaces are available in [2–5]. In the following sentences we define the basic convexity notions in a global NPC space.

Definition 2. A set $C \subset E$ is called *convex* if $\gamma([0,1]) \subset C$ for each geodesic γ: $[0,1] \to C$ joining $\gamma(0), \gamma(1) \in C$.

A function $\varphi: C \to \mathbb{R}$ is called concave if the function $\varphi \circ \gamma : [0,1] \to \mathbb{R}$ is concave for each geodesic $\gamma : [0,1] \to C, \gamma(t) = \gamma_t$, that is;

$$\varphi(\gamma_t) \geq (1-t)\varphi(\gamma_0) + t\varphi(\gamma_1), \tag{3}$$

for all $t \in [0,1]$. If $-\gamma$ is concave we say that γ is convex.

Note that the functions $-d^2(z,\cdot)$ are concave (more precisely, uniformly concave). Moreover, one can prove that the distance function $-d$ is concave in each of its variables (and also with respect to both variables).

In what follows $\mathscr{P}^1(E)$ means the set of all Borel probability measures μ on E with separable support, which verify the condition $\int_E d(x,y)d\mu(y) < \infty, \forall x \in E$. The set $\mathscr{P}^2(E)$ is given by the family of all square integrable probability measures with separable support; that is, $\int_E d^2(x,y)d\mu(y) < \infty$.

We define the barycenter of a measure $\mu \in \mathscr{P}^1(E)$ as the unique point $z \in E$ which minimizes the uniformly convex function:

$$F_y(x) = \int_E \left[d^2(z,x) - d(y,x)\right] d\mu(x). \tag{4}$$

We can remark that this minimizer is independent of $y \in E$ and it denotes $b(\mu)$. In the case of a square integrable measure Sturm [5] proves that the barycenter can be also seen as

$$b(\mu) = \arg\min_{z\in E} \int_E d^2(z,x)\,d\mu(x). \tag{5}$$

The main idea in a global NPC spaces is the fact that, in general, the minimizer may fail to exist or is not unique, but the existence and the uniqueness of a barycenter always hold for NPC spaces, since the metric is uniformly convex.

Note that if the support of μ is included in a convex closed set K, then $b(\mu) \in K$.

Definition 3. Given $x = (x_1,\ldots,x_n) \in E^n$ and some positive real weights $\lambda = (\lambda_1,\ldots,\lambda_n)$ with $\sum_{i=1}^n \lambda_i = 1$, we define

$$\mathscr{M}_n(\lambda,x) := \arg\min_{z\in E} \sum_{i=1}^n \lambda_i d^2(z,x_i). \tag{6}$$

Remark 4. If we consider $\mu = \sum_{i=1}^n \lambda_i \delta_{x_i}$ then

$$b(\mu) = \arg\min_{z\in E} \int_E d^2(z,x)\,d\mu(x) = \mathscr{M}_n(\lambda,x). \tag{7}$$

Note that, for each $x_0, x_1 \in E$, the point $y \in E$ and the point $y' = (x_1 + x_2)/2$ are the same.

We recall also a result of Lawson and Lim [6] which shows that, for each $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n) \in E^n$ and some positive real weight $\lambda = (\lambda_1,\ldots,\lambda_n)$ with $\sum_{i=1}^n \lambda_i = 1$, we have

$$d(\mathscr{M}_n(\lambda,x), \mathscr{M}_n(\lambda,y)) \leq \sum_{i=1}^n \lambda_i d(x_i,y_i), \tag{8}$$

where $\mu = \sum_{i=1}^n \lambda_i \delta_{x_i}$ and $\nu = \sum_{i=1}^n \lambda_i \delta_{y_i}$ are two finitely supported probability measures. Interesting connections and applications obtained into this field can be found in [4, 7].

The outline of the paper is the following: Section 1 is devoted to some preliminaries on the convexity/concavity properties in metric spaces of global nonpositive curvature. In Section 2 we study a characterization of concave functions defined on a K-spider and we are dealing with the localization of minimum and maximum of a concave function defined on a K-spider. In Section 3 we present a nice engineering application in the area of the behaviour of tension forces appearing in a metallic roof supported on seismic isolators.

2. Extremum Points and Convexity/Concavity Properties on K-Spiders

The aim of section is the study of the minimizers and maximizers of a convex function defined on a K-spider.

We consider K an arbitrary set and let $N_i = \{(i,r) : r \in \mathbb{R}\}, \forall i \in K$. We define the K-spider (N,d) as

$$N = \frac{\{(i,r) : i \in K, r \in \mathbb{R}\}}{\sim}, \qquad \text{where } (i,0) \sim (j,0) \quad (i,j \in K), \tag{9}$$

with the corresponding distance given by

$$d((i,r),(j,s)) = \begin{cases} |r-s|, & \text{if } i = j \\ |r| + |s|, & \text{otherwise.} \end{cases} \tag{10}$$

The intersection of each of the two sets N_i and N_j, with $i \neq j$, is given by the origin $o := (i,0) = (j,0)$. The K-spider N depends only on the cardinality of K. If the set $K = \{1,2,\ldots,k\}$ the K-spider is called k-spider. The tripod is a 3-spider. Note also that the sets N_i can be seen as closed subsets of N. More details on graph theory can be found in [8, 9].

Proposition 5 (see [5]). *Each K-spider (N,d) endowed with the metric given by (10) is a global NPC space.*

We introduce that the notion of convex hull is introduced via the formula

$$\text{co } F = \bigcup_{n=0}^{\infty} F_n, \tag{11}$$

where $F_0 = F$ and for $n \geq 1$ the set F_n consists of all points in E which lie on geodesics which start and end in F_{n-1}.

A simple consequence is that the convex hull of subset of a K-spider N is also a K-spider included in N. Based on the fact that the closed convex hull of every nonempty finite family of points of E has the fixed point property in [4] the Schauder fixed point theorem has been proved.

Definition 6. We say that $x \in A$ is an extremal point for the convex set $A \subset N$ if x does not belong to the interior of some geodesic segment with the ends in A.

Naturally, Minkowski's theorem can be easily extended in the framework of global NPC spaces; that is, each point from a closed convex set can be written as a convex combination of extremal points. More precisely, we can say that each point belonging to a convex set belongs to the convex hull of the extremal points. For more details, see [10].

We are now in position to prove our first result of this section. See also [11], for some other dual results and more details.

Theorem 7. *Let (N, d) be a K-spider and let $f\colon A \to \mathbb{R}$ be a continuous concave function defined on a closed convex subset $A \subset N$. Then f attains its infimum in an extremal point. As a consequence, if f attains its infimum in an interior point, then f is a constant function.*

Proof. Firstly, let us consider a point $x_m \in A$ such that

$$f(x_m) = \inf_{x\in A} f(x). \tag{12}$$

Applying Minkowski's theorem we infer the existence of some extremal points $e_1, \ldots, e_n$ of the set A and $\lambda_i > 0$, $i = 1, \ldots, n$, with $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$x_m = \arg\min_{z\in A} \sum_{i=1}^{n} \lambda_i d^2(z, e_i). \tag{13}$$

Secondly, by using Jensen's inequality we obtain that

$$f(e_k) \geq f(x_m) \geq \sum_{i=1}^{n} \lambda_i f(e_i) \geq \max_{i=1,\ldots,n} f(e_i) \qquad (k = 1, \ldots, n). \tag{14}$$

Finally, we have that $f(x_m) = \min_{i=1,\ldots,n} f(e_i)$ and the proof is finished. □

The following results are devoted to the minimizing properties of a convex function defined on a K-spider.

Theorem 8. *Let $f\colon A \to \mathbb{R}$ be a concave function defined on a convex subset $A \subset N$ of a K-spider (N, d). If $x_M \in A$ is a local maximum for the function f then x_M is a global maximum of f; that is,*

$$f(x_M) = \sup_{x\in A} f(x). \tag{15}$$

Proof. The definition of x_M as a point of local maximum gives the existence of a radius $r > 0$ such that

$$f(x_M) \geq f(x) \qquad (x \in B_r(x_M) = \{x \in A \mid d(x, x_M) \leq r\}). \tag{16}$$

Let $x \in A \setminus B_r(x_M)$; for any point $\tilde{x} \neq x_M$, $\tilde{x} \in B_r(x_M) \cap \mathrm{co}\{x, x_M\}$ there exists $\lambda \in (0, 1)$ such that

$$\tilde{x} = \arg\min_{z\in N} \left((1-\lambda) d^2(z, x) + \lambda d^2(z, x_M)\right). \tag{17}$$

Jensen's inequality gives that

$$f(x_M) \geq f(\tilde{x}) \geq (1-\lambda) f(x) + \lambda f(x_M), \tag{18}$$

and we deduce that $f(x_M) \geq f(x)$ and the proof is finished. □

We end this section by recalling a problem consisting in finding the conditions which need to be satisfied by three convex functions defined on each arm of a 3-spider in order to obtain a convex function defined on the entire 3-spider.

We consider a 3-spider N, with the arms given by N_1, N_2, N_3 and let $f\colon N \to \mathbb{R}$ be a convex function. The restrictions of f to each arm are convex functions, denoted by f_1, f_2, f_3. Let us consider $f_1, f_2, f_3\colon [0, \infty) \to \mathbb{R}$ convex functions which satisfy the properties

$$\begin{aligned} f'_{1d}(0) + f'_{3d}(0) &\geq 0, \\ f'_{1d}(0) + f'_{2d}(0) &\geq 0, \\ f'_{2d}(0) + f'_{3d}(0) &\geq 0. \end{aligned} \tag{19}$$

Without loosing the generality we can suppose that $f_1(0) = f_2(0) = f_3(0) = 0$. In [12] was proved that conditions (19) (which are equivalent with the property that the functions $f_1 + f_2$, $f_1 + f_3$, $f_2 + f_3$ are nondecreasing) imply that f is convex on the entire 3-spider.

3. An Application for the Best Earthquake Concave Behaviour of a Seismic Isolator Which Support a Metallic Roof

As an application we consider a mathematical model for an engineering problem. More precisely, we are trying to model the behaviour of tension forces appearing in a metallic roof supported on seismic isolators. In fact, we are looking to find the properties of a good seismic isolators in order to reduce the destructive power of an earthquake.

The answer consists of a concave behaviour in terms of the displacements on the 3D axes. The reason for which we choose to consider a concave function in order to describe good behaviour of the seismic isolators is the following: the concavity geometric properties says that the graph of a concave function is under the tangent line on each point belonging to the graph (see Figure 1). For example, if we consider the concave function $f(x) = cx^{\alpha}$, $x > 0$, $\alpha \in (0, 1)$, $c > 0$, Figure 1 shows that for small numbers x the function has almost no effect, but for large numbers (displacements of an earthquake) we can reduce drastically the largest displacements under the action of a concave function f. The entering data in the function are depending on the earthquake displacements over the 3D axes of a Cartesian system in $\mathbb{R}^3$. Of course, the angles between the seismic isolators and the axes of the Cartesian system are very important. In fact, the resultant forces appearing in the seismic isolator are determined by the sum of the projections of the displacements over the direction of the seismic isolator.

Once we have established the usefulness of a good concave behaviour of the seismic isolators we can proceed

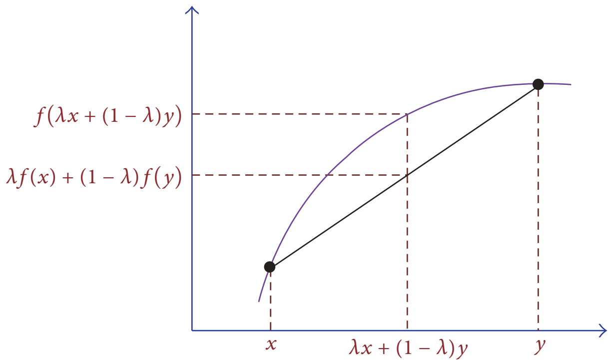

FIGURE 1: The graph of a concave function.

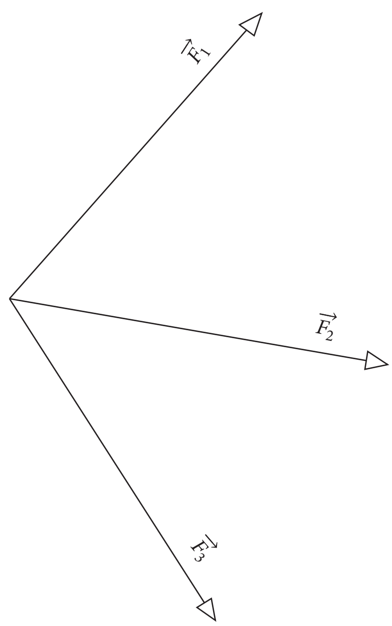

FIGURE 2: The tension forces acting in a single point.

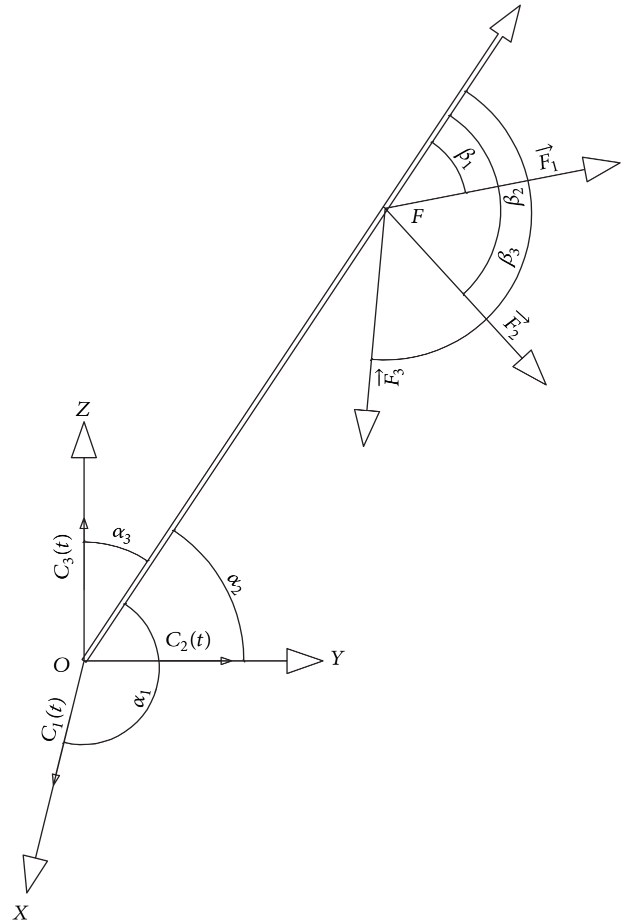

FIGURE 3: The resultant tension forces.

to model the action of an earthquake in a single point of a metallic roof. In the following we describe the mathematical model of the forces acting in a single point.

In Figure 2 from below we draw the initial state of the tension forces acting in a point F which are connected with three other points by some arms. Let us consider F_1, F_2, F_3 the forces acting in each of the three arms.

We assume that the point is supported on a seismic isolator with a concave behaviour. Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a concave function.

We consider now a seismic isolator fixed with the two ends in O and F, which forms an angle α with the plane xOy. We denote by $\alpha_1, \alpha_2, \alpha_3$ the angles between the direction of the seismic isolator with each of the three positive senses of the axes Ox, Oy, Oz. We denote also by $\beta_1, \beta_2, \beta_3$ the angles between the direction of the seismic isolator with each of the three arms. Let us consider the displacements under the action of an earthquake in the point O in each of the directions Ox, Oy, Oz at the moment of time $t > 0$, described by the functions $c_1(t)$, $c_2(t)$, $c_3(t)$.

Hence, the seismic isolator receive, as entering data in the starting point O, the displacement functions $c_1(t)$, $c_2(t)$, $c_3(t)$ and after "a good concave behaviour" of the seismic isolator, we obtain at the ending point F the displacement field into the direction of the seismic isolator, as a tension force given by

$$F(t) = \operatorname{smn}(t) \cdot f\left(\left|c_1(t)\cos\alpha_1 + c_2(t)\cos\alpha_2 + c_3(t)\cos\alpha_3\right|\right), \tag{20}$$

where $\operatorname{smn}(t) = \operatorname{sgn}(c_1(t)\cos\alpha_1 + c_2(t)\cos\alpha_2 + c_3(t)\cos\alpha_3)$ and $\operatorname{sgn}(a) = 1$, if $a \geq 0$, and $\operatorname{sgn}(a) = -1$, if $a < 0$.

Now, taking into account the angles between the seismic isolator and the arms starting from the point F and the initial forces F_1, F_2, F_3, we obtain the following resultant tension forces (as we can see in Figure 3):

$$\begin{aligned} F_1 &\longrightarrow F_1 + F(t)\cos\beta_1, \\ F_2 &\longrightarrow F_2 + F(t)\cos\beta_2, \\ F_3 &\longrightarrow F_3 + F(t)\cos\beta_3. \end{aligned} \tag{21}$$

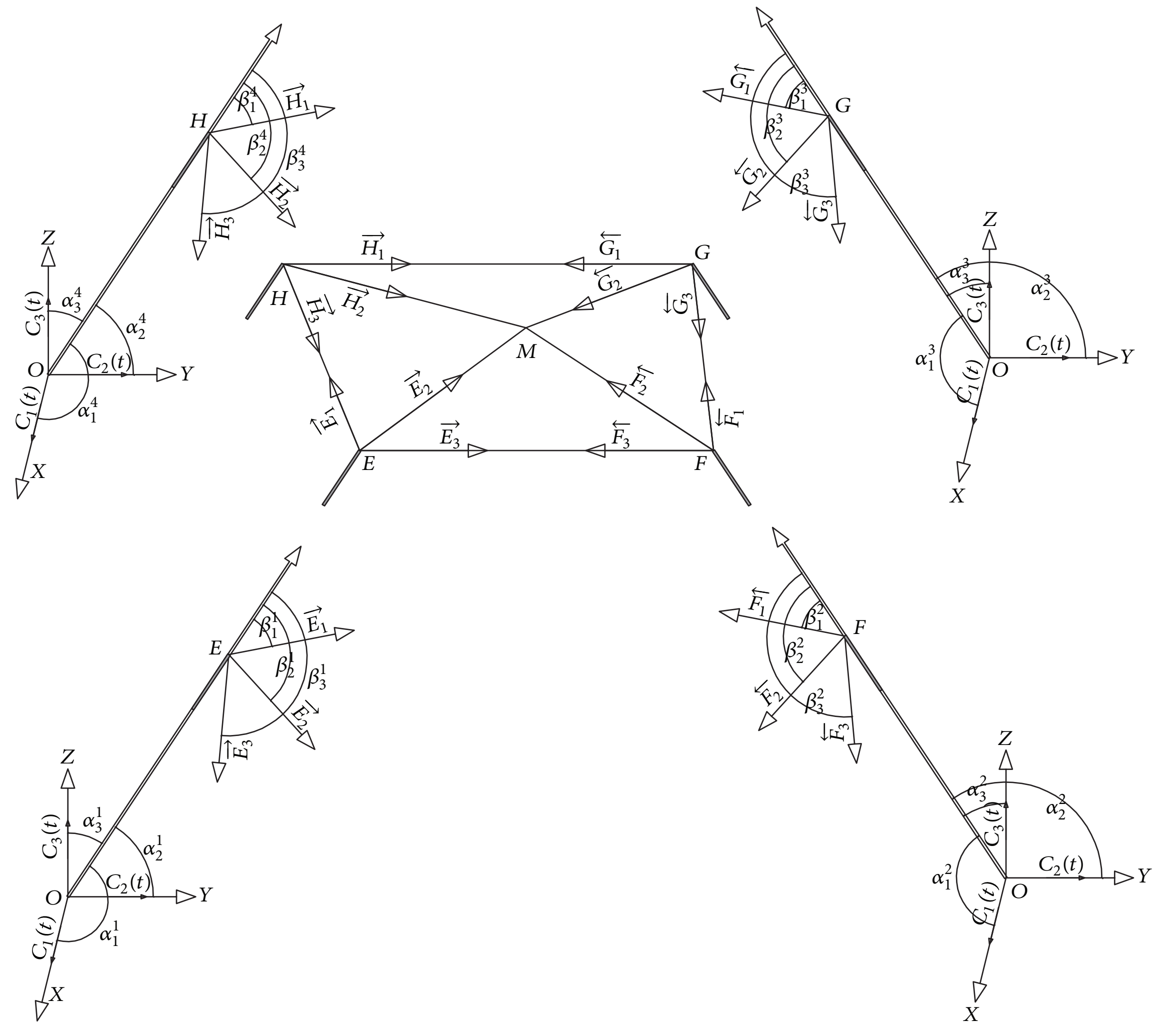

Figure 4: A simple model of a metallic roof.

In order to extend our idea to a more complex case of a metallic roof we can consider a model with 4 points (E, F, G, and H) where the seismic isolators act (see Figure 4). We can do all the computation in order to compute all the tension forces in the roof arms. Our aim is to compute the points with maximal tension forces and to see the dependence of tension forces in terms of the angles between the seismic isolators and the roof and between the arms of the roof. Since we know the maximal tension forces which can be supported by the arms, the second aim is to choose the best angles and the best concave behaviour of the seismic isolators under the action of an earthquake. Of course, the final aim is to minimize the maximum of the tension forces in the metallic roof.

Firstly, we need to remark that the earthquake produces the same displacements along the same axes for each of the points E, F, G, H described by the same displacement functions $c_1(t), c_2(t), c_3(t), t > 0$.

Hence, we consider 4 seismic isolators which support the roof at the points E, F, G, H. We denote by $\alpha_1^i, \alpha_2^i, \alpha_3^i$ the angles between the direction of the seismic isolators and each of the three positive senses of the taxes Ox, Oy, Oz and by $\beta_1^i, \beta_2^i, \beta_3^i$ the angles between the direction of the seismic isolators and each of the three arms. The index $i = 1, \ldots, 4$ used for the angles, denotes the corresponding angles for the seismic isolator fixed in E, F, G, and H, respectively.

In a similar way, we compute the resultant forces acting on each arms starting from E, F, G, and H, at each moment of time, and we obtain the following estimates:

$$E_1 \longrightarrow E_1 + E(t)\cos\beta_1^1,$$

$$E_2 \longrightarrow E_2 + E(t)\cos\beta_2^1,$$

$$E_3 \longrightarrow E_3 + E(t)\cos\beta_3^1,$$

$$F_1 \longrightarrow F_1 + F(t)\cos\beta_1^2,$$

$$F_2 \longrightarrow F_2 + F(t)\cos\beta_2^2,$$

$$F_3 \longrightarrow F_3 + F(t)\cos\beta_3^2,$$

$$G_1 \longrightarrow G_1 + G(t)\cos\beta_1^3,$$

$$G_2 \longrightarrow G_2 + G(t)\cos\beta_2^3,$$
$$G_3 \longrightarrow G_3 + G(t)\cos\beta_3^3,$$
$$H_1 \longrightarrow H_1 + H(t)\cos\beta_1^4,$$
$$H_2 \longrightarrow H_2 + H(t)\cos\beta_2^4,$$
$$H_3 \longrightarrow H_3 + H(t)\cos\beta_3^4, \tag{22}$$

where

$$\begin{aligned}
E(t) &= \mathrm{smn}^1(t) \cdot f\left(\left|c_1(t)\cos\alpha_1^1 + c_2(t)\cos\alpha_2^1 + c_3(t)\cos\alpha_3^1\right|\right),\\
\mathrm{smn}^1(t) &= \mathrm{sgn}\left(c_1(t)\cos\alpha_1^1 + c_2(t)\cos\alpha_2^1 + c_3(t)\cos\alpha_3^1\right),\\
F(t) &= \mathrm{smn}^2(t) \cdot f\left(\left|c_1(t)\cos\alpha_1^2 + c_2(t)\cos\alpha_2^2 + c_3(t)\cos\alpha_3^2\right|\right),\\
\mathrm{smn}^2(t) &= \mathrm{sgn}\left(c_1(t)\cos\alpha_1^2 + c_2(t)\cos\alpha_2^2 + c_3(t)\cos\alpha_3^2\right),\\
G(t) &= \mathrm{smn}^3(t) \cdot f\left(\left|c_1(t)\cos\alpha_1^3 + c_2(t)\cos\alpha_2^3 + c_3(t)\cos\alpha_3^3\right|\right),\\
\mathrm{smn}^3(t) &= \mathrm{sgn}\left(c_1(t)\cos\alpha_1^3 + c_2(t)\cos\alpha_2^3 + c_3(t)\cos\alpha_3^3\right),\\
H(t) &= \mathrm{smn}^4(t) \cdot f\left(\left|c_1(t)\cos\alpha_1^4 + c_2(t)\cos\alpha_2^4 + c_3(t)\cos\alpha_3^4\right|\right),\\
\mathrm{smn}^4(t) &= \mathrm{sgn}\left(c_1(t)\cos\alpha_1^4 + c_2(t)\cos\alpha_2^4 + c_3(t)\cos\alpha_3^4\right),
\end{aligned} \tag{23}$$

and $\mathrm{sgn}(a) = 1$, if $a \geq 0$ and $\mathrm{sgn}(a) = -1$, if $a < 0$.

Now, by adding the tension forces with act in different senses on each arm, we obtain the total tension forces in each arm. The unique point of maximal tension forces can be now easily computed by taking into account the estimates from above. For example, if we want to compute the tension force at the time $t > 0$ in the arm EF we need to estimate the value of the following expression:

$$E_3 + E(t)\cos\beta_3^1 + F_3 + F(t)\cos\beta_3^2. \tag{24}$$

In order to avoid the fact that the above expression is bigger than the maxim tension force which can be supported by the metallic arm EF, we need to manage all the angles and the concave function f. This problem is now purely mathematics and differential calculus can be used to find the best angles, depending on which concave function we choose, in order to minimize the tension forces appearing in the arm EF.

We recall that the properties and existence results for the points of maximum/minimum tension forces in a very complex roof is based on Theorems 7 and 8.

Another interesting aim is to minimize the total tension forces in the roof. More precisely, another idea can be to choose in a smart way all the angles in order to reduce all the amount of tension forces. This can be done, by considering minimizing problems in each part (subset) of the roof and of course, in the entire roof.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The second author of this paper is supported by UEFISCDI Research Project PN-II-RU-TE-2014-4-1109.

References

[1] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, vol. 319 of *Grundlehren der Mathematischen Wissenschaften*, Springer, 1999.

[2] W. Ballmann, *Lectures on Spaces with Nonpositive Curvature*, DMV Seminar Band 25, Birkhausser, Basel, Switzerland, 2005.

[3] J. Jost, *Nonpositive Curvature: Geometric and Analytic Aspects*, Lectures in Mathematics ETH Zurich, Birkhauser, Basel, Switzerland, 1997.

[4] C. P. Niculescu and I. Rovenţa, "Fan's inequality in geodesic spaces," *Applied Mathematics Letters*, vol. 22, no. 10, pp. 1529–1533, 2009.

[5] K. T. Sturm, "Probability measures on metric spaces of non-positive curvature," in *Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces, Lecture Notes from a Quarter Program on Heat Kernels, Random Walks, and Analysis on Manifolds and Graphs, April 16–July 13, 2002, Paris, France*, P. Auscher, T. Coulhon, and A. Grigoryan, Eds., vol. 338 of *Contemporary Mathematics*, pp. 357–390, American Mathematical Society, 2003.

[6] J. Lawson and Y. Lim, "Monotonic properties of the least squares mean," *Mathematische Annalen*, vol. 351, no. 2, pp. 267–279, 2011.

[7] C. P. Niculescu and I. Rovenţa, "An approach of majorization in spaces with a curved geometry," *Journal of Mathematical Analysis and Applications*, vol. 411, no. 1, pp. 119–128, 2014.

[8] N. L. Biggs, E. K. Lloyd, and R. J. Wilson, *Graph Theory*, Oxford University Press, Oxford, UK, 1998.

[9] K. B. Reid, "Balance vertices in trees," *Networks*, vol. 34, no. 4, pp. 264–271, 1999.

[10] C. P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications. A contemporary Approach*, vol. 23 of *CMS Books in Mathematics*, Springer, New York, NY, USA, 2006.

[11] M. Mălin and I. Rovenţa, "Some remarks on convex network flows for K-spiders," *Mathematical Problems in Engineering*, vol. 2015, Article ID 710516, 6 pages, 2015.

[12] I. Rovenţa, "The convex functions on the spiders," in *Proceedings of the Joint International Conference of Doctoral and Post-Doctoral Researchers*, Craiova, Romania, September 2014.