Internat. J. Math. & Math. Sci.
Vol. 6 No. 3 (1983) 483-486

UNIVALENT FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVES

S.L. SHUKLA and VINOD KUMAR

Department of Mathematics
Janta College, Bakewar
Etawah 206124, India

(Received September 28, 1982 and in revised form August 8, 1983)

ABSTRACT. We study some radii problems concerning the integral operator

$$F(z) = \frac{\gamma+1}{z^{\gamma}} \int_0^z u^{\gamma-1} f(u)\, du$$

for certain classes, namely K_n and $M_n(\alpha)$, of univalent functions defined by Ruscheweyh derivatives. Infact, we obtain the converse of Ruscheweyh's result and improve a result of Goel and Sohi for complex γ by a different technique. The results are sharp.

KEY WORDS AND PHRASES. Hadamard product, starlike, univalent.
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 30C45.

1. INTRODUCTION.

Let S denote the class of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are regular in the unit disc $U = \{z : |z| < 1\}$.

A function f of S is said to belong to the class K_n if

$$\operatorname{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \frac{1}{2}, \text{ where } z \in U,\ n \in N_0 = \{0,1,2,\ldots\},$$

and

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z),$$

and the operation (*) stands for the convolution or Hadamard product of the power series.

Ruscheweyh [1] introduced the classes K_n and showed, via the inclusion relation $K_{n+1} \subset K_n$, that the functions in these classes are starlike of order 1/2 and hence are univalent. He also observed that

$$D^n f(z) = z(z^{n-1} f(z))^{(n)}/n!\ .$$

Following Al-Amiri [2], we also refer to $D^n f$ as the nth order Ruscheweyh derivative of f.

A function f of S is said to belong to the class $M_n(\alpha)$, $0 \le \alpha < 1$, if

$$\operatorname{Re}\left\{\frac{D^{n+1} f(z)}{z}\right\} > \alpha, \quad z \in U,\ n \in N_0.$$

Goel and Sohi [3] introduced the classes $M_n(\alpha)$ and showed, via the inclusion relation $M_{n+1}(\alpha) \subset M_n(\alpha)$, that the functions in these classes are univalent.

Ruscheweyh [1] proved that the function F defined by

$$F(z) = \frac{\gamma+1}{z^\gamma} \int_0^z u^{\gamma-1} f(u)\, du$$

belongs to K_n if $f \in K_n$ and γ is a complex number such that $\operatorname{Re}(\gamma)>(n-1)/2$. Goel and Sohi [3] obtained an analogous result for the class $M_n(\alpha)$. Conversely, they [3, Theorem 4] determined the radius of the disc in which $f \in M_n(\alpha)$ when $F \in M_n(\alpha)$ and γ is a real number such that $\gamma > -1$.

In the present paper we obtain the converse of Ruscheweyh's [1] result. We also obtain the above mentioned result of Goel and Sohi [3, Theorem 4], by using a different technique, for complex γ . The results are shown to be sharp.

2. PRELIMINARY LEMMA.

Let P_0 denote the class of functions of the form $P(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$ which are regular in U and satisfy $\operatorname{Re}\{p(z)\} > 0$ for $z \in U$.

We require the following lemma which follows from a result of Yoshikawa and Yoshikai [4, Theorem 1]:

LEMMA 2.1. Let $p \in P_0$. If b is a non-negative real number and c is a complex number such that $c+b \neq 0$, then

$$\operatorname{Re}\{p(z) + zp'(z)/(c+bp(z))\} > 0$$

holds in $|z| < R(c,b) = \{|c|^2+2+4b+b^2-\sqrt{E}\}^{1/2}/|c-b|$, where $E = 2(2+4b+b^2)\,|c|^2 + 2b^2 \operatorname{Re}(c^2)+4(1+b^2)(1+2b)$. The result is sharp with the extremal function $p(z) = (1+z)/(1-z)$.

3. MAIN RESULTS.

In the following theorem we study the converse of Ruscheweyh's [1] result.

THEOREM 3.1. Let γ be a complex number such that $\operatorname{Re}(\gamma) > -1$. If $F \in K_n$, then the function f defined by

$$F(z) = \frac{\gamma+1}{z^\gamma} \int_0^z u^{\gamma-1} f(u)\, du \tag{3.1}$$

satisfies $\operatorname{Re}\{D^{n+1} f(z)/D^n f(z)\} > 1/2$ in $|z| < R(c,b)$ where $c = (\gamma-n) + (n+1)/2$, $b = (n+1)/2$, and R(c,b) is given by Lemma 2.1. The result is sharp.

For the existence of the integral in (3.1), the power represents principle branch. We note that the integral operator under consideration can also be written as

$$F(z) = (\gamma+1) \int_0^1 t^{\gamma-1} f(tz)\, dt$$

which solves the question of principal branch.

PROOF. It is easy to verify the identity

$$z(D^n F(z))' = (n+1)\, D^{n+1} F(z) - nD^n F(z). \tag{3.2}$$

Also, from the definition of F it can be verified that

$$z(D^n F(z))' = (\gamma+1)\, D^n f(z) - \gamma D^n F(z). \tag{3.3}$$

Since $F \in K_n$, there exists a function p in P_o such that

$$\frac{D^{n+1} F(z)}{D^n F(z)} = \frac{1}{2}\,(1 + p(z)). \tag{3.4}$$

Using (3.2), (3.3), and (3.4), we get

$$\begin{aligned}(\gamma+1)\, D^{n+1} f(z) &= \gamma D^{n+1} F(z) + z(D^{n+1} F(z))' \\ &= \frac{\gamma}{2}\,(1+p(z))\, D^n F(z) + \frac{1}{2}\, zp'(z)\, D^n F(z) \\ &\quad + \frac{1}{2}\,(1+p(z))\, z(D^n F(z))' \\ &= \frac{\gamma}{2}\,(1+p(z)) D^n F(z) + \frac{1}{2}\, zp'(z) D^n F(z) \\ &\quad + \frac{1}{2}\,(1+p(z))\{(n+1)\, D^{n+1} F(z) - nD^n F(z)\}.\end{aligned}$$

Thus,

$$(\gamma+1) D^{n+1} f(z) = \frac{1}{2}\,[(\gamma-n)(1+p(z)) + zp'(z) + (\tfrac{n+1}{2})(1+p(z))^2]\, D^n F(z). \tag{3.5}$$

Also,

$$\begin{aligned}(\gamma+1)\, D^n f(z) &= \gamma D^n F(z) + z(D^n F(z))' \\ &= \gamma D^n F(z) + (n+1) D^{n+1} F(z) - nD^n F(z) \\ &= [(\gamma-n) + \tfrac{1}{2}(n+1)(1+p(z))]\, D^n F(z).\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$\left[\frac{D^{n+1} f(z)}{D^n f(z)} - \frac{1}{2}\right]/(1/2) = p(z) + \frac{zp'(z)}{c+bp(z)}$$

where $c = (\gamma-n) + (n+1)/2$ and $b = (n+1)/2$.

The required result now follows by using Lemma 2.1.

To establish sharpness, we take $F(z) = z/(1-z)$.

Then,

$$\frac{D^{n+1} F(z)}{D^n F(z)} = \frac{z/(1-z)^{n+2}}{z/(1-z)^{n+1}} = \frac{1}{2}\,(1 + \frac{1+z}{1-z}). \tag{3.7}$$

From (3.4) and (3.7), we get $p(z) = (1+z)/(1-z)$; hence, the sharpness of the result follows from that of Lemma 2.1.

In the following theorem, we obtain the converse of the result of Goel and Sohi [3, Theorem 2] for complex γ.

THEOREM 3.2. Let $F \in M_n(\alpha)$ and γ be a complex number such that $\text{Re}(\gamma) > -1$.

If f is defined by (3.1), then Re $\{\frac{D^{n+1}f(z)}{z}\} > \alpha$ in $|z| < R^* = \frac{\sqrt{(|\gamma+1|^2+1)} - 1}{|\gamma+1|}$.

The result is sharp.

PROOF. Since $F \in M_n(\alpha)$, there exists a function p in P_o such that

$$D^{n+1} F(z) = \alpha z + (1-\alpha)zp(z). \tag{3.8}$$

Differentiating (3.8) and using (3.3), we get

$$\frac{D^{n+1} f(z)/z-\alpha}{1-\alpha} = p(z) + \frac{zp'(z)}{\gamma+1} . \tag{3.9}$$

Using Lemma 2.1 for $c = \gamma+1$ and $b = 0$, we find that the real part of right hand side of (3.9) is greater than zero in $|z| < R^* = \frac{\sqrt{(|\gamma+1|^2+1)} - 1}{|\gamma+1|}$. Hence, Re$\{\frac{D^{n+1}f(z)}{z}\} >$ α in $|z| < R^*$.

The sharpness of the result follows easily by taking the function F defined by

$$D^{n+1} F(z) = \alpha z + (1-\alpha)\ z(\frac{1+z}{1-z}).$$

Goel and Sohi [3, Theorem 2] proved that, if $f \in M_n(\alpha)$, then the function F defined by (3.1) also belongs to $M_n(\alpha)$, provided that Re$(\gamma) > -1$. In this direction, the following theorem provides a better result for suitable choices of γ .

THEOREM 3.3. If $f \in M_n(\alpha)$ and γ is a real number such that $-1 < \gamma \leq n+1$, then the function F defined by (3.1) belongs to $M_{n+1}(\alpha)$.

PROOF. Since

$$z(D^{n+1} F(z))' = (n+2)\ D^{n+2} F(z) - (n+1)\ D^{n+1} F(z)$$

and, by the definition of F,

$$z(D^{n+1}F(z))' = (\gamma+1)D^{n+1}f(z) - \gamma D^{n+1}F(z),$$

we have

$$\text{Re}\{(n+2)\ \frac{D^{n+2}F(z)}{z} - (n+1-\gamma)\ \frac{D^{n+1}F(z)}{z}\} = (\gamma+1)\text{Re}\ \{\frac{D^{n+1}f(z)}{z}\} > (\gamma+1)\alpha.$$

Since $F \in M_n(\alpha)$, the above inequality leads us to

$$(n+2)\ \text{Re}\{\frac{D^{n+2}F(z)}{z}\} > (n+1-\gamma)\ \text{Re}\{\frac{D^{n+1}F(z)}{z}\} + (\gamma+1)\alpha$$

$$\geq (n+1-\gamma)\alpha + (\gamma+1)\alpha = (n+2)\alpha.$$

Hence, $F \in M_{n+1}(\alpha)$.

ACKNOWLEDGEMENT. The authors are thankful to the referee for his valuable suggestions about the earlier version of this paper.

REFERENCES

1. RUSCHEWEYH, S. New Criteria for Univalent Functions, Proc. Amer. Math. Soc. 49 (1975), 109-115.

2. AL-AMIRI, H.S. On Ruscheweyh Derivatives, Ann. Polon. Math. 38 (1980), 87-94.

3. GOEL, R.M. and SOHI, N.S. Subclasses of Univalent Functions, Tamkang J. Math. 11 (1980), 77-81.

4. YOSHIKAWA, H. and YOSHIKAI, T. Some Notes on Bazilevic Functions, J. London Math. Soc. (2) 20 (1979), 79-85.