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Research Article

A Simple Repairable System with Warning Device

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This paper considers a simple repairable system with a warning device which can signal an alarm when the system is not in good condition and a repairman who can have delayed-multiple vacations. By using Markov renewal process theory and the probability analysis method, the system is first described into a group of integrodifferential equations. Then the unique existence and asymptotic stability, especially the exponential stability of the system dynamic solution, are studied by using the strongly continuous semigroup theory or C_0 semigroup theory and the spectrum theory. The reliability indices and some applications (such as the comparisons of some indexes and profit of systems with and without warning device) as well as numerical examples are presented at the end of the paper.

1. Introduction

A repairable system is a system which, after failing to perform one or more of its functions satisfactorily, can be restored to fully satisfactory performance by any method rather than the replacement of the entire system. Since the 1960s, various repairable system models have been established and researched, such as one-unit, series, parallel, series-parallel, redundance, k-out-of-n, multiple-state, human-machine, and software systems.

In traditional repairable systems, it is assumed that the repairman or server remains idle until a failed component presents. However, as Mobley [1] pointed out, one-third of all maintenance costs were wasted as the result of unnecessary or improper maintenance activities. Today, the role of maintenance tends to be a "profit contributor." Therefore, much more profit can be produced when the repairman in a system might take a sequence of vacations in the idle time. During vacation, the repairman is not in the system or may take another assigned job. From the perspective of rational use of human resources, the introduction of repairman's vacation makes modeling of the repairable system more realistic and flexible. This is due to the fact that, in practice, the vast majority of small- and medium-sized enterprisers (SMEs) cannot afford to hire a full-time repairman. So, the repairman in SMEs usually plays two roles: one for looking after the units and the other for other duties. Under normal circumstances, if the system is found to be failed, the repairman repairs it immediately after the end of vacation; otherwise, the repairman leaves the system for other duties or for another vacation.

Vacation model originally arises in queueing theory and has been well studied in the past three decades and successfully applied in many areas such as manufacturing/service and computer/communication network systems. Excellent surveys on the earlier works of vacation models have been reported by Doshi [2], Takagi [3], and Tian and Zhang [4]. Ke et al. [5] provided a summary of the most recent research works on vacation queueing systems in the past 10 years, in which a wide class of vacation policies for governing the vacation mechanism is presented.

In the past decade, inspired by the vacation queueing theory, some researchers introduced vacation model into repairable systems. The available references concerning repairman vacation in repairable systems can be classified into two categories: one focuses on system indices and the other deals with optimization problems.

For the first category, Jain and Rakhee [6] considered a bilevel control policy for a machining system with two repairmen. One starts to work when queue size of failed units reaches a preassigned level. The other's provision in case of long queue of failed units may help to reduce the backlog. The steady state queue size distribution is obtained by applying the recursive method. Hu et al. [7] studied the steady-state availability and the mean uptime of a series-parallel repairable system, which consists of one master control unit and two slave units and a single repairman runs single vacation via the method of supplementary variable and the vector Markov process theory. Q. T. Wu and S. M. Wu [8] analyzed some reliability indices of a cold standby system consisting of two repairable units, a switch and a repairman who may not always be on the job site while taking vacation. Yuan [9] and Yuan and Cui [10] studied a k-out-of-n: G system and a consecutive-k-out-of-n: F system, respectively, with R repairmen who can take multiple vacations and by using Markov model, the analytical solution of some reliability indexes was discussed. Yuan and Xu [11] studied a deteriorating system with one repairman who can have multiple vacations. By means of the geometric process and the supplementary variable techniques, a group of partial differential equations of the system was presented and some reliability indices were derived. Ke and Wu [12] studied a multiserver machine repair model with standbys and synchronous multiple vacation, and the stationary probability vectors were obtained by using the matrix-analytical approach and the technique of matrix recursive.

For the second category, Ke and Wang [13] studied a machine repair problem consisting of M operating machines with two types of spare machines and R servers (repairmen) who can take different vacation policies. The steady-state probabilities of the number of failed machines in the system as well as the performance measures were derived by using the matrix geometric theory and a direct search algorithm was used to determine the optimal values of the number of two types of spares and the number of servers while maintaining a minimum specified level of system availability. Jia and Wu [14] considered a replacement policy for a repairable system that cannot be repaired "as good as new" with a repairman who can have multiple vacations. By using geometric processes, the explicit expression of the expected cost rate was derived, and the corresponding optimal policy was determined analytically or numerically. Yuan and Xu [15, 16] considered, respectively, a deteriorating repairable system and a cold standby repairable system with two different components of different priority in use, both with one repairman who can take multiple vacations. The explicit expression of the expected cost rate was given and an optimal replacement policy was discussed. Yu et al. [17] analyzed a phase-type geometric process repair model with spare device procurement lead time and repairman's multiple vacations. Employing the theory of renewal reward process, the explicit expression of the long-run average profit rate for the system was derived, and the optimal maintenance policy was also numerically determined.

A survey of the current research effort suggests that steady behavior (the steady-state indices or the steady-state optimization problems) of the systems is widely explored, which is because the transient solution of a system is difficult or sometimes impossible to obtain. Therefore, researchers usually substitute the steady-state solution for the instantaneous one of a system since the steady-state solution can be easily obtained by Laplace transform and a limit theorem. Whereas, Laplace transform is based on two hypotheses; namely, the instantaneous solution of the interested system exists and the instantaneous solution is stable. Whether the hypotheses held or not is still an open question and should be justified. Moreover, the substitution of the steady-state solution for the instantaneous one is not always rational. Readers are referred to [18, 19] for detailed information or explanations. Thus the study of time-dependent solution of a system as well as its stability is indispensable.

Warning systems emerging in the background of repairable systems are stepping into the times of requiring both advanced warning and real-time fault detection. The so-called warning system is able to send emergency signals and report dangerous situations prior to disasters, catastrophes, and/or other dangers which need to watch out based on previous experiences and/or observed possible omens. Real-time warning systems play an important role in fault management in banking, telecommunications, securities, electric power, and other industries. If the warning is prompted during system operation, operating staff can choose whether to shut down the system, operate carefully, or repair the system. Warning systems can help users to achieve the 24-hour uninterrupted real-time monitoring and alerting during running of various types of network infrastructure sand application services. Accordingly, the study of repairable systems with warning device is important in both theory and practice. However, repairable systems with warning device are seldom reported in the current literatures.

To this end, this paper considers a simple repairable system (which includes one unit and a repairman) with a warning device. The warning device can signal an alarm once the system fails. Considering the practice situation, we also assume that it may signal an alarm when not necessary, which is called a false alarm. The repairman in the considered system follows delayed-multiple vacations policy. The delayed-multiple vacation means that the repairman will not leave for a vacation immediately if no component failed. However, there is a stochastic vacation-preparing period in which if a failed component appears the repairman will stop the vacation preparing and serve it immediately; otherwise the repairman will take a rest at the end of the vacation-preparing period. When a vacation ends, the repairman will either deal with the failed components waiting in the system or prepare for another vacation. In this paper, we are devoted to studying the asymptotic behavior of the system by strongly continuous semigroup theory and make comparisons of reliability indexes (such as reliability, availability, and the probability of the repairman' vacation) and profit of the two systems with and without warning device.

The paper is structured as follows. The coming section introduces the system model specifically and expresses it as a group of integrodifferential equations by Markov renewal process theory and the probability analysis method. Section 3 discusses the asymptotic behavior of the system by strongly continuous semigroup theory or C_0 semigroup theory. Section 4 presents some reliability indices of the system, and the steady-state indexes are discussed from the viewpoint of eigenfunction of the system operator. In Section 5, comparisons of indexes and profit of systems with and without warning device are made. A brief conclusion is offered in Section 6.

2. System Formulation

The system model of interest is a simple repairable system (i.e., a repairable system with a unit and a repairman) with repairman vacation and a warning device. It is described specifically as follows: at the initial time $t = 0$, the unit is new; the system begins to work and the repairman starts to prepare for the vacation. The warning device may signal true alarms (when the system needs to be repaired) or false alarms (when the system does not need to be repaired). In order to distinguish true or false alarms, the system is inspected after the warning device sends an alarm. The distributions of time duration of the distinguishing process for false and true alarm follow exponential distributions with parameters λ_1 and λ_2, respectively. If the warning device sends an alarm in the delayed-vacation period, the repairman will either deal with the unit if it is a true alarm and the delayed-vacation is terminated or leave for a vacation after the delayed-vacation period ends if it is a false alarm. Whenever the repairman returns from a vacation, he or she either prepares for the next vacation if the unit is working or deals with the failed unit immediately or stays in the system if the warning device has sent an alarm. And assume that the time repairman returning to the system cannot be late than the time warning device sending next alarm. The repair facility is neither failed nor deteriorated. The unit is repaired as good as new. Further we assume the following:

(1) The distribution function of the delayed-vacation time length of the repairman is $D(t) = 1 - e^{-\varepsilon t}$, $t \geq 0$, and ε is a positive constant and the distribution function of its vacation time length is $V(t) = 1 - e^{-\int_0^t r(x)\mathrm{d}x}$, and $\int_0^\infty t\mathrm{d}V(t) = 1/r$.

(2) The distribution function of time interval between warning device's beginning to work and its first sending alarm as well as the distribution function of time interval of two successive alarms is $U(t) = 1 - e^{-\alpha_0 t}$, $t \geq 0$, and α_0 is a positive constant.

(3) The distribution function of the repair time length of the system is $G(t) = 1 - e^{-\int_0^t \mu(x)\mathrm{d}x}$, and $\int_0^\infty t\mathrm{d}G(t) = 1/\mu$.

(4) The above stochastic variables are independent of each other.

Set all the possible states at time t as follows.

0: the system is working and the repairman is preparing for the vacation.

1: the system is working and the repairman is on vacation.

2: the system is warning and the repairman is in the system.

21: the warning is distinguished as a false alarm and the repairman is in the system.

22: the warning is distinguished as a true alarm and the unit is repaired by the repairman.

3: the system is warning and the repairman is on vacation.

31: the warning is distinguished as a false alarm while the repairman is on vacation.

32: the warning is distinguished as a true alarm and the unit needs to be repaired while the repairman is on vacation.

Then by using stochastic process theory and probability analysis method, the repairable system model described above can be expressed by a group of integrodifferential equations with integral boundaries as below:

$$\begin{aligned}
&\left(\frac{\mathrm{d}}{\mathrm{d}t} + \varepsilon + \alpha_0\right) P_0(t) \\
&\quad = \int_0^\infty r(x) P_1(t,x)\,\mathrm{d}x + \int_0^\infty \mu(y) P_{22}(t,y)\,\mathrm{d}y, \\
&\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \alpha_0 + r(x)\right] P_1(t,x) = 0, \\
&\left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda_1 + \lambda_2\right) P_2(t) \\
&\quad = \alpha_0 P_0(t) + \int_0^\infty r(x) P_3(t,x)\,\mathrm{d}x, \\
&\left(\frac{\mathrm{d}}{\mathrm{d}t} + \varepsilon\right) P_{21}(t) = \lambda_1 P_2(t) + \int_0^\infty r(x) P_{31}(t,x)\,\mathrm{d}x, \\
&\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu(y)\right] P_{22}(t,y) = 0, \\
&\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_1 + \lambda_2 + r(x)\right] P_3(t,x) = \alpha_0 P_1(t,x), \\
&\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + r(x)\right] P_{3i}(t,x) = \lambda_i P_3(t,x), \quad i = 1,2.
\end{aligned} \tag{1}$$

The boundary conditions are

$$\begin{aligned}
&P_1(t,0) = \varepsilon\left(P_0(t) + P_{21}(t)\right), \\
&P_{22}(t,0) = \lambda_2 P_2(t) + \int_0^\infty r(x) P_{32}(t,x)\,\mathrm{d}x, \\
&P_3(t,0) = P_{3i}(t,0) = 0, \quad i = 1,2.
\end{aligned} \tag{2}$$

The initial conditions are

$$P_0(0) = 1, \text{ the others equal to } 0. \tag{3}$$

Here $P_i(t)$ represents the probability that the system is in state i at time t, $i = 0, 2, 21$. $P_j(t, x)\mathrm{d}x$ represents the probability that the system is in state j with elapsed vacation time lying in $[x, x+\mathrm{d}x)$ at time t, $j = 1, 3, 31, 32$. $P_{22}(t, y)\mathrm{d}y$ represents the probability that the system is in state 22 with elapsed repair time lying in $[y, y + \mathrm{d}y)$ at time t.

Concerning the practical background, we can assume that $r(x)$ and $\mu(y)$ are nonnegative functions satisfying

$$\tilde{r} = \sup_{x\in[0,\infty)} r(x) < \infty, \qquad \tilde{\mu} = \sup_{y\in[0,\infty)} \mu(y) < \infty. \tag{4}$$

3. Stability of System Solution

In this section, we will discuss the stability, especially the exponential stability of the system solution by C_0 semigroup theory. For this purpose, we first translate the system equations (1)–(3) into an abstract Cauchy problem in a suitable Banach space. Then some primary properties of system operator and its adjoint operator are presented. With the preparation, the unique existence and asymptotic stability of system solution can be derived readily. Further, the exponential stability of the system solution is also studied by constructing proper operators.

3.1. System Transformation. In this section, we translate the system equations (1)–(3) into an abstract Cauchy problem in a suitable Banach space.

Firstly, choose the state space to be

$$X = \{P = (P_0, P_1(x), P_2, P_{21}, P_{22}(y), P_3(x), P_{31}(x), P_{32}(x))^{\mathrm{T}} : P_i \in \mathbb{R}, P_j \in L^1(\mathbb{R}^+),\ i = 0, 2, 21; j = 1, 22, 3, 31, 32\} \tag{5}$$

endowed with norm

$$\|P\| = \sum_{i=0,2,21} |P_i| + \sum_{j=1,22,3,31,32} \|P_j\|_{L^1(\mathbb{R}^+)}, \tag{6}$$

where $\mathbb{R}^+$ denotes the set of nonnegative real numbers. It is obvious that X is a Banach space.

Next, define system operator in state space X as

$$AP = \begin{pmatrix} -(\varepsilon+\alpha_0)P_0 + \int_0^\infty r(x)P_1(x)\,\mathrm{d}x + \int_0^\infty \mu(y)P_{22}(y)\,\mathrm{d}y \\ -P_1'(x) - [\alpha_0 + r(x)]P_1(x) \\ -(\lambda_1+\lambda_2)P_2 + \alpha_0 P_0 + \int_0^\infty r(x)P_3(x)\,\mathrm{d}x \\ -\varepsilon P_{21} + \lambda_1 P_2 + \int_0^\infty r(x)P_{31}(x)\,\mathrm{d}x \\ -P_{22}'(y) - \mu(y)P_{22}(y) \\ -P_3'(x) - [\lambda_1+\lambda_2+r(x)]P_3(x) + \alpha_0 P_1(x) \\ -P_{31}'(x) - r(x)P_{31}(x) + \lambda_1 P_3(x) \\ -P_{32}'(x) - r(x)P_{32}(x) + \lambda_2 P_3(x) \end{pmatrix} \tag{7}$$

with domain

$$D(A) = \{P = (P_0, P_1, P_2, P_{21}, P_{22}, P_3, P_{31}, P_{32})^{\mathrm{T}} \in X : P_j' \in L^1(\mathbb{R}^+) \text{ are absolutely continuous functions satisfying } P_1(0) = \varepsilon(P_0 + P_{21}), P_{22}(0) = \lambda_2 P_2 + \int_0^\infty r(x)P_{32}(x)\,\mathrm{d}x, P_3(0) = P_{3i}(0) = 0,\ j = 1, 22, 3, 3i,\ i = 1, 2\}. \tag{8}$$

Thus the system equations (1)–(3) can be rewritten as an abstract Cauchy problem in the Banach space X:

$$\begin{aligned} &\frac{\mathrm{d}P(t,\cdot)}{\mathrm{d}t} = AP(t,\cdot), \quad t \ge 0, \\ &P(t,\cdot) = (P_0(t), P_1(t,x), P_2(t), P_{21}(t), P_{22}(t,y), P_3(t,x), P_{31}(t,x), P_{32}(t,x))^{\mathrm{T}}, \\ &P(0,\cdot) = (1, 0, 0, \ldots, 0)^{\mathrm{T}}_{1\times 8}. \end{aligned} \tag{9}$$

3.2. Properties of System Operator A. In this section, we present some concerned properties of system operator A including the distribution of its spectrum.

Lemma 1. *The system operator A is a densely closed dissipative operator.*

Lemma 2. *For any $\gamma \in \mathbb{C}$ satisfying* $\operatorname{Re}\gamma > 0$ *or $\gamma = ia$, $a \in \mathbb{R} \setminus \{0\}$, γ is a regular point of the system operator A.*

Lemma 3. *0 is an eigenvalue of the system operator A with algebraic multiplicity one.*

3.3. Properties of Adjoint Operator A^*. In this section, we present some properties of A^*, the adjoint operator of system operator A, including its spectrum distribution.

The dual space of X is

$$X^* = \mathbb{R} \times L^\infty(\mathbb{R}^+) \times \mathbb{R}^2 \times (L^\infty(\mathbb{R}^+))^4. \tag{10}$$

For $Q = (Q_0, Q_1(x), Q_2, Q_{21}, Q_{22}(y), Q_3(x), Q_{31}(x), Q_{32}(x))^{\mathrm{T}} \in X^*$, its norm is defined by

$$\|Q\| = \sup\{|Q_i|, \|Q_j\|_{L^\infty(\mathbb{R}^+)},\ i = 0, 2, 21,\ j = 1, 22, 3, 31, 32\}. \tag{11}$$

For any $P \in D(A)$ and $Q \in X^*$, the equality $\langle AP, Q\rangle = \langle P, A^*Q\rangle$ follows the expression of A^*, the adjoint operator of system operator A, and its domain as below:

$$A^{*}Q=\begin{pmatrix} -(\varepsilon+\alpha_0)Q_0+\varepsilon Q_1(0)+\alpha_0 Q_2 \\ Q_1'(x)-[\alpha_0+r(x)]Q_1(x)+r(x)Q_0+\alpha_0 Q_3(x) \\ -(\lambda_1+\lambda_2)Q_2+\lambda_1 Q_{21}+\lambda_2 Q_{22}(0) \\ -\varepsilon Q_{21}+\varepsilon Q_1(0) \\ Q_{22}'(y)-\mu(y)Q_{22}(y)+\mu(y)Q_0 \\ Q_3'(x)-(\lambda_1+\lambda_2)Q_3(x)+\lambda_1 Q_{31}(x)+\lambda_2 Q_{32}(x)+r(x)[Q_2-Q_3(x)] \\ Q_{31}'(x)-r(x)Q_{31}(x)+r(x)Q_{21} \\ Q_{32}'(x)-r(x)Q_{32}(x)+r(x)Q_{22}(0) \end{pmatrix} \triangleq (C+D)Q \tag{12}$$

$$D(A^{*})=\{Q=(Q_0,Q_1,Q_2,Q_{21},Q_{22},Q_3,Q_{31},Q_{32})^{\mathrm{T}}\in X^{*}:Q_j' \in L^{\infty}(\mathbb{R}^{+}) \text{ are absolutely continuous functions satisfying } Q_j(\infty)<\infty,\ j=1,22,3,31,32\}.$$

Here

$$C=\operatorname{diag}\left(-(\varepsilon+\alpha_0),\frac{\mathrm{d}}{\mathrm{d}x}-[\alpha_0+r(x)],-(\lambda_1+\lambda_2),-\varepsilon,\frac{\mathrm{d}}{\mathrm{d}y}-\mu(y),\frac{\mathrm{d}}{\mathrm{d}x}-[\lambda_1+\lambda_2+r(x)],\frac{\mathrm{d}}{\mathrm{d}x}-r(x),\frac{\mathrm{d}}{\mathrm{d}x}-r(x)\right),$$

$$D=\begin{pmatrix} 0 & \varepsilon\theta_1(\cdot) & \alpha_0 & 0 & 0 & 0 & 0 & 0 \\ r(x) & 0 & 0 & 0 & 0 & \alpha_0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & \lambda_2\theta_{22}(\cdot) & 0 & 0 & 0 \\ 0 & \varepsilon\theta_1(\cdot) & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu(y) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r(x) & 0 & 0 & 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & 0 & r(x) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r(x)\theta_{22}(\cdot) & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

and $\theta_k(\cdot)=L^{\infty}(\mathbb{R}^{+})\to\mathbb{C}$ satisfying $\theta_k(f)=f(0)$, $k=1,22$.

Lemma 4. *For any $\gamma\in\mathbb{C}$ satisfying*

$$\sup\left\{\frac{\varepsilon+\alpha_0}{|\gamma+\varepsilon+\alpha_0|},\frac{\alpha_0+M}{\operatorname{Re}\gamma+\alpha_0+M},\frac{\lambda_1+\lambda_2}{|\gamma+\lambda_1+\lambda_2|},\frac{\varepsilon}{|\gamma+\varepsilon|},\frac{\lambda_1+\lambda_2+M}{\operatorname{Re}\gamma+\lambda_1+\lambda_2+M},\frac{M}{\operatorname{Re}\gamma+M}\right\}<1, \tag{14}$$

$\gamma\in\rho(A^{*})$, the resolvent set of A^{*}, where $M=\sup\{\tilde{r},\tilde{\mu}\}$ and $\tilde{r},\tilde{\mu}$ are defined in (4).

The following result of eigenvalue 0 of A^{*} can also be obtained with the same method of Lemma 3.

Lemma 5. *0 is an eigenvalue of operator A^{*} with algebraic multiplicity one.*

3.4. Stability of System Solution. In this section, we will present the asymptotic stability, especially the exponential stability of the system solution, by using C_0 semigroup theory.

According to Phillips Theorem (see [20]) combining Lemmas 1 and 2, we can obtain the following result.

Theorem 6. *The system operator A generates a positive C_0 semigroup of contraction $T(t)$.*

Theorem 6 can readily derive the existence and uniqueness of system solution according to [21].

Theorem 7. *The system (9) has a unique nonnegative time-dependent solution $P(t,\cdot)$ with expression as*

$$P(t,\cdot)=T(t)P_0,\quad \forall t\in[0,\infty). \tag{15}$$

Remark 8. Because the initial condition P_0 of system (9) is not in the domain of system operator A, then the solution $P(t,\cdot)$ obtained by Theorem 7 is the mild solution of system (9). However, it can be proved that it is the classical solution of system (9) for $t>0$ with pure analysis method [22].

Noting that the C_0 semigroup $T(t)$ generated by A is uniformly bounded because it is contractive according to Theorem 6, then according to [21] combining Lemmas 2, 3, 4, and 5, the asymptotic stability of system (9) can be deduced readily as follows.

Theorem 9. *Let $\widehat{P}$ be the nonnegative eigenfunction corresponding to eigenvalue 0 of the system operator A satisfying $\|\widehat{P}\|=1$ and $Q^{*}=(1,1,1,1,1,1,1,1)^{\mathrm{T}}\in X^{*}$; then the time-dependent solution $P(t,\cdot)$ of system (9) converges to the nonnegative steady-state solution $\widehat{P}$. That is*

$$\lim_{t\to\infty}P(t,\cdot)=\langle P_0,Q^{*}\rangle\widehat{P}=\widehat{P}, \tag{16}$$

where P_0 is the initial value of the system.

Theorem 9 presented the asymptotic stability of the system solution. In the following, we will study a better stability behavior, that is, the exponential stability of the system solution, which is helpful to settle problems such that the convergence rate and the behavior of system solution.

For simplicity, we will divide the system operator A into two operators. The first one is a compact operator B, and the other $\overline{A}$ generates a quasi-compact C_0 semigroup. Then, by the perturbation of compact operator, the system operator A also generates a quasi-compact C_0 semigroup. Therefore, the system solution is exponentially stable.

For convenience, we first introduce several operators as follows:

$$\begin{aligned} BP &= \left(\int_0^\infty r(x) P_1(x)\,dx + \int_0^\infty \mu(y) P_{22}(y)\,dy, 0, \alpha_0 P_0 + \int_0^\infty r(x) P_3(x)\,dx, \lambda_1 P_2 \right. \\ &\quad \left. + \int_0^\infty r(x) P_{31}(x)\,dx, 0, \alpha_0 P_1(x), \lambda_1 P_3(x), \lambda_2 P_3(x)\right)^{\mathrm{T}} \quad \text{with } D(B) = X \\ \overline{A} &= A - B \\ &\text{with } D(\overline{A}) = \left\{P = (P_0, P_1, P_2, P_{21}, P_{22}, P_3, P_{31}, P_{32})^{\mathrm{T}} \in X : P_j \in L^1(\mathbb{R}^+) \text{ are absolutely continuous functions}, j = 1, 22, 3, 31, 32\right\}. \\ A_0 &= A \quad \text{with } D(A_0) = \left\{P \in D(\overline{A}) \mid P_j(0) = 0,\ j = 1, 22, 3, 31, 32\right\}. \end{aligned} \tag{17}$$

It is not difficult to know that $\overline{A}$ and A_0 are both closed operators with dense domains in X. And by the perturbation of C_0 semigroup, it can be known that $\overline{A}$ also generates a C_0 semigroup $S(t)$ from Theorem 6.

Lemma 10. *Assume that*

$$\begin{aligned} 0 < \widehat{r} &= \lim_{x\to\infty} \frac{1}{x}\int_0^x r(s)\,ds, \\ \widehat{\mu} &= \lim_{y\to\infty} \frac{1}{y}\int_0^y \mu(s)\,ds < \infty. \end{aligned} \tag{18}$$

Then A_0 generates a quasi-compact semigroup $T_0(t)$.

To get the desired result of this section, we need a little preparation. For $\gamma > 0$, $P \in X$, let

$$\begin{aligned} (\Phi_\gamma(P))(x,y) &= \Big[\operatorname{diag}\Big(0, \varepsilon P_0 + \varepsilon P_{21}, 0, 0, \lambda_2 P_2 \\ &\quad + \int_0^\infty r(s) P_{32}(s)\,ds, 0, 0, 0\Big)\Big] \cdot E_\gamma(x,y), \end{aligned} \tag{19}$$

where $E_\gamma(x,y) = (0, e^{-\int_0^x (\gamma+\alpha_0+r(s))ds}, 0, e^{-\int_0^x (\gamma+\lambda_1+\lambda_2+r(s))ds}, e^{-\int_0^y (\gamma+\mu(s))ds}, e^{-\int_0^x (\gamma+r(s))ds}, e^{-\int_0^x (\gamma+r(s))ds})^{\mathrm{T}} \in \operatorname{Ker}(\gamma I - \overline{A})$. It is not hard to see that Φ_γ is a compact operator with the property that $I + \Phi_\gamma$ is a bijection from $D(A_0)$ to $D(A)$ and

$$[\gamma I - (A - B)](I + \Phi_\gamma) = \gamma I - A_0. \tag{20}$$

Lemma 11. *$S(t) - T_0(t)$ is a nonnegative compact operator, for any $t \geq 0$.*

With the above preparation, the main results of this section will be presented as follows.

Theorem 12. *C_0 semigroup $T(t)$ generated by the system operator A is quasi-compact.*

Theorem 13. *The time-dependent solution of the system (1)–(3) strongly converges to its steady-state solution, and there exist $C > 0$ and $\delta > 0$ such that*

$$\left\|P(t,\cdot) - \widehat{P}\right\| \leq Ce^{-\delta t}. \tag{21}$$

Here $\widehat{P}$ is defined in Theorem 9.

4. Reliability Indices

In this section, we will discuss some reliability indices of the system, namely, the reliability and failure frequency of the system, the probabilities of the repairman in vacation, and the system in warning state. Noting that the eigenfunction corresponding to eigenvalue 0 of the system operator A is just the steady-state solution of system (1)–(3), the corresponding steady-state indices of the system can be presented from the point of eigenfunction.

Let $\phi = \int_0^\infty \phi(\alpha)d\alpha$; then from (A.28), we can deduce that

$$\begin{aligned} P_1 &= \int_0^\infty P_1(x)\,dx = \frac{[\varepsilon(\lambda_1+\lambda_2) + \alpha_0\lambda_1]\,g}{\lambda_1+\lambda_2-\alpha_0\lambda_1 g} P_0, \\ P_2 &= \frac{\alpha_0[(\lambda_1+\lambda_2)(1+\varepsilon h) + \alpha_0\lambda_1(h-g) - \alpha_0(1+\varepsilon g)]}{(\lambda_1+\lambda_2-\alpha_0)(\lambda_1+\lambda_2-\alpha_0\lambda_1 g)} \\ &\quad \cdot P_0, \\ P_3 &= \int_0^\infty P_3(x)\,dx = \frac{\alpha_0(g-h)[\varepsilon(\lambda_1+\lambda_2)+\alpha_0\lambda_1]}{(\lambda_1+\lambda_2-\alpha_0)(\lambda_1+\lambda_2-\alpha_0\lambda_1 g)} \\ &\quad \cdot P_0, \\ P_{21} &= \frac{\alpha_0\lambda_1(1+\varepsilon g)}{\varepsilon(\lambda_1+\lambda_2-\alpha_0\lambda_1 g)} P_0, \\ P_{22} &= \int_0^\infty P_{22}(y)\,dy = \frac{\alpha_0\lambda_2 k(1+\varepsilon g)}{\lambda_1+\lambda_2-\alpha_0\lambda_1 g} P_0, \end{aligned}$$

$$P_{3i} = \int_0^\infty P_{3i}(x)\,\mathrm{d}x = \frac{\alpha_0\lambda_i[\varepsilon(\lambda_1+\lambda_2)+\alpha_0\lambda_1]}{(\lambda_1+\lambda_2-\alpha_0)(\lambda_1+\lambda_2-\alpha_0\lambda_1 g)}\left(\frac{f-g}{\alpha_0}+\frac{h-f}{\lambda_1+\lambda_2}\right)P_0, \quad i=1,2, \tag{22}$$

where

$$\begin{aligned} f &= \int_0^\infty e^{-\int_0^x r(s)\mathrm{d}s}\mathrm{d}x, \\ g &= \int_0^\infty e^{-\int_0^x [\alpha_0+r(s)]\mathrm{d}s}\mathrm{d}x, \\ h &= \int_0^\infty e^{-\int_0^x [\lambda_1+\lambda_2+r(s)]\mathrm{d}s}\mathrm{d}x, \\ k &= \int_0^\infty e^{-\int_0^y \mu(s)\mathrm{d}s}\mathrm{d}y. \end{aligned} \tag{23}$$

And set

$$\begin{aligned} S &= \sum_{i=0}^{3}P_i + \sum_{j=1}^{2}\left(P_{2j}+P_{3j}\right) \\ &= \left[\frac{(1+\varepsilon g)[\varepsilon(\lambda_1+\lambda_2)+\alpha_0(\varepsilon+\lambda_1+\varepsilon k\lambda_2)]}{\varepsilon(\lambda_1+\lambda_2-\alpha_0\lambda_1 g)} + \frac{\varepsilon(\lambda_1+\lambda_2)+\alpha_0\lambda_1}{(\lambda_1+\lambda_2-\alpha_0)(\lambda_1+\lambda_2-\alpha_0\lambda_1 g)}[(\lambda_1+\lambda_2)\cdot(f-g)+\alpha_0(h-f)]\right]P_0. \end{aligned} \tag{24}$$

Theorem 14. *The steady-state availability of the system is*

$$A_v = \frac{1}{N}\left[(\lambda_1+\lambda_2-\alpha_0)(1+\varepsilon g)\cdot[\varepsilon(\lambda_1+\lambda_2)+\alpha_0(\varepsilon+\lambda_1)] + \alpha_0\lambda_1\varepsilon[\varepsilon(\lambda_1+\lambda_2)+\alpha_0\lambda_1]\cdot\left(\frac{f-g}{\alpha_0}+\frac{h-f}{\lambda_1+\lambda_2}\right)\right], \tag{25}$$

where $N = (1+\varepsilon g)(\lambda_1+\lambda_2-\alpha_0)[\varepsilon(\lambda_1+\lambda_2)+\alpha_0(\varepsilon+\lambda_1+\varepsilon k\lambda_2)] + \varepsilon[\varepsilon(\lambda_1+\lambda_2)+\alpha_0\lambda_1][(\lambda_1+\lambda_2)(f-g)+\alpha_0(h-f)]$.

Theorem 15. *The steady-state probability of the repairman in vacation is*

$$P_v = \frac{\varepsilon f(\lambda_1+\lambda_2-\alpha_0)[\varepsilon(\lambda_1+\lambda_2)+\alpha_0\lambda_1]}{N}, \tag{26}$$

where N is defined in Theorem 14.

Theorem 16. *The steady-state probability of the system in warning state is*

$$P_w = \frac{\alpha_0\varepsilon(1+\varepsilon g)(\lambda_1+\lambda_2-\alpha_0)}{N}, \tag{27}$$

where N is defined in Theorem 14.

Theorem 17. *The steady-state failure frequency of the system is*

$$W_f = \frac{\lambda_2\alpha_0\varepsilon(1+\varepsilon g)(\lambda_1+\lambda_2-\alpha_0)}{N}, \tag{28}$$

where N is defined in Theorem 14.

5. Applications and Numerical Examples

In [23], we have discussed the effects of repairman vacation policies on a system. That is, the longer the delayed-vacation time and the shorter the vacation time, the larger the reliability and failure frequency of a system. In this section, we mainly concentrate on that how the warning device will affect the system. Specifically, we will compare the availability, failure frequency, and profit of the system with warning device and those of the system without warning device and present corresponding numerical examples.

With the method of Section 4, the steady-state indices (specifically, the reliability $\widetilde{A}_v$, failure frequency $\widetilde{W}_f$, and the probability of the repairman in vacation $\widetilde{P}_v$) of the system without warning device corresponding to system (1)–(3) can be obtained as follows:

$$\widetilde{A}_v = \frac{1+\varepsilon m}{1+\varepsilon f+(1+\varepsilon m)k\lambda}, \tag{29}$$

$$\widetilde{W}_f = \frac{\lambda[(1+\varepsilon m)k\lambda+\varepsilon(f-m)]}{1+\varepsilon f+(1+\varepsilon m)k\lambda}, \tag{30}$$

$$\widetilde{P}_v = \frac{\varepsilon f}{1+\varepsilon f+(1+\varepsilon m)k\lambda}, \tag{31}$$

where $m = \int_0^\infty e^{-\int_0^x[\lambda+r(s)]\mathrm{d}s}\mathrm{d}x$.

For simplicity, we assume that $r(x) \equiv r$ and $\mu(y) \equiv \mu$, where r and μ can be found in Section 2. Then by comparing the two groups of (25) and (29) and (28) and (30), we can deduce the following results:

(i) The steady-state availability of the system with warning device (i.e., systems (1)–(3)) A_v is larger than that of the system without warning device $\widetilde{A}_v$.

(ii) The steady-state failure frequency of the system with warning device W_f is less than that of the system without warning device $\widetilde{W}_f$.

Let I and $\widetilde{I}$, respectively, be the total profit of the system with and without warning device in steady state. That is

$$\begin{aligned} I &= c_1A_v - c_2W_f + c_3P_v, \\ \widetilde{I} &= c_1\widetilde{A}_v - c_2\widetilde{W}_f + c_3\widetilde{P}_v. \end{aligned} \tag{32}$$

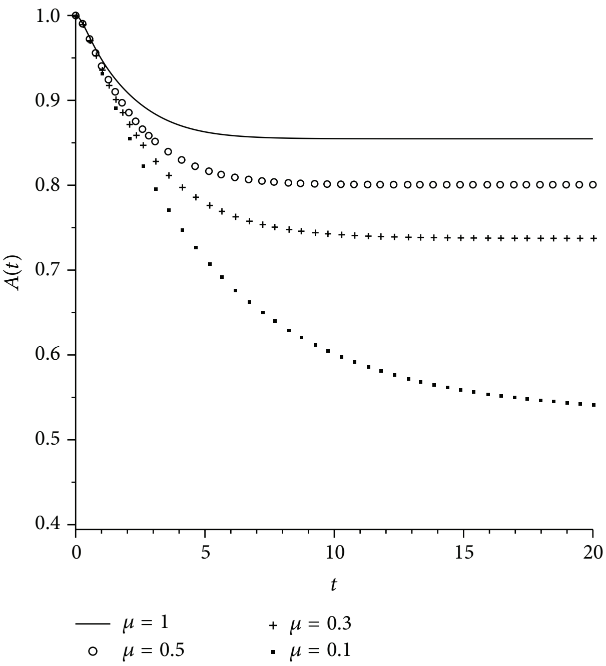

Figure 1: Instantaneous availabilities of the system with warning device with different μ.

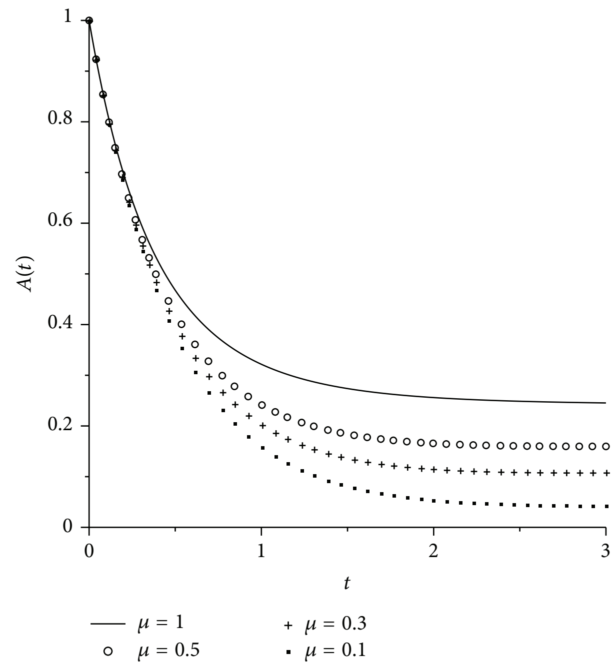

Figure 2: Instantaneous availabilities of the system without warning device with different μ.

Here c_1, c_2, and c_3 represent the income of the system for working unit per unit time, the loss of the system for failed unit per unit time, and the income of the system for the repairman vacation per unit time, respectively. We can deduce the estimation that $D = I - \widetilde{I} > 0$ for $c_1 > c_2\alpha_0$ and $c_2\lambda^2 > (c_1 + c_2\lambda)\mu$.

In the following, we will present some numerical examples to compare availabilities, failure frequencies, and total profits of systems with and without warning device by choosing $\varepsilon = 1$, $\lambda = 0.01$, $\lambda_1 = \lambda_2 = 2$, $r = 0.5$.

(1) Figures 1 and 2, respectively, present the instantaneous availabilities of the systems with and without warning device with $\alpha_0 = 0.2$ and $\mu = 1, 0.5, 0.3, 0.1$. It can be seen that both the availabilities of the systems with and without warning device are decreasing with the decreasing of μ. Moreover, the availabilities of system with warning device are greater than that of the system without warning device.

(2) Figures 3 and 4, respectively, present the instantaneous failure frequencies of the systems with and without warning device with $\alpha_0 = 0.2$ and $\mu = 1, 0.5, 0.3, 0.1$. It can be seen that both the failure frequencies of the systems with and without warning device are decreasing with the decreasing of μ. However the failure frequencies of system with warning device are less than that of the system without warning device.

(3) Figures 5 and 6, respectively, present the instantaneous availabilities and failure frequencies of the systems with device with $\mu = 0.5$ and $\alpha_0 = 1, 0.5, 0.3, 0.1$. It can be seen that the availabilities of the system

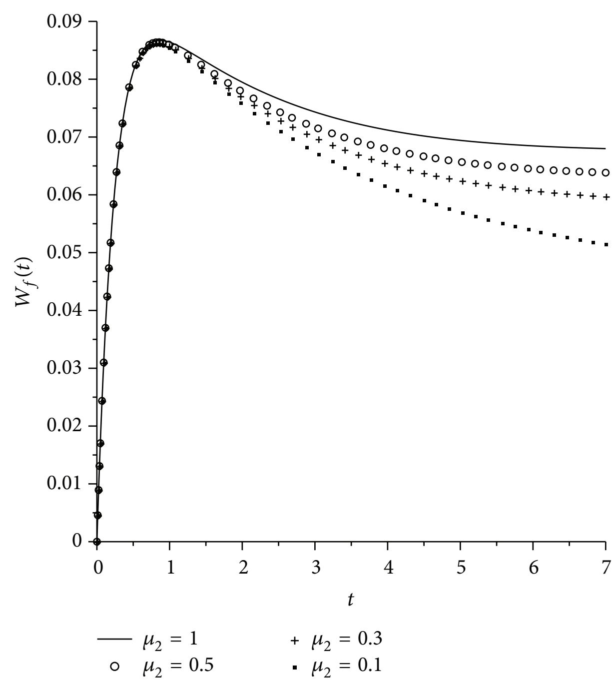

Figure 3: Instantaneous failure frequencies of the system with warning device with different μ.

with warning device are increasing while the its failure frequencies are decreasing with the decreasing of α_0.

(4) Choose $c_1 = c_2 = c_3 = 1$. Figures 7 and 8 present the total profit differences of the systems with and without warning device in steady state with variables λ and

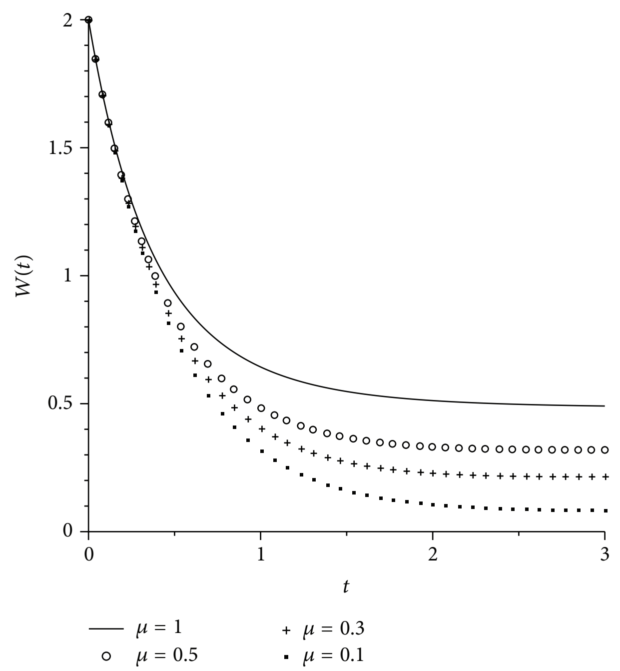

FIGURE 4: Instantaneous failure frequencies of the system without warning device with different μ.

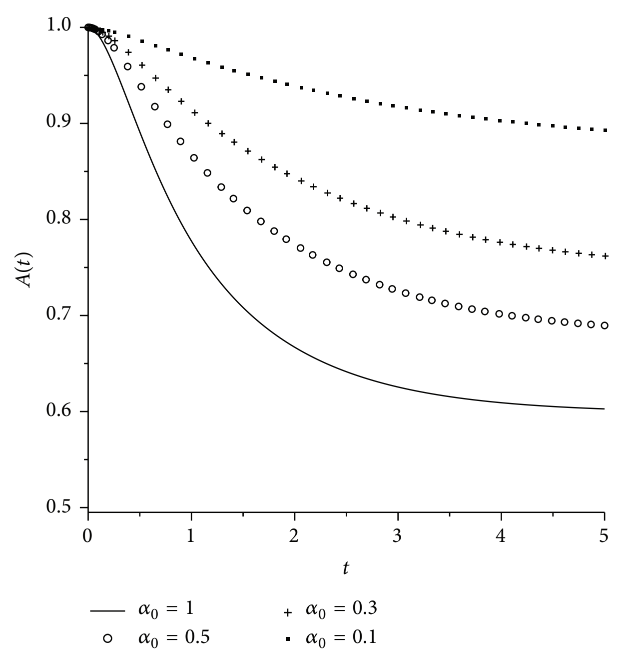

FIGURE 5: Instantaneous availabilities of the system with warning device with different α_0.

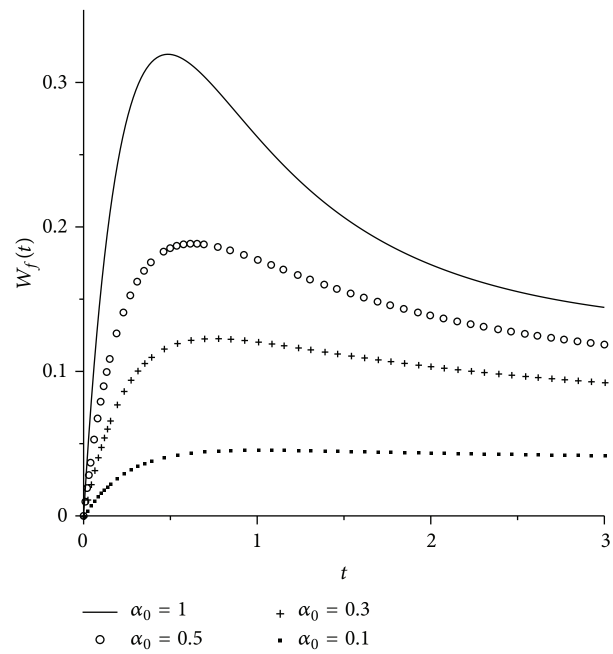

FIGURE 6: Instantaneous failure frequencies of the system with warning device with different α_0.

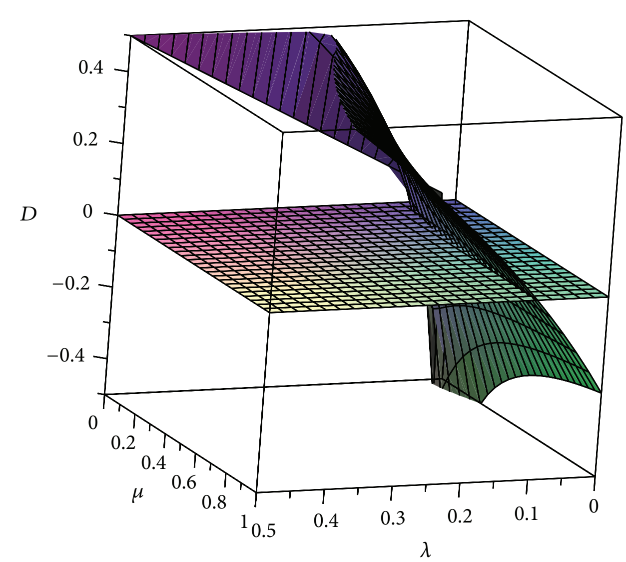

FIGURE 7: Profit difference of systems with and without warning device with $\alpha_0 = 0.2$.

μ and with $\alpha_0 = 0.2$ and 0.1, respectively. It can be deduced from the figures that the profit of system with warning device can be more than that of system without warning device by giving suitable parameters.

6. Conclusion

In this paper, we propose a simple repairable system with a warning device which can send alarms when the system does not work properly. Because the study of well-posedness of the time-dependent solution of a system is in demand in terms of theory and practice due to the two hypotheses used for Laplace transform in order to obtain the steady-state solution of a repairable system in traditional reliability research that needs to be verified, and the substitution of steady-state solution for the dynamic one should be based on some conditions, we then discuss and obtain the unique existence and the stability, especially the exponential stability of the system solution by C_0 semigroup theory. Because

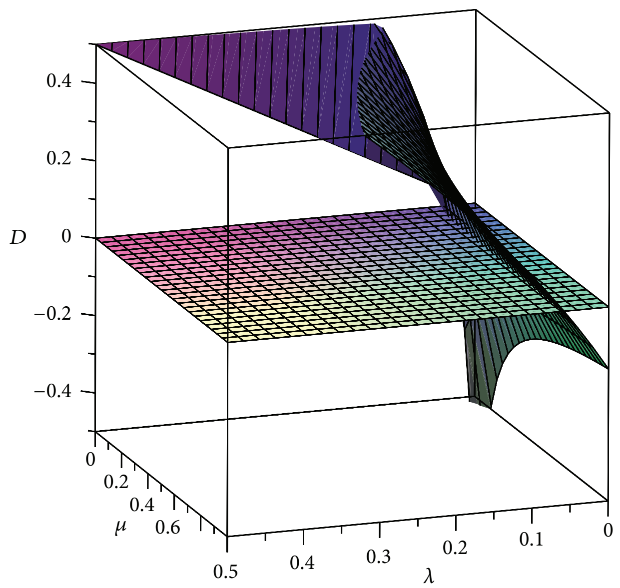

Figure 8: Profit difference of systems with and without warning device with $\alpha_0 = 0.1$.

the stable solution of the system is just the eigenfunction corresponding to eigenvalue 0 of the system operator, we also present some reliability indices, especially steady state indexes, such as reliability, failure frequency, probabilities of repairman in vacation, and system in warning state of the system in the viewpoint of eigenfunction. At the end of the paper, we also discuss the advantages and disadvantages of the systems with and without warning device theoretically and numerically. Because the availability or failure frequency of the system with warning device is greater or less than those of the system without warning device, and it can be controlled to ensure that the total profit of the system with warning device is more than that of the system without warning device, we get the conclusion that the system with a warning device is better than the corresponding system without warning device in practice.

Appendix

Proof of Lemma 1. Firstly, we prove that A is a closed operator. Choose $P^n = (P_0^n, P_1^n, P_2^n, P_{21}^n, P_{22}^n, P_3^n, P_{31}^n, P_{32}^n)^{\mathrm{T}} \in D(A)$. Let $P^n \to P = (P_0, P_1, P_2, P_{21}, P_{22}, P_3, P_{31}, P_{32})^{\mathrm{T}}$, $AP^n \to Q = (Q_0, Q_1, Q_2, Q_{21}, Q_{22}, Q_3, Q_{31}, Q_{32})^{\mathrm{T}}$, $n \to \infty$. According to Proposition 1 ([24, II.2.10]), the differential operator $\mathscr{D}$ is the infinitesimal generator of a left translation semigroup $\{T_l(t)\}_{t\geq 0}$ defined on

$$D(\mathscr{D}) = \left\{ f \in L^1(\mathbb{R}^+) \mid f \text{ is absolutely continuous satisfying } f' \in L^1(\mathbb{R}^+) \right\}. \tag{A.1}$$

Then $P_j \in D(\mathscr{D})$ due to $D(\mathscr{D})$ is closed and $P_j^n \in D(\mathscr{D})$, which is equivalent to $P_j' \in L^1(\mathscr{R}^+)$, is absolutely continuous, $j = 1, 3, 22, 31, 32$. Furthermore, $P_1^n(0) = \varepsilon P_0^n + \varepsilon P_{21}^n \to \varepsilon P_0 + \varepsilon P_{21} = P_1(0)$, $P_{22}^n(0) = \lambda_2 P_2^n + \int_0^\infty r(x) P_{32}^n(x) dx \to \lambda_2 P_2 + \int_0^\infty r(x) P_{32}(x) dx = P_{22}(0)$, $n \to \infty$. Thus $P \in D(A)$. Noting the bounded measurable of $r(x), \mu(y)$, it is not hard to deduce that $AP = Q$. This implies that A is a closed operator.

Next, we prove that $D(A)$, the domain of operator A, is dense in X. Choose $F = (F_0, F_1, F_2, F_{21}, F_{22}, F_3, F_{31}, F_{32})^{\mathrm{T}} \in X$. Let $P_0 = F_0$, $P_2 = F_2$, $P_{21} = F_{21}$. Because $F_j \in L^1(\mathbb{R}^+)$, then for any $\sigma > 0$, there exist $\delta_j > 0$ and $G_j > 0$, $j = 1, 3, 22, 31, 32$ such that

$$\begin{aligned} &\int_0^{\delta_j} \left| F_j(\alpha) \right| d\alpha < \frac{\sigma}{30}, \\ &\int_{G_j}^{\infty} \left| F_j(\alpha) \right| d\alpha < \frac{\sigma}{15}. \end{aligned} \tag{A.2}$$

Take $\delta = \min\{\delta_1, \delta_3, \delta_{22}, \delta_{31}, \delta_{32}, \sigma/6[\varepsilon P_0 + \varepsilon P_{21} + \lambda_2 P_2 + \int_0^\infty r(x) P_{32}(x) dx]\}$, and define

$$\begin{aligned} P_1(x) &= \begin{cases} \varepsilon P_0 + \varepsilon P_{21}, & 0 \leq x < \delta \\ g_1(x), & \delta \leq x \leq G_1 \\ 0, & G_1 < x < \infty, \end{cases} \\ P_3(x) &= \begin{cases} 0, & 0 \leq x < \delta \\ g_3(x), & \delta \leq x \leq G_3 \\ 0, & G_3 < x < \infty \end{cases} \\ P_{22}(y) &= \begin{cases} \lambda_2 P_2 + \int_0^\infty r(x) P_{32}(x) dx, & 0 \leq y < \delta \\ g_{22}(y), & \delta \leq y \leq G_{22} \\ 0, & G_{22} < y < \infty, \end{cases} \\ P_{3i}(x) &= \begin{cases} 0, & 0 \leq x < \delta \\ g_{3i}(x), & \delta \leq x \leq G_{3i} \\ 0, & G_{3i} < x < \infty. \end{cases} \quad i = 1, 2. \end{aligned} \tag{A.3}$$

Here g_j are continuously differentiable functions on $[\delta, G_j]$, $j = 1, 3, 22, 31, 32$ satisfying

$$\begin{aligned} &g_j(G_j) = 0, \\ &g_1(\delta) = \varepsilon P_0 + \varepsilon P_{21}, \\ &g_3(\delta) = g_{3i}(\delta) = 0, \quad i = 1, 2 \\ &g_{22}(\delta) = \lambda_2 P_2 + \int_0^\infty r(x) P_{32}(x) dx, \\ &\int_\delta^{G_j} \left| P_j(x) - g_j(x) \right| dx < \frac{\sigma}{15}. \end{aligned} \tag{A.4}$$

Take $P = (P_0, P_1, P_2, P_{21}, P_{22}, P_3, P_{31}, P_{32})^{\mathrm{T}}$. Then $P \in D(A)$, and

$$\begin{aligned}\|F-P\| &= \int_0^\infty |F_1(x)-P_1(x)|\,\mathrm{d}x + \int_0^\infty |F_3(x) \\ &- P_3(x)|\,\mathrm{d}x + \int_0^\infty |F_{22}(y)-P_{22}(y)|\,\mathrm{d}y \\ &+ \sum_{i=1}^{2}\int_0^\infty |F_{3i}(x)-P_{3i}(x)|\,\mathrm{d}x \le \int_0^\delta |F_1(x)|\,\mathrm{d}x \\ &+ \int_0^\delta |P_1(x)|\,\mathrm{d}x + \int_\delta^{G_1} |F_1(x)-P_1(x)|\,\mathrm{d}x \\ &+ \int_{G_1}^\infty |F_1(x)|\,\mathrm{d}x + \int_0^\delta |F_3(x)|\,\mathrm{d}x + \int_0^\delta |P_3(x)|\,\mathrm{d}x \\ &+ \int_\delta^{G_3} |F_3(x)-P_3(x)|\,\mathrm{d}x + \int_{G_3}^\infty |F_3(x)|\,\mathrm{d}x \\ &+ \int_0^\delta |F_{22}(y)|\,\mathrm{d}y + \int_0^\delta |P_{22}(y)|\,\mathrm{d}y \\ &+ \int_\delta^{G_{22}} |F_{22}(y)-P_{22}(y)|\,\mathrm{d}y + \int_{G_{22}}^\infty |F_{22}(y)|\,\mathrm{d}y \\ &+ \sum_{i=1}^{2}\left(\int_0^\delta |F_{3i}(x)|\,\mathrm{d}x + \int_0^\delta |P_{3i}(x)|\,\mathrm{d}x\right. \\ &+ \left.\int_\delta^{G_{3i}} |F_{3i}(x)-P_{3i}(x)|\,\mathrm{d}x + \int_{G_{3i}}^\infty |F_{3i}(x)|\,\mathrm{d}x\right) \\ &< \frac{5\sigma}{6} + \delta\left[\varepsilon P_0 + \varepsilon P_{21} + \lambda_2 P_2\right. \\ &+ \left.\int_0^\infty r(x) P_{32}(x)\,\mathrm{d}x\right] < \sigma. \end{aligned} \tag{A.5}$$

This implies that $D(A)$ is dense in X.

It remains to be proven that A is a dissipative operator. In fact, For any $P = (P_0, P_1, P_2, P_{21}, P_{22}, P_3, P_{31}, P_{32})^{\mathrm{T}} \in D(A)$, set $Q_k = \|P\|\mathrm{sgn}(P_k)$, $k = 0, 1, 2, 21, 22, 31, 32$, and take $Q = (Q_0, Q_1, Q_2, Q_{21}, Q_{22}, Q_3, Q_{31}, Q_{32})^{\mathrm{T}}$. Clearly, $Q \in X^* = \mathbb{R} \times L^\infty(\mathbb{R}^+) \times \mathbb{R} \times (L^\infty(\mathbb{R}^+))^4$, the duel space of X. Moreover, it is easy to know that $\langle P, Q\rangle = \|P\|^2 = \|Q\|^2$ and $\langle AP, Q\rangle \le 0$. This manifests that A is a dissipative operator. □

Proof of Lemma 2. For any $G = (G_0, G_1, G_2, G_{21}, G_{22}, G_3, G_{31}, G_{32})^{\mathrm{T}} \in X$, consider the resolvent equation $(\gamma I - A)P = G$. That is

$$\begin{aligned}(\gamma+\varepsilon+\alpha_0) P_0 &= G_0 + \int_0^\infty r(x) P_1(x)\,\mathrm{d}x \\ &\quad + \int_0^\infty \mu(y) P_{22}(y)\,\mathrm{d}y,\end{aligned} \tag{A.6}$$

$$P_1'(x) + [\gamma+\alpha_0+r(x)] P_1(x) = G_1(x), \tag{A.7}$$

$$(\gamma+\lambda_1+\lambda_2) P_2 = G_2 + \alpha_0 P_0 + \int_0^\infty r(x) P_3(x)\,\mathrm{d}x, \tag{A.8}$$

$$(\gamma+\varepsilon) P_{21} = G_{21} + \lambda_1 P_2 + \int_0^\infty r(x) P_{31}(x)\,\mathrm{d}x, \tag{A.9}$$

$$P_{22}'(y) + (\gamma+\mu_i(y)) P_{22}(y) = G_{22}(y), \tag{A.10}$$

$$\begin{aligned}&P_3'(x) + [\gamma+\lambda_1+\lambda_2+r(x)] P_3(x) \\ &\quad = G_3(x) + \alpha_0 P_1(x),\end{aligned} \tag{A.11}$$

$$\begin{aligned}&P_{3i}'(x) + (\gamma+r(x)) P_{3i}(x) = G_{3i}(x) + \lambda_i P_3(x), \\ &\qquad i = 1, 2,\end{aligned} \tag{A.12}$$

$$\begin{aligned}&P_1(0) = \varepsilon P_0 + \varepsilon P_{21}, \\ &P_3(0) = P_{3i}(0) = 0, \quad i = 1, 2,\end{aligned} \tag{A.13}$$

$$P_{22}(0) = \lambda_2 P_2 + \int_0^\infty r(x) P_{32}(x)\,\mathrm{d}x. \tag{A.14}$$

Solving (A.7) and (A.10)–(A.12) with the help of (A.13) and (A.14) yields

$$P_1(x) = (\varepsilon P_0 + \varepsilon P_{21}) e^{-\int_0^x (\gamma+\alpha_0+r(s))\mathrm{d}s} + Y_1(x), \tag{A.15}$$

$$\begin{aligned}P_3(x) &= \frac{\alpha_0(\varepsilon P_0 + \varepsilon P_{21})\left(1 - e^{-(\alpha_0-\lambda_1-\lambda_2)x}\right)}{\alpha_0-\lambda_1-\lambda_2} \\ &\quad \cdot e^{-\int_0^x (\gamma+\lambda_1+\lambda_2+r(s))\mathrm{d}s} + Y_3(x),\end{aligned} \tag{A.16}$$

$$\begin{aligned}P_{3i}(x) &= \frac{\lambda_i\alpha_0(\varepsilon P_0 + \varepsilon P_{21})}{\alpha_0-\lambda_1-\lambda_2} \\ &\quad \cdot e^{-\int_0^x (\gamma+r(s))\mathrm{d}s}\left[\frac{1-e^{-(\lambda_1+\lambda_2)x}}{\lambda_1+\lambda_2} - \frac{1-e^{-\alpha_0 x}}{\alpha_0}\right] \\ &\quad + Y_{3i}(x), \quad i = 1, 2,\end{aligned} \tag{A.17}$$

$$\begin{aligned}P_{22}(0) &= \lambda_2 P_2 + \lambda_2\alpha_0(\varepsilon P_0 + \varepsilon P_{21}) \\ &\quad \cdot \int_0^\infty f_\gamma(x)\phi(x)\,\mathrm{d}x + Y_{22},\end{aligned} \tag{A.18}$$

$$P_{22}(y) = P_{22}(0) e^{-\int_0^y (\gamma+\mu(s))\mathrm{d}s} + Y_{22}(y). \tag{A.19}$$

Here

$$Y_1(x) = \int_0^x G_1(\tau) e^{-\int_\tau^x (\gamma+\alpha_0+r(s))\mathrm{d}s}\mathrm{d}\tau,$$

$$\begin{aligned}&Y_3(x) \\ &\quad = \int_0^x [\alpha_0 Y_1(\tau) + G_3(\tau)] e^{-\int_\tau^x (\gamma+\lambda_1+\lambda_2+r(s))\mathrm{d}s}\mathrm{d}\tau,\end{aligned}$$

$$Y_{22}(y) = \int_0^y G_{22}(\tau) e^{-\int_\tau^y (\gamma+\mu(s))\mathrm{d}s}\mathrm{d}\tau,$$

$$Y_{3i}(x) = \int_0^x [\lambda_i Y_3(\tau) + G_{3i}(\tau)] e^{-\int_\tau^x (\gamma+r(s))\mathrm{d}s}\mathrm{d}\tau,$$

$$Y_{22} = \int_0^\infty r(x) Y_{32}(x)\,dx,$$

$$f_\gamma(x) = r(x) e^{-\int_0^x (\gamma + r(s))ds},$$

$$\phi(x) = \frac{1}{\alpha_0 - \lambda_1 - \lambda_2}\left[\frac{1 - e^{-(\lambda_1+\lambda_2)x}}{\lambda_1 + \lambda_2} - \frac{1 - e^{-\alpha_0 x}}{\alpha_0}\right]. \tag{A.20}$$

Substituting (A.15)–(A.17) and (A.19) into (A.6), (A.8), and (A.9), respectively, yields

$$(\gamma + \varepsilon + \alpha_0) P_0 = (\varepsilon P_0 + \varepsilon P_{21}) \int_0^\infty g_\gamma(x)\,dx + P_{22}(0) \int_0^\infty k_{\gamma 2}(y)\,dy + Y_0, \tag{A.21}$$

$$(\gamma + \lambda_1 + \lambda_2) P_2 = \alpha_0 P_0 + \frac{\alpha_0 (\varepsilon P_0 + \varepsilon P_{21})}{\alpha_0 - \lambda_1 - \lambda_2} \int_0^\infty \left(h_\gamma(x) - g_\gamma(x)\right) dx + Y_2, \tag{A.22}$$

$$(\gamma + \varepsilon) P_{21} = \lambda_1 P_2 + \lambda_1 \alpha_0 (\varepsilon P_0 + \varepsilon P_{21}) \int_0^\infty f_\gamma(x) \phi(x)\,dx + Y_{22}, \tag{A.23}$$

where

$$g_\gamma(x) = r(x) e^{-\int_0^x (\gamma + \alpha_0 + r(s))ds},$$

$$h_\gamma(x) = r(x) e^{-\int_0^x (\gamma + \lambda_1 + \lambda_2 + r(s))ds},$$

$$k_\gamma(y) = \mu(y) e^{-\int_0^y (\gamma + \mu(s))ds},$$

$$Y_0 = G_0 + \int_0^\infty r(x) Y_1(x)\,dx + \int_0^\infty \mu(y) Y_{22}(y)\,dy, \tag{A.24}$$

$$Y_2 = G_2 + \int_0^\infty r(x) Y_3(x)\,dx,$$

$$Y_{21} = G_{21} + \int_0^\infty r(x) Y_{31}(x)\,dx.$$

Combing (A.21)–(A.23) and (A.18) yields the following matrix equation:

$$\begin{pmatrix} \gamma + \varepsilon + \alpha_0 - \varepsilon \int_0^\infty g_\gamma(x)\,dx & 0 & -\varepsilon \int_0^\infty g_\gamma(x)\,dy & -\int_0^\infty k_\gamma(y)\,dy \\ -\alpha_0 - \frac{\alpha_0 \varepsilon}{\alpha_0 - \lambda_1 - \lambda_2} \int_0^\infty \left(h_\gamma(x) - g_\gamma(x)\right) dx & \gamma + \lambda_1 + \lambda_2 & -\frac{\alpha_0 \varepsilon}{\alpha_0 - \lambda_1 - \lambda_2} \int_0^\infty \left(h_\gamma(x) - g_\gamma(x)\right) dx & 0 \\ -\lambda_1 \alpha_0 \varepsilon \int_0^\infty f_\gamma(x) \phi(x)\,dx & -\lambda_1 & \gamma + \varepsilon - \lambda_1 \alpha_0 \varepsilon \int_0^\infty f_\gamma(x) \phi(x)\,dx & 0 \\ -\lambda_2 \alpha_0 \varepsilon \int_0^\infty f(x) \phi(x)\,dx & -\lambda_2 & -\lambda_2 \alpha_0 \varepsilon \int_0^\infty f_\gamma(x) \phi(x)\,dx & 1 \end{pmatrix}, \tag{A.25}$$

$$\begin{pmatrix} P_0 \\ P_2 \\ P_{21} \\ P_{22}(0) \end{pmatrix} = \begin{pmatrix} Y_0 \\ Y_2 \\ Y_{21} \\ Y_{22} \end{pmatrix}.$$

For $\operatorname{Re}\gamma > 0$ or $\gamma = ia$, $a \in \mathbb{R}$, $a \neq 0$, and noting the assumption (4), it is not hard to know that

$$|\gamma + \lambda_1 + \lambda_2| > \lambda_1 + \lambda_2,$$

$$\int_0^\infty k_\gamma(y)\,dy < 1,$$

$$\left|-\alpha_0 - \frac{\alpha_0 \varepsilon}{\alpha_0 - \lambda_1 - \lambda_2} \int_0^\infty \left(h_\gamma(x) - g_\gamma(x)\right) dx\right| + \sum_{i=1}^{2} \left|-\lambda_i \alpha_0 \varepsilon \int_0^\infty f_\gamma(x) \phi(x)\,dx\right| = \alpha_0$$

$$+ \frac{\alpha_0 \varepsilon}{\alpha_0 - \lambda_1 - \lambda_2} \int_0^\infty \left(h_\gamma(x) - g_\gamma(x)\right) dx + \sum_{i=1}^{2} \lambda_i \alpha_0 \varepsilon \int_0^\infty f_\gamma(x) \phi(x)\,dx = \alpha_0$$

$$+ \frac{\alpha_0 \varepsilon}{\alpha_0 - \lambda_1 - \lambda_2} \int_0^\infty \left(h_\gamma(x) - g_\gamma(x)\right) dx + \varepsilon \int_0^\infty f_\gamma(x)\,dx - \frac{\alpha_0 \varepsilon}{\alpha_0 - \lambda_1 - \lambda_2} \int_0^\infty h_\gamma(x)\,dx$$

$$+\frac{\varepsilon(\lambda_1+\lambda_2)}{\alpha_0-\lambda_1-\lambda_2}\int_0^\infty g_\gamma(x)\,dx=\alpha_0$$
$$+\varepsilon\int_0^\infty f_\gamma(x)\,dx-\varepsilon\int_0^\infty g_\gamma(x)\,dx<\alpha_0+\varepsilon$$
$$-\varepsilon\int_0^\infty g_\gamma(x)\,dx$$
$$<\left|\gamma+\varepsilon+\alpha_0-\varepsilon\int_0^\infty g_\gamma(x)\,dx\right|. \tag{A.26}$$

Similarly

$$\begin{aligned}&\left|-\frac{\alpha_0\varepsilon}{\alpha_0-\lambda_1-\lambda_2}\int_0^\infty\left(h_\gamma(x)-g_\gamma(x)\right)dx\right|\\&\quad+\left|-\lambda_2\alpha_0\varepsilon\int_0^\infty f_\gamma(x)\phi(x)\,dx\right|\\&\quad+\left|-\varepsilon\int_0^\infty g_\gamma(x)\,dx\right|\\&<\left|\gamma+\varepsilon-\lambda_1\alpha_0\varepsilon\int_0^\infty f_\gamma(x)\phi(x)\,dx\right|.\end{aligned} \tag{A.27}$$

This means that the coefficient matrix of (A.25) is a column strictly diagonal dominant matrix. So it is inverse and the matrix equation (A.25) has a unique solution $(P_0, P_2, P_{21}, P_{22}(0))^T$. Then from the expressions (A.15)–(A.19), we can conclude that (A.6)–(A.14) have a unique solution $P=(P_0, P_1, P_2, P_{21}, P_{22}, P_3, P_{31}, P_{32})^T$.

Moreover, according to [25], it can be derived that, for any $t\geq 0$, $\int_t^\infty r(x)e^{-\int_t^x r(s)ds}dx$ and $\int_t^\infty \mu(y)e^{-\int_t^y r(s)ds}dy$ are all bounded. Then from (A.15)–(A.19), it is easy to see that the solution P of (A.6)–(A.14) belongs to the domain of the system operator A. This manifests that the operator equation $(\gamma I-A)$ is surjective. Because $(\gamma I-A)$ is closed and $D(A)$ is dense in X, then for any $\gamma\in\mathbb{C}$ satisfying $\operatorname{Re}\gamma>0$ or $\gamma=ia$, $a\in\mathbb{R}\setminus\{0\}$, $(\gamma I-A)^{-1}$ exists and is bounded by the inverse operator theorem. □

Proof of Lemma 3. Consider the operator equation $AP=0$. That is repeating the proof process of Lemma 2 with $\gamma=0$ and $G=0$, we can obtain

$$P_1(x)=(\varepsilon P_0+\varepsilon P_{21})\,e^{-\int_0^x(\alpha_0+r(s))ds}, \tag{A.28}$$

$$\begin{aligned}(\lambda_1+\lambda_2)P_2&=\alpha_0P_0+\frac{\alpha_0(\varepsilon P_0+\varepsilon P_{21})}{\alpha_0-\lambda_1-\lambda_2}\\&\quad\cdot\int_0^\infty\left(h_0(x)-g_0(x)\right)dx,\end{aligned} \tag{A.29}$$

$$\begin{aligned}\varepsilon P_{21}&=\lambda_1P_2+\lambda_1\alpha_0(\varepsilon P_0+\varepsilon P_{21})\\&\quad\cdot\int_0^\infty f_\gamma(x)\phi(x)\,dx,\end{aligned} \tag{A.30}$$

$$P_{22}(y)=P_{22}(0)\,e^{-\int_0^y\mu(s)ds}, \tag{A.31}$$

$$\begin{aligned}P_{22}(0)&=\lambda_2P_2+\lambda_2\alpha_0(\varepsilon P_0+\varepsilon P_{21})\\&\quad\cdot\int_0^\infty f_0(x)\phi(x)\,dx,\end{aligned} \tag{A.32}$$

$$\begin{aligned}P_3(x)&=\frac{\alpha_0(\varepsilon P_0+\varepsilon P_{21})\left(1-e^{-(\alpha_0-\lambda_1-\lambda_2)x}\right)}{\alpha_0-\lambda_1-\lambda_2}\\&\quad\cdot e^{-\int_0^x(\lambda_1+\lambda_2+r(s))ds},\end{aligned} \tag{A.33}$$

$$\begin{aligned}P_{3i}(x)&=\frac{\lambda_i\alpha_0(\varepsilon P_0+\varepsilon P_{21})}{\alpha_0-\lambda_1-\lambda_2}\\&\quad\cdot e^{-\int_0^x r(s)ds}\left[\frac{1-e^{-(\lambda_1+\lambda_2)x}}{\lambda_1+\lambda_2}-\frac{1-e^{-\alpha_0x}}{\alpha_0}\right],\\&\qquad i=1,2.\end{aligned} \tag{A.34}$$

This follows readily that 0 is an eigenvalue of the system operator A with geometric multiplicity one. Then by recalling [26], it only needs to be proven that the algebraic index of eigenvalue 0 is one, which can be easily obtained by using the reduction to absurdity. □

Proof of Lemma 4. For any $W=(W_0, W_1, W_2, W_{21}, W_{22}, W_3, W_{31}, W_{32})^T\in X^*$, consider the operator equation $(\gamma I-C)Q=DW$. That is

$$(\gamma+\varepsilon+\alpha_0)Q_0=\varepsilon W_1(0)+\alpha_0W_2, \tag{A.35}$$

$$\begin{aligned}\frac{dQ_1(x)}{dx}&=\left[\gamma+\alpha_0+r(x)\right]Q_1(x)\\&\quad-r(x)W_0-\alpha_0W_3(x),\end{aligned} \tag{A.36}$$

$$(\gamma+\lambda_1+\lambda_2)Q_2=\lambda_1W_{21}+\lambda_2W_{22}(0), \tag{A.37}$$

$$(\gamma+\varepsilon)Q_{21}=\varepsilon W_1(0), \tag{A.38}$$

$$\frac{dQ_{22}(y)}{dy}=\left[\gamma+\mu(y)\right]Q_{22}(y)-\mu(y)W_0, \tag{A.39}$$

$$\begin{aligned}\frac{dQ_3(x)}{dx}&=\left[\gamma+\lambda_1+\lambda_2+r(x)\right]Q_3(x)\\&\quad-r(x)W_2-\lambda_1W_{31}(x)\\&\quad-\lambda_2W_{32}(x),\end{aligned} \tag{A.40}$$

$$\frac{dQ_{31}(x)}{dx}=\left[\gamma+r(x)\right]Q_{31}(x)-r(x)W_{21}, \tag{A.41}$$

$$\begin{aligned}\frac{dQ_{32}(x)}{dx}&=\left[\gamma+r(x)\right]Q_{32}(x)\\&\quad-r(x)W_{22}(0).\end{aligned} \tag{A.42}$$

(A.35), (A.37), and (A.38) derive the following estimations:

$$\left|Q_0\right| < \frac{\varepsilon+\alpha_0}{\left|\gamma+\varepsilon+\alpha_0\right|}\|W\|,$$
$$\left|Q_2\right| \le \frac{\lambda_1+\lambda_2}{\left|\gamma+\lambda_1+\lambda_2\right|}\|W\|, \tag{A.43}$$
$$\left|Q_{21}\right| < \frac{\varepsilon}{\left|\gamma+\varepsilon\right|}\|W\|.$$

Solving (A.39) yields

$$Q_{22}(y) = e^{\int_0^y(\gamma+\mu(s))ds}\left(Q_{22}(0) - \int_0^y \mu(\tau) W_0 e^{-\int_0^\tau(\gamma+\mu(s))ds} d\tau\right). \tag{A.44}$$

Noting $Q_{22}(\infty) < \infty$, multiply $e^{-\int_0^y(\gamma+\mu(s))ds}$ in the two sides of (A.44) and let $y \to \infty$; we can get

$$Q_{22}(0) = \int_0^\infty W_0\mu(\tau) e^{-\int_0^\tau(\gamma+\mu(s))ds} d\tau. \tag{A.45}$$

Substituting (A.45) into (A.44) yields

$$Q_{22}(y) = e^{\int_0^y(\gamma+\mu(s))ds}\int_y^\infty W_0\mu(\tau) e^{-\int_0^\tau(\gamma+\mu(s))ds} d\tau. \tag{A.46}$$

Then the following estimation is immediate:

$$\begin{aligned}\left\|Q_{22}\right\|_{L^\infty[0,\infty)} &= \sup_{y\in[0,\infty)}\left|e^{\int_0^y(\gamma+\mu(s))ds}\int_y^\infty W_0\mu(\tau) \cdot e^{-\int_0^\tau(\gamma+\mu(s))ds}d\tau\right| \le \|W\| \\ &\cdot \sup_{y\in[0,\infty)} e^{\int_0^y(\operatorname{Re}\gamma+\mu(s))ds}\int_y^\infty\left(-e^{-\operatorname{Re}\gamma\tau}\right)de^{-\int_0^\tau\mu(s)ds} \\ &= \|W\|\sup_{y\in[0,\infty)}\left(1-\operatorname{Re}\gamma\int_y^\infty e^{-\int_y^\tau(\operatorname{Re}\gamma+\mu(s))ds}d\tau\right) \\ &\le \|W\|\sup_{y\in[0,\infty)}\left(1-\operatorname{Re}\gamma\int_y^\infty e^{-\int_y^\tau(\operatorname{Re}\gamma+M)ds}d\tau\right) \\ &= \frac{M}{\operatorname{Re}\gamma+M}\|W\|,\end{aligned} \tag{A.47}$$

where $M = \sup\{\tilde{r},\tilde{\mu}\}$. With the same method, the following estimations can be also obtained

$$\left\|Q_1\right\|_{L^\infty[0,\infty)} \le \frac{\alpha_0+M}{\operatorname{Re}\gamma+\alpha_0+M}\|W\|,$$
$$\left\|Q_{3i}\right\|_{L^\infty[0,\infty)} \le \frac{M}{\operatorname{Re}\gamma+M}\|W\|, \quad i=1,2, \tag{A.48}$$
$$\left\|Q_3\right\|_{L^\infty[0,\infty)} \le \frac{\lambda_1+\lambda_2+M}{\operatorname{Re}\gamma+\lambda_1+\lambda_2+M}\|W\|.$$

Then for $\gamma \in \mathbb{C}$ satisfying $\sup\{(\varepsilon+\alpha_0)/|\gamma+\varepsilon+\alpha_0|, (\alpha_0+M)/(\operatorname{Re}\gamma+\alpha_0+M), (\lambda_1+\lambda_2)/|\gamma+\lambda_1+\lambda_2|, \varepsilon/|\gamma+\varepsilon|, (\lambda_1+\lambda_2+M)/(\operatorname{Re}\gamma+\lambda_1+\lambda_2+M), M/(\operatorname{Re}\gamma+M)\} < 1$, we have

$$\begin{aligned}\|Q\| &= \left|Q_0\right| + \left\|Q_1\right\| + \left|Q_2\right| + \left\|Q_3\right\| + \sum_{i=1}^2\left(\left\|Q_{2i}\right\| + \left\|Q_{3i}\right\|\right) \le \sup\left\{\frac{\varepsilon+\alpha_0}{\left|\gamma+\varepsilon+\alpha_0\right|}, \frac{\alpha_0+M}{\operatorname{Re}\gamma+\alpha_0+M},\right. \\ &\frac{\lambda_1+\lambda_2}{\left|\gamma+\lambda_1+\lambda_2\right|}, \frac{\varepsilon}{\left|\gamma+\varepsilon\right|}, \frac{\lambda_1+\lambda_2+M}{\operatorname{Re}\gamma+\lambda_1+\lambda_2+M}, \\ &\left.\frac{M}{\operatorname{Re}\gamma+M}\right\}\|W\| < \|W\|.\end{aligned} \tag{A.49}$$

This implies that $\|(\gamma I - C)^{-1}D\| < 1$. Then $[I-(\gamma I-C)^{-1}D]$ is invertible. Therefore $\gamma I - A^*$ is invertible and $(\gamma I - A^*)^{-1} = [\gamma I - (C+D)]^{-1} = [I-(\gamma I-C)^{-1}D]^{-1}(\gamma I-C)^{-1}$. □

Proof of Lemma 10. We divide the proof into two steps.

Step 1. We prove that A_0 generates a C_0 semigroup $T_0(t)$. Consider the following abstract Cauchy problem:

$$\begin{aligned}\frac{dP(t,\cdot)}{dt} &= A_0P(t,\cdot), \quad t\ge 0 \\ P(0,\cdot) &= \Phi,\end{aligned} \tag{A.50}$$

where $\Phi = (\Phi_0, \Phi_1(x), \Phi_2, \Phi_{21}, \Phi_{22}(y), \Phi_3(x), \Phi_{31}(x), \Phi_{32}(x))^T \in X$. That is

$$\left(\frac{d}{dt}+\varepsilon+\alpha_0\right)P_0(t) = 0, \tag{A.51}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\alpha_0+r(x)\right]P_1(t,x) = 0, \tag{A.52}$$

$$\left(\frac{d}{dt}+\lambda_1+\lambda_2\right)P_2(t) = 0, \tag{A.53}$$

$$\left(\frac{d}{dt}+\varepsilon\right)P_{21}(t) = 0, \tag{A.54}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\mu(y)\right]P_{22}(t,y) = 0, \tag{A.55}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\lambda_1+\lambda_2+r(x)\right]P_3(t,x) = 0, \tag{A.56}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+r(x)\right]P_{3i}(t,x)=0,\quad i=1,2, \tag{A.57}$$

$$P_j(t,0)=0,\quad j=1,3,22,31,32, \tag{A.58}$$

$$P_i(0)=\varphi_i,\quad i=0,2,21, \tag{A.59}$$

$$P_j(0,\xi)=\varphi_j(\xi),\quad j=1,3,22,31,32. \tag{A.60}$$

Solving (A.51), (A.53), and (A.54) with the help of (A.59) yields

$$\begin{aligned} P_0(t)&=\varphi_0 e^{-(\varepsilon+\alpha_0)t},\\ P_2(t)&=\varphi_2 e^{-(\lambda_1+\lambda_2)t},\\ P_{21}(t)&=\varphi_{21}e^{-\varepsilon t}. \end{aligned} \tag{A.61}$$

Solving (A.52) and (A.55)–(A.57) with the help of (A.58) and (A.60) by the method of characteristics yields

$$\begin{aligned} P_1(t,x)&=\begin{cases}0, & x<t\\ \varphi_1(x-t)e^{-\int_{x-t}^{x}(\alpha_0+r(s))\mathrm{d}s}, & x\ge t\end{cases}\\ P_3(t,x)&=\begin{cases}0, & x<t\\ \varphi_3(x-t)e^{-\int_{x-t}^{x}(\lambda_1+\lambda_2+r(s))\mathrm{d}s}, & x\ge t\end{cases}\\ P_{22}(t,y)&=\begin{cases}0, & y<t\\ \varphi_{22}(y-t)e^{-\int_{y-t}^{y}\mu(s)\mathrm{d}s}, & y\ge t\end{cases}\\ P_{3i}(t,x)&=\begin{cases}0, & x<t\\ \varphi_{3i}(x-t)e^{-\int_{x-t}^{x}r(s)\mathrm{d}s}, & x\ge t,\end{cases}\\ &\qquad i=1,2 \end{aligned} \tag{A.62}$$

therefore, operator A_0 generates a C_0 semigroup $T_0(t)$ given by

$$\begin{aligned} &(T_0(t)\Phi)(x,y)\\ &=\begin{cases}(\Phi_0,0,\Phi_2,\Phi_{21},0,0,0,0)^{\mathrm{T}}, & x,y<t\\ (\Phi_0,\Phi_1,\Phi_2,\Phi_{21},\Phi_{22},\Phi_3,\Phi_{31},\Phi_{32})^{\mathrm{T}}, & x,y\ge t,\end{cases} \end{aligned} \tag{A.63}$$

where

$$\begin{aligned} \Phi_0&=\varphi_0 e^{-(\varepsilon+\alpha_0)t},\\ \Phi_1&=\varphi_1(x-t)e^{-\int_{x-t}^{x}(\alpha_0+r(s))\mathrm{d}s},\\ \Phi_2&=\varphi_2 e^{-(\lambda_1+\lambda_2)t},\\ \Phi_3&=\varphi_3(x-t)e^{-\int_{x-t}^{x}(\lambda_1+\lambda_2+r(s))\mathrm{d}s},\\ \Phi_{21}&=\varphi_{21}e^{-\varepsilon t},\\ \Phi_{22}&=\varphi_{22}(y-t)e^{-\int_{y-t}^{y}\mu(s)\mathrm{d}s},\\ \Phi_{3i}&=\varphi_{3i}(x-t)e^{-\int_{x-t}^{x}r(s)\mathrm{d}s},\quad i=1,2. \end{aligned} \tag{A.64}$$

Step 2. We prove that $T_0(t)$ is quasi-compact. It needs to be proven that the essential growth bound $W_{\mathrm{ess}}(A_0)$ of A_0 is less than zero. The assumption condition (18) implies that, for any $\sigma>0$, there exists $t_0>0$ such that

$$\begin{aligned} &\frac{1}{t}\int_{x-t}^{x}r(s)\,\mathrm{d}s>\hat{r}-\sigma,\\ &\frac{1}{t}\int_{y-t}^{y}\mu(s)\,\mathrm{d}s>\hat{\mu}-\sigma,\\ &\qquad x,y\ge t\ge t_0. \end{aligned} \tag{A.65}$$

Then the following estimation can be deduced:

$$\begin{aligned} \|T_0(t)\Phi\|&=|\varphi_0|e^{-(\varepsilon+\alpha_0)t}+|\varphi_2|e^{-(\lambda_1+\lambda_2)t}\\ &\quad+|\varphi_{21}|e^{-\varepsilon t}\\ &\quad+\int_t^{\infty}|\varphi_1(x-t)|e^{-\int_{x-t}^{x}(\alpha_0+r(s))\mathrm{d}s}\mathrm{d}x\\ &\quad+\int_t^{\infty}|\varphi_3(x-t)|e^{-\int_{x-t}^{x}(\lambda_1+\lambda_2+r(s))\mathrm{d}s}\mathrm{d}x\\ &\quad+\int_t^{\infty}|\varphi_{22}(y-t)|e^{-\int_{y-t}^{y}\mu(s)\mathrm{d}s}\mathrm{d}y\\ &\quad+\sum_{i=1}^{2}\int_t^{\infty}|\varphi_{3i}(x-t)|e^{-\int_{x-t}^{x}r(s)\mathrm{d}s}\mathrm{d}x\\ &\le|\varphi_0|e^{-(\varepsilon+\alpha_0)t}+|\varphi_2|e^{-(\lambda_1+\lambda_2)t}\\ &\quad+|\varphi_{21}|e^{-\varepsilon t} \end{aligned}$$

$$+e^{-(\alpha_0+\hat{r}-\sigma)t}\int_0^\infty |\varphi_1(x)|\,dx$$
$$+e^{-(\lambda_1+\lambda_2+\hat{r}-\sigma)}\int_0^\infty |\varphi_3(x)|\,dx$$
$$+e^{-(\hat{\mu}-\sigma)t}\int_0^\infty |\varphi_{22}(y)|\,dy$$
$$+\sum_{i=1}^{2}e^{-(\hat{r}-\sigma)t}\int_0^\infty |\varphi_{3i}(x)|\,dx$$
$$\le e^{-\min\{\varepsilon,\varepsilon+\alpha_0,\alpha_0+\hat{r}-\sigma,\lambda_1+\lambda_2+\hat{r}-\sigma,\hat{\mu}-\sigma\}t}\|\Phi\|. \tag{A.66}$$

Therefore

$$W_{\text{ess}}(A_0)\le W(A_0)=\lim_{t\to\infty}\frac{\ln\|T_0(t)\|}{t}<0. \tag{A.67}$$

This implies that the C_0 semigroup $T_0(t)$ generated by operator A_0 is quasi-compact. □

Proof of Lemma 11. From (20) we know that $R(\gamma, A-B)\ge R(\gamma, A_0)$, $\gamma>0$, which follows $S(t)\ge T_0(t)$, $t\ge 0$ immediately, where $R(\gamma, A)$ denotes the resolvent of operator A. For $P\in D(A_0)$, set $\Psi(s)P=S(t-s)(I+\Phi_\gamma)T_0(s)P$, $0\le s\le t$, $\gamma>0$. Then with the help of (20) we can obtain

$$\begin{aligned}\Psi'(s)P &= -S(t-s)(A-B)(I+\Phi_\gamma)T_0(s)P\\ &\quad+S(t-s)(I+\Phi_\gamma)A_0T_0(s)P\\ &= S(t-s)[\gamma I-(A-B)](I+\Phi_\gamma)T_0(s)P\\ &\quad+S(t-s)(I+\Phi_\gamma)[-\gamma I+A_0]T_0(s)P\\ &= S(t-s)(\gamma I-A_0)T_0(s)P\\ &\quad+S(t-s)(I+\Phi_\gamma)(-\gamma I+A_0)T_0(s)P\\ &= S(t-s)\Phi_\gamma(-\gamma I+A_0)T_0(s)P.\end{aligned} \tag{A.68}$$

Noting $[\Psi(t)-\Psi(0)]P=\int_0^t\Psi'(s)P\,ds$, then

$$\begin{aligned}&[\Psi(t)-\Psi(0)]P\\ &=\int_0^t S(t-s)\Phi_\gamma(-\gamma I+A_0)T_0(s)P\,ds.\end{aligned} \tag{A.69}$$

That is

$$\begin{aligned}&S(t)P-T_0(t)P\\ &\quad=-\int_0^t S(t-s)\Phi_\gamma(-\gamma I+A_0)T_0(s)P\,ds\\ &\quad\quad+\Phi_\gamma T_0(t)P-S(t)\Phi_\gamma P.\end{aligned} \tag{A.70}$$

Therefore, $S(t)-T_0(t)$ $(t\ge 0)$ is compact because the right-hand side of the above equation is the sum of three compact operators for the compactness of Φ_γ. □

Proof of Theorem 12. According to Proposition 9.20 (see [27]) combing Lemmas 10 and 11, we can deduce that

$$W_{\text{ess}}(\overline{A})\le W(A_0)<0. \tag{A.71}$$

Because B is a compact operator, then according to [28], it is evident that

$$W_{\text{ess}}(A)=W_{\text{ess}}(A-B)=W_{\text{ess}}(\overline{A})<0. \tag{A.72}$$

This implies that $T(t)$ is quasi-compact. □

Proof of Theorem 13. From Theorem 12 and the results in Section 3.2 with the help of Theorem 2.10 (see [28]), we can decompose the C_0 semigroup $T(t)$ generated by system operator A as $T(t)=\overline{P}_0+R(t)$, where $\overline{P}_0$ is the residue corresponding to eigenvalue 0 and $\|R(t)\|\le Ce^{-\varepsilon t}$ for suitable constants $C>0$ and $\delta>0$.

However, by Theorem 7, the nonnegative solution of the system (1)–(3) can be expressed as $P(t,\cdot)=T(t)P_0$, $t\in[0,\infty)$. Then combining Theorem 12.3 in [29], we can derive that

$$\begin{aligned}P(t,\cdot)&=T(t)P_0=(\overline{P}_0+R(t))P_0\\ &=\langle P_0,Q^*\rangle\hat{P}+R(t)P_0=\hat{P}+R(t)P_0,\end{aligned} \tag{A.73}$$

where Q^* is defined in Theorem 9. Hence we can get $\|P(t,\cdot)-\hat{P}\|\le Ce^{-\delta t}$. □

Proof of Theorem 14. The instantaneous availability of the system at time t is

$$A_v(t)=\sum_{i=0}^{3}P_i(t)+\sum_{i=2}^{3}P_{i1}(t). \tag{A.74}$$

Let $t\to\infty$; then the steady-state availability of the system is obtained as follows:

$$\begin{aligned}A_v&=\frac{\sum_{i=0}^{3}P_i+\sum_{i=2}^{3}P_{i1}}{S}\\ &=\frac{1}{S}\left[\frac{(1+\varepsilon g)[\varepsilon(\lambda_1+\lambda_2)+\alpha_0(\varepsilon+\lambda_1)]}{\varepsilon(\lambda_1+\lambda_2-\alpha_0\lambda_1 g)}+\frac{\alpha_0\lambda_1[\varepsilon(\lambda_1+\lambda_2)+\alpha_0\lambda_1]((f-g)/\alpha_0+(h-f)/(\lambda_1+\lambda_2))}{(\lambda_1+\lambda_2-\alpha_0)(\lambda_1+\lambda_2-\alpha_0\lambda_1 g)}\right]P_0\\ &=\frac{1}{N}\left[(\lambda_1+\lambda_2-\alpha_0)(1+\varepsilon g)[\varepsilon(\lambda_1+\lambda_2)+\alpha_0(\varepsilon+\lambda_1)]+\alpha_0\lambda_1\varepsilon[\varepsilon(\lambda_1+\lambda_2)+\alpha_0\lambda_1]\left(\frac{f-g}{\alpha_0}+\frac{h-f}{\lambda_1+\lambda_2}\right)\right].\end{aligned} \tag{A.75}$$

□

Proof of Theorem 15. The instantaneous probability of the repairman in vacation at time t is

$$P_v(t) = P_1(t) + P_3(t) + P_{31}(t) + P_{32}(t). \quad \text{(A.76)}$$

Letting $t \to \infty$ derives the steady-state probability of the repairman in vacation

$$\begin{aligned} P_v &= \frac{P_1 + P_3 + P_{31} + P_{32}}{S} \\ &= \frac{1}{S}\frac{[\varepsilon(\lambda_1+\lambda_2)+\alpha_0\lambda_1] f}{\lambda_1+\lambda_2-\alpha_0\lambda_1 g}P_0 \\ &= \frac{\varepsilon f(\lambda_1+\lambda_2-\alpha_0)[\varepsilon(\lambda_1+\lambda_2)+\alpha_0\lambda_1]}{N}. \end{aligned} \quad \text{(A.77)}$$

□

Proof of Theorem 16. The instantaneous probability of the system in warning state at time t is

$$P_w(t) = P_2(t) + P_3(t). \quad \text{(A.78)}$$

Letting $t \to \infty$ derives the steady-state probability of the system in warning state

$$\begin{aligned} P_w &= \frac{P_2 + P_3}{S} \\ &= \frac{(\alpha_0(1+\varepsilon g)/(\lambda_1+\lambda_2-\alpha_0\lambda_1 g))P_0}{S} \\ &= \frac{\alpha_0\varepsilon(1+\varepsilon g)(\lambda_1+\lambda_2-\alpha_0)}{N}. \end{aligned} \quad \text{(A.79)}$$

□

Proof of Theorem 17. Let $P_i(t) = \int_0^\infty P_i(t,\alpha)d\alpha$, $i = 1, 3, 22, 31, 32$; $r_j(t) = \int_0^\infty r(x)P_j(t,x)dx/P_j(t)$, $j = 1, 3, 31, 32$; and $\mu_{22}(t) = \int_0^\infty \mu(y)P_{22}(t,y)dy/P_{22}(t)$. Then the matrix of the transition probability of the system equations (1)–(3) can be obtained as follows:

$$T = \begin{pmatrix} -\varepsilon-\alpha_0 & r_1(t) & 0 & 0 & 0 & \mu_{22}(t) & 0 & 0 \\ \varepsilon & -\alpha_0 - r_1(t) & 0 & 0 & \varepsilon & 0 & 0 & 0 \\ \alpha_0 & 0 & -\lambda_1-\lambda_2 & r_3(t) & 0 & 0 & 0 & 0 \\ 0 & \alpha_0 & 0 & -\lambda_1-\lambda_2-r_3(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & -\varepsilon & 0 & r_{31}(t) & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & -\mu_{22}(t) & 0 & r_{32}(t) \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & -r_{31}(t) & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & -r_{32}(t) \end{pmatrix}. \quad \text{(A.80)}$$

Thus by [27] the instantaneous failure frequency of the system at time t can be derived as

$$W_f(t) = \lambda_2(P_2(t) + P_3(t)). \quad \text{(A.81)}$$

Let $t \to \infty$; then the steady-state failure frequency is immediate

$$\begin{aligned} W_f &= \frac{\lambda_2(P_2+P_3)}{S} = \lambda_2 P_w \\ &= \frac{\lambda_2\alpha_0\varepsilon(1+\varepsilon g)(\lambda_1+\lambda_2-\alpha_0)}{N}. \end{aligned} \quad \text{(A.82)}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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