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Research Article

Generalized Lacunary Statistical Difference Sequence Spaces of Fractional Order

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We generalize the lacunary statistical convergence by introducing the generalized difference operator Δ_ν^α of fractional order, where α is a proper fraction and $\nu = (\nu_k)$ is any fixed sequence of nonzero real or complex numbers. We study some properties of this operator and investigate the topological structures of related sequence spaces. Furthermore, we introduce some properties of the strongly Cesaro difference sequence spaces of fractional order involving lacunary sequences and examine various inclusion relations of these spaces. We also determine the relationship between lacunary statistical and strong Cesaro difference sequence spaces of fractional order.

1. Introduction

By ω, we denote the space of all real valued sequences and any subspace of w is called a *sequence space*. Let ℓ_∞, c, and c_0 be the linear spaces of bounded, convergent, and null sequences $x = (x_k)$ with real or complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N}$, the set of positive integers. With this norm, it is proved that these are all Banach spaces. Also by bs, cs, ℓ_1, and ℓ_p, we denote the spaces of all bounded, convergent, absolutely summable, and p-absolutely summable series, respectively.

The concept of difference sequence space was determined by Kızmaz [1]. Et and Colak generalized difference sequence spaces [2]. Later on Et and Esi [3] generalized these sequence spaces to the following sequence spaces. Let $\nu = (\nu_k)$ be any fixed sequence of nonzero complex numbers and let m be a nonnegative integer. Then, $\Delta_\nu^m(X) = \{x = (x_k) : (\Delta_\nu^m x) \in X\}$ for $X = \ell_\infty, c$ or c_0, where $m \in \mathbb{N}$, $\Delta_\nu^0 x = (\nu_k x_k)$, $\Delta_\nu^m x = (\Delta_\nu^{m-1} x_k - \Delta_\nu^{m-1} x_{k+1})$, and so $\Delta_\nu^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} \nu_{k+i} x_{k+i}$. These are Banach spaces with the norm defined by $\|x\|_\Delta = \sum_{i=1}^m |\nu_i x_i| + \sup_k |\Delta_\nu^m x_k|$. Furthermore Et and Basarir [4] have generalized difference sequence spaces. Aydın and Başar [5] have introduced some new difference sequence spaces. Also the notion of difference sequence has been extended by Mursaleen [6], Mursaleen and Noman [7], Malkowsky et al. [8], and Bektaş et al. [9]. Also Tripathy et al. [10, 11] have generalized difference sequences by Orlicz functions.

Let $\theta = (k_r)$ be the sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$, and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. Then θ is called a lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. Das and Mishra [12] introduced lacunary strong almost convergence. Colak et al. [13] have studied lacunary strongly summable sequences. Also some geometric properties of sequence spaces involving lacunary sequence have been examined by Karakaya [14]. Et has generalized Cesaro difference sequence spaces involving lacunary sequences [15].

Besides the Cesaro sequence spaces Ces_p and Ces_∞ have been introduced by Shiue [16]. Jagers [17] has determined the Köthe duals of the sequence space Ces_p $(1 < p < \infty)$. Later on the Cesaro sequence spaces X_p and X_∞ of nonabsolute type are defined by Ng and Lee [18, 19].

The main focus of the present paper is to generalize strong Cesaro and lacunary statistical difference sequence spaces and investigate their topological structures as well as some interesting results concerning the operator Δ_ν^α.

The rest of this paper is organized, as follows: in Section 2, some required definitions and consequences related to the difference operator Δ^α are given. Also some new classes of difference sequences of fractional order involving lacunary sequences are determined and some topological properties are investigated. Section 3 is devoted to the strong Cesaro difference sequence spaces of fractional order. Prior to stating and proving the main results concerning these spaces, we give some theorems about the notion of linearity and BK-space. In final section, we present some theorems related to the lacunary statistical convergence of difference sequences of fractional order and examine some inclusion relations of these spaces.

2. Some New Difference Sequence Spaces with Fractional Order

By $\Gamma(\alpha)$, we denote the *Euler gamma function* of a real number α. Using the definition, $\Gamma(\alpha)$ with $\alpha \notin \{0, -1, -2, -3, \ldots\}$ can be expressed as an improper integral as follows:

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt. \quad (1)$$

For a positive proper fraction α, Baliarsingh and Dutta [20, 21] (also see [22]) have defined the generalized fractional difference operator Δ^α as

$$\Delta^\alpha (x_k) = \sum_{i=0}^\infty (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}. \quad (2)$$

In particular, we have

(i) $\Delta^{1/2}(x_k) = x_k - (1/2)x_{k+1} - (1/8)x_{k+2} - (1/16)x_{k+3} - (5/128)x_{k+4} - (7/256)x_{k+5} - (21/1024)x_{k+6} - \cdots$,

(ii) $\Delta^{-1/2}(x_k) = x_k + (1/2)x_{k+1} + (3/8)x_{k+2} + (5/16)x_{k+3} + (35/128)x_{k+4} + (63/256)x_{k+5} + (231/1024)x_{k+6} + \cdots$,

(iii) $\Delta^{2/3}(x_k) = x_k - (2/3)x_{k+1} - (1/9)x_{k+2} - (4/81)x_{k+3} - (7/243)x_{k+4} - (14/729)x_{k+5} - (91/6561)x_{k+6} - \cdots$.

Theorem 1 (see [20]). *(a) For proper fraction α, $\Delta^\alpha : \omega \to \omega$ defined by (2) is a linear operator.*

(b) For $\alpha, \beta > 0$, $\Delta^\alpha(\Delta^\beta(x_k)) = \Delta^{\alpha+\beta}(x_k)$ and $\Delta^\alpha(\Delta^{-\alpha}(x_k)) = x_k$.

Now, we determine the new classes of difference sequence spaces $\Delta^\alpha_\nu(X)$ as follows:

$$\Delta^\alpha_\nu (X) := \{x = (x_k) \in \omega : (\Delta^\alpha_\nu x) \in X\}, \quad (3)$$

where $\Delta^\alpha_\nu(x_k) = \sum_{i=0}^\infty (-1)^i (\Gamma(\alpha+1)/i!\Gamma(\alpha-i+1))\nu_{k+i}x_{k+i}$ and X is any sequence spaces.

Theorem 2. *For a proper fraction α, if X is a linear space, then $\Delta^\alpha_\nu(X)$ is also a linear space.*

Proof. The proof is straightforward (see [20]). □

Theorem 3. *If X is a Banach space with the norm $\|\cdot\|_\infty$, then $\Delta^\alpha_\nu(X)$ is also a Banach space with the norm $\|\cdot\|_{\Delta^\alpha_\nu}$ defined by*

$$\|x\|_{\Delta^\alpha_\nu(X)} = \sum_{i=0}^\infty |\nu_i x_i| + \|\Delta^\alpha_\nu x\|_\infty, \quad (4)$$

where $\|\Delta^\alpha_\nu x\|_\infty = \sup_k |\Delta^\alpha_\nu(x_k)|$.

Proof. Proof of this theorem is a routine verification, hence omitted. □

Remark 4. Without loss of generality, we assume throughout that each series given in (4) is convergent. Furthermore, if α is a positive integer, then these infinite sums in (4) reduce to finite sums; that is, $\sum_{i=0}^\alpha (-1)^i (\Gamma(\alpha+1)/i!\Gamma(\alpha-i+1))\nu_{k+i}x_{k+i}$ and $\sum_{i=0}^\alpha |\nu_i x_i|$.

Lemma 5. *Let α be a proper fraction. If $X \subset Y$, then $\Delta^\alpha_\nu(X) \subset \Delta^\alpha_\nu(Y)$.*

Proof. Let $X \subset Y$ and $x = (x_k) \in \Delta^\alpha_\nu(X)$. It is trivial that $(\Delta^\alpha_\nu x) \in X$ implies $(\Delta^\alpha_\nu x) \in Y$. Hence $x \in \Delta^\alpha_\nu(Y)$ and $\Delta^\alpha_\nu(X) \subset \Delta^\alpha_\nu(Y)$. □

Theorem 6. *Let X be a Banach space and K a closed subset of X. Then $\Delta^\alpha_\nu(K)$ is also closed subset of $\Delta^\alpha_\nu(X)$.*

Proof. By using Lemma 5, it is trivial that $\Delta^\alpha_\nu(K) \subset \Delta^\alpha_\nu(X)$. Now we prove that $\overline{\Delta^\alpha_\nu(K)} = \Delta^\alpha_\nu(\overline{K})$. Let $x \in \overline{\Delta^\alpha_\nu(K)}$; then there exists a sequence $(x^n) \in \Delta^\alpha_\nu(K)$ such that

$$\begin{aligned} &\|x^n - x\|_{\Delta^\alpha_\nu(K)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\ &\text{i.e. } \sum_{i=0}^\infty |\nu_i (x_i^n - x_i)| \\ &\quad + \sup_k \left| \sum_{i=0}^\infty (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i} (x_{k+i}^n - x_{k+i}) \right| \\ &\quad \longrightarrow 0 \\ &\text{i.e. } \lim_n \sup_k \left| \nu_k (x_k^n - x_k) - \alpha \nu_{k+1} (x_{k+1}^n - x_{k+1}) \right. \\ &\quad \left. + \frac{\alpha(\alpha-1)}{2} \nu_{k+2} (x_{k+2}^n - x_{k+2}) + \cdots \right| = 0. \end{aligned} \quad (5)$$

Thus, we observe that

$$\begin{aligned} \|x_j^n - x_j\|_{\Delta^\alpha_\nu(K)} &= \sum_{i=0}^\infty |\nu_i (x_i^n - x_i)| \\ &\quad + \|\Delta^\alpha_\nu (x_j^n) - \Delta^\alpha_\nu (x_j)\|_\infty \longrightarrow 0, \end{aligned} \quad (6)$$

as $n \to \infty$ in K. This implies that $x \in \Delta^\alpha_\nu(\overline{K})$.

Conversely, let $x \in \Delta^\alpha_\nu(\overline{K})$; then $x \in \Delta^\alpha_\nu(K)$. Since K is closed $\overline{\Delta^\alpha_\nu(K)} = \Delta^\alpha_\nu(K)$, hence $\Delta^\alpha_\nu(K)$ is a closed subset of $\Delta^\alpha_\nu(X)$. □

Theorem 7. *If X is a BK-space with the norm $\|\cdot\|_\infty$, then $\Delta^\alpha_\nu(X)$ is also a BK-space with the norm given in (4).*

Proof. It is clear that $\Delta_\nu^\alpha(X)$ is a Banach space (see Theorem 3). Suppose that $\|x_j^n - x_j\|_{\Delta_\nu^\alpha(X)} \to 0$ for each $j \in \mathbb{N}$. One can conclude that $|\nu_j(x_j^n - x_j)| \to 0$ as $n \to \infty$ for each $j \in \mathbb{N}$ and implies $\|\Delta_\nu^\alpha(x_j^n) - \Delta_\nu^\alpha(x_j)\|_\infty \to 0$ as $n \to \infty$. This follows from the fact that

$$\sup_k \left|\sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i}\left(x_{k+i}^n - x_{k+i}\right)\right| \longrighto 0, \quad (\nu_{k+i} \neq 0), \tag{7}$$

where $P = \sum_{i=0}^{\infty}(-1)^i(\Gamma(\alpha+1)/i!\Gamma(\alpha-i+1))$; since $\alpha > 0$ the series represented by P is finite. Hence $|x_j^n - x_j| \to 0$ as $n \to \infty$ for each $j \in \mathbb{N}$. Therefore $\Delta_\nu^\alpha(X)$ is a Banach space with the continuous coordinates. This completes the proof. □

Definition 8. A sequence $x = (x_k)$ is said to be Δ_ν^α-strongly Cesaro convergent if there is a real or complex number L such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} \left|\sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i}x_{k+i} - L\right|^p = 0 \quad \text{for some } L, \tag{8}$$

where p is a fixed positive number and α is a proper fraction. The number L is unique when it exists. By $\Delta_\nu^\alpha(\omega_p)$, one denotes the set of all strongly Δ_ν^α-Cesaro convergent sequences. In this case, one writes $x_k \to L(\Delta_\nu^\alpha(\omega_p))$.

Theorem 9. *The sequence space $\Delta_\nu^\alpha(\omega_p)$ is a Banach space for $1 \le p < \infty$ normed by*

$$\|x\|_{\Delta_{\nu_1}} = \sum_{i=0}^{\infty} |\nu_i x_i| + \sup_{n\in\mathbb{N}_1} \left(\frac{1}{n}\sum_{k=0}^{n} \left|\sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i}x_{k+i}\right|^p\right)^{1/p} \tag{9}$$

and a complete p-normed space with the p-norm

$$\|x\|_{\Delta_{\nu_2}} = \sum_{i=0}^{\infty} |\nu_i x_i|^p + \sup_{n\in\mathbb{N}_1} \frac{1}{n}\sum_{k=0}^{n} \left|\sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i}x_{k+i}\right|^p \tag{10}$$

for $0 < p < 1$.

Proof. Proof follows by using Theorem 3. □

3. Cesaro Difference Sequence Spaces of Fractional Order

In this section by using the operator Δ_ν^α, we introduce some new sequence spaces $C(\Delta_\nu^\alpha, p)_\theta$, $C[\Delta_\nu^\alpha, p]_\theta$, $C_\infty(\Delta_\nu^\alpha, p)_\theta$, $C_\infty[\Delta_\nu^\alpha, p]_\theta$, and $N(\Delta_\nu^\alpha, p)_\theta$ involving lacunary sequences θ and arbitrary sequence $p = (p_r)$ of strictly positive real numbers.

We define the sequence spaces as follows:

$$\begin{aligned}
C(\Delta_\nu^\alpha, p)_\theta &:= \left\{x = (x_k) \in \omega : \sum_{r=1}^{\infty} \left|\frac{1}{h_r}\sum_{k\in I_r}\sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i}x_{k+i}\right|^{p_r} < \infty\right\}, \\
C[\Delta_\nu^\alpha, p]_\theta &:= \left\{x = (x_k) \in \omega : \sum_{r=1}^{\infty} \left(\frac{1}{h_r}\sum_{k\in I_r}\left|\sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i}x_{k+i}\right|\right)^{p_r} < \infty\right\}, \\
C_\infty(\Delta_\nu^\alpha, p)_\theta &:= \left\{x = (x_k) \in \omega : \sup_r \left|\frac{1}{h_r}\sum_{k\in I_r}\sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i}x_{k+i}\right|^{p_r} < \infty\right\}, \\
C_\infty[\Delta_\nu^\alpha, p]_\theta &:= \left\{x = (x_k) \in \omega : \sup_r \frac{1}{h_r} \cdot \sum_{k\in I_r}\left|\sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i}x_{k+i}\right|^{p_r} < \infty\right\}, \\
N(\Delta_\nu^\alpha, p)_\theta &:= \left\{x = (x_k) \in \omega : \lim_r \frac{1}{h_r} \cdot \sum_{k\in I_r}\left|\sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i}x_{k+i} - L\right|^{p_r} = 0\right\},
\end{aligned} \tag{11}$$

where $\nu = (\nu_k)$ is a fixed sequence of nonzero real or complex numbers.

Theorem 10. *Assume that (p_r) is a bounded sequence. Then the sequence spaces $C(\Delta_\nu^\alpha, p)_\theta$, $C[\Delta_\nu^\alpha, p]_\theta$, $C_\infty(\Delta_\nu^\alpha, p)_\theta$, $C_\infty[\Delta_\nu^\alpha, p]_\theta$, and $N(\Delta_\nu^\alpha, p)_\theta$ are linear spaces.*

Proof. Because the linearity may be proved in a similar way for each of the sets of sequences, hence it is omitted. □

Theorem 11. *If $p = p_r$ for all $r \in \mathbb{N}$, then the sequence space $C[\Delta_\nu^\alpha, p]_\theta$ is a BK-space with the norm defined by*

$$\|x\|_1 = \sum_{i=0}^{\infty} |\nu_i x_i| + \left\{ \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \cdot \sum_{k\in I_r} \left| \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i} x_{k+i} \right| \right)^p \right\}^{1/p}, \quad (1 \le p). \tag{12}$$

Also if $p_r = 1$ for all $r \in \mathbb{N}$, then the sequence spaces $C_\infty[\Delta_\nu^\alpha, p]_\theta$ and $N(\Delta_\nu^\alpha, p)_{2^r}$ are BK-spaces with the norm defined by

$$\|x\|_2 = \sum_{i=0}^{\infty} |\nu_i x_i| + \sup_r \frac{1}{h_r} \sum_{k\in I_r} \left| \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i} x_{k+i} \right|. \tag{13}$$

Proof. We give the proof for the space $C_\infty[\Delta_\nu^\alpha, p]_\theta$ and that of others followed by using similar techniques.

Suppose (x^n) is a Cauchy sequence in $C_\infty[\Delta_\nu^\alpha, p]_\theta$, where $x^n = (x_j^n)$ and $x^m = (x_j^m)$ are two elements in $C_\infty[\Delta_\nu^\alpha, p]_\theta$. Then there exists a positive integer $n_0(\epsilon)$ such that $\|x^n - x^m\|_2 \to 0$ as $m, n \to \infty$ for all $m, n \ge n_0(\epsilon)$ and for each $j \in \mathbb{N}$. Therefore $(x_i^1, x_i^2, \ldots)$ and $(\Delta_\nu^\alpha(x_j^1), \Delta_\nu^\alpha(x_j^2), \ldots)$ are Cauchy sequences in complex field $\mathbb{C}$ and $C_\infty[\Delta_\nu^\alpha, p]_\theta$, respectively. By using the completeness of $\mathbb{C}$ and $C_\infty[\Delta_\nu^\alpha, p]_\theta$, we have that they are convergent and suppose that $x_i^n \to x_i$ in $\mathbb{C}$ and $(\Delta_\nu^\alpha(x_j^n)) \to y_j$ in $C_\infty[\Delta_\nu^\alpha, p]_\theta$ for each $j \in \mathbb{N}$ as $n \to \infty$. Then we can find a sequence (x_j) such that $y_j = \Delta_\nu^\alpha(x_j)$ for each $j \in \mathbb{N}$. These x_j's can be interpreted as

$$\begin{aligned} x_j &= \frac{1}{\nu_j} \sum_{i=1}^{j-k} (-1)^r \frac{\Gamma(j-i)}{(k-1)!\Gamma(j-i-k+1)} y_i \\ &= \frac{1}{\nu_j} \sum_{i=1}^{j} \frac{(-1)^r}{(k-1)!} \frac{\Gamma(j+k-i)}{\Gamma(j-i+1)} y_{i-k}, \\ &\qquad (y_{1-k} = y_{2-k} = \cdots = y_0 = 0) \end{aligned} \tag{14}$$

for sufficiently large j; that is, $j > 2k$. Then $(\Delta_\nu^\alpha(x^n)) = (\Delta_\nu^\alpha(x_j^1), \Delta_\nu^\alpha(x_j^2), \ldots)$ converges to $(\Delta_\nu^\alpha(x_j))$ for each $j \in \mathbb{N}$ as $n \to \infty$. Thus $\|x^n - x\|_2 \to 0$ as $m \to \infty$. Since $C_\infty[\Delta_\nu^\alpha, p]_\theta$ is a Banach space with continuous coordinates, that is, $\|x^n - x\|_2 \to 0$ implies $|x_j^n - x_j| \to 0$ for each $j \in \mathbb{N}$, as $n \to \infty$, this shows that $C_\infty[\Delta_\nu^\alpha, p]_\theta$ is a *BK*-space. □

Theorem 12. *If $p = p_r$ for all $r \in \mathbb{N}$, then the sequence space $C(\Delta_\nu^\alpha, p)_\theta$ is a BK-space with the norm defined by*

$$\|x\|_3 = \sum_{i=0}^{\infty} |\nu_i x_i| + \left\{ \sum_{r=1}^{\infty} \left| \frac{1}{h_r} \cdot \sum_{k\in I_r} \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i} x_{k+i} \right|^p \right\}^{1/p}, \quad (1 \le p). \tag{15}$$

Also if $p_r = 1$ for all $r \in \mathbb{N}$, then the sequence space $C_\infty(\Delta_\nu^\alpha, p)_\theta$ is a BK-space with the norm defined by

$$\|x\|_4 = \sum_{i=0}^{\infty} |\nu_i x_i| + \sup_r \left| \frac{1}{h_r} \sum_{k\in I_r} \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \nu_{k+i} x_{k+i} \right|. \tag{16}$$

Proof. The proof follows from Theorem 11. □

Now, we can present the following theorem, determining some inclusion relations without proof, since it is a routine verification.

Theorem 13. *Let two positive proper fractions $\alpha > \beta > 0$ and $p_r = p$ for each $r \in \mathbb{N}$ be given. Then the following inclusions are satisfied:*

(i) $C(\Delta_\nu^\beta, p)_\theta \subset C(\Delta_\nu^\alpha, p)_\theta$,

(ii) $C[\Delta_\nu^\beta, p]_\theta \subset C[\Delta_\nu^\alpha, p]_\theta$,

(iii) $C(\Delta_\nu^\alpha, p)_\theta \subset C(\Delta_\nu^\alpha, q)_\theta$, $(0 < p < q)$.

4. Lacunary Statistical Convergence of Difference Sequences

The concept of statistical convergence from different aspects has been studied by various mathematicians. The notion of statistical convergence was independently introduced by Fast [23] and Schoenberg [24].

Let K be a subset of the set of natural numbers $\mathbb{N}$. Then the asymptotic density of K denoted by $\delta(K)$ is defined as $\delta(K) = \lim_n (1/n)|\{k \le n : k \in K\}|$, where the vertical bars denote the cardinality of the enclosed set.

A number sequence $x = (x_k)$ said to be statistically convergent to the number L if, for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \le n : |x_k - L| > \varepsilon\}$ has asymptotic density zero; that is, $\lim_n (1/n)|\{k \le n : |x_k - L| \ge \varepsilon\}| = 0$.

Definition 14. Let $\theta = (k_r)$ be a lacunary sequence and α a proper fraction. Then the sequence $x = (x_k)$ is said to be Δ_ν^α-lacunary statistically convergent to a real or complex number L if, for each $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in (k_{r-1}, k_r] : |\Delta_\nu^\alpha(x_k) - L| \ge \varepsilon\}\right| = 0, \tag{17}$$

where the vertical bars denote the cardinality of the enclosed set. The set of Δ_ν^α-lacunary statistically convergent sequences will be denoted by $\Delta_\nu^\alpha(S_\theta)$. In this case we write $x_k \to L(\Delta_\nu^\alpha(S_\theta))$.

In particular, the Δ_ν^α-lacunary statistical convergence includes many special cases; that is, in the case $\alpha = m \in \mathbb{N}$, Δ_ν^α-lacunary statistical convergence reduces to the Δ_ν^m-lacunary statistical convergence defined by [15].

Theorem 15. *Let $0 < p < \infty$. If $x_k \to L(N(\Delta_\nu^\alpha, p)_\theta)$ for $p_r = p$, then $x_k \to L(\Delta_\nu^\alpha(S_\theta))$. If $x \in \Delta_\nu^\alpha(\ell_\infty)$ and $x_k \to L(\Delta_\nu^\alpha(S_\theta))$, then $x_k \to L(N(\Delta_\nu^\alpha, p)_\theta)$.*

Proof. Let $x = (x_k) \in L(N(\Delta_\nu^\alpha, p)_\theta)$ and $\epsilon > 0$ and let $\sum_*$ denote the sum over $k \in (k_{r-1}, k_r]$ such that $|\Delta_\nu^\alpha(x_k) - L| \geq \varepsilon$. We have that

$$\frac{1}{h_r}\sum_* \left|\Delta_\nu^\alpha x_k - L\right|^p \geq \frac{1}{h_r}\left|\left\{k \in (k_{r-1}, k_r] : \left|\Delta^\alpha(x_k) - L\right| \geq \varepsilon\right\}\right| \epsilon^p. \tag{18}$$

So we observe, by passing to limit as $r \to \infty$, in (18)

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in (k_{r-1}, k_r] : \left|\Delta_\nu^\alpha(x_k) - L\right| \geq \varepsilon\right\}\right| \leq \frac{1}{\epsilon^p}\left(\lim_{r\to\infty}\frac{1}{h_r}\sum_*\left|\Delta_\nu^\alpha x_k - L\right|^p\right) = 0 \tag{19}$$

which implies that $x_k \to L(\Delta_\nu^\alpha(S_\theta))$.

Suppose that $x \in \Delta_\nu^\alpha(\ell_\infty)$ and $x_k \to L(\Delta_\nu^\alpha(S))$. Then it is obvious that $(\Delta_\nu^\alpha x) \in \ell_\infty$ and $(1/h_r)|\{k \in (k_{r-1}, k_r] : |\Delta_\nu^\alpha(x_k) - L| \geq \varepsilon\}| \to 0$ as $r \to \infty$. Let $\varepsilon > 0$ be given and there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\left|\left\{k \in (k_{r-1}, k_r] : \left|\Delta^\alpha(x_k) - L\right| \geq (\varepsilon/2)^{1/p}\right\}\right|}{h_r} \leq \frac{\varepsilon}{2\left(\left\|\Delta_\nu^\alpha x\right\|_\infty + L\right)^p}, \tag{20}$$

where $\sum_{i=1}^\infty |\nu_i x_i| = L$, for all $r > n_0$. Furthermore, we can write

$$\left|\Delta_\nu^\alpha(x_k) - L\right| \leq \left\|\Delta_\nu^\alpha(x_k) - L\right\|_{\Delta_\nu^\alpha} \leq L + \left\|\Delta_\nu^\alpha x\right\|_\infty = \|x\|_{\Delta_\nu^\alpha(X)}. \tag{21}$$

For $r > n_0$,

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\left|\Delta_\nu^\alpha x_k - L\right|^p &= \frac{1}{h_r}\left(\sum_{k\in L_r}\left|\Delta^\alpha x_k - L\right|^p + \sum_{k\notin L_r}\left|\Delta^\alpha x_k - L\right|^p\right) \\ &< \frac{1}{h_r}\left(h_r\frac{\varepsilon}{2} + h_r\frac{\varepsilon\|x\|_{\Delta_\nu^\alpha(X)}^p}{2\|x\|_{\Delta_\nu^\alpha(X)}^p}\right) = \varepsilon,\end{aligned} \tag{22}$$

where $L_r = \{k \in (k_{r-1}, k_r] : |\Delta^\alpha(x_k) - L| \geq (\varepsilon/2)^{1/p}\}$. Hence $x_k \to L(N(\Delta_\nu^\alpha, p)_\theta)$. This completes the proof. □

Corollary 16. *The following statements hold:*

(a) $S \cap \ell_\infty \subset \Delta_\nu^\alpha(S_\theta) \cap \Delta_\nu^\alpha(\ell_\infty)$,

(b) $\Delta_\nu^\alpha(S_\theta) \cap \Delta_\nu^\alpha(\ell_\infty) = \Delta_\nu^\alpha(\omega_p)$.

Definition 17. Let θ be a lacunary sequence and α a proper fraction. Then a sequence $x = (x_k)$ is said to be Δ_ν^α-lacunary statistically Cauchy if there exists a number $N = N(\varepsilon)$ such that

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in (k_{r-1}, k_r] : \left|\Delta_\nu^\alpha(x_k - x_N)\right| \geq \varepsilon\right\}\right| = 0 \tag{23}$$

for every $\varepsilon > 0$.

Theorem 18. *If $x = (x_k)$ is a Δ_ν^α-lacunary statistically convergent sequence, then x is a Δ_ν^α-lacunary statistically Cauchy sequence.*

Proof. Assume that $x_k \to L(\Delta_\nu^\alpha(S_\theta))$ and $\varepsilon > 0$. Then $|\Delta_\nu^\alpha(x_k) - L| < \varepsilon/2$ for almost all k, and if we select N, then $|\Delta_\nu^\alpha(x_N) - L| < \varepsilon/2$ holds. Now, we have

$$\left|\Delta_\nu^\alpha(x_k) - \Delta_\nu^\alpha(x_N)\right| < \left|\Delta_\nu^\alpha(x_k) - L\right| + \left|\Delta_\nu^\alpha(x_N) - L\right| < \varepsilon \tag{24}$$

for almost all k. Hence x is a Δ_ν^α-lacunary statistically Cauchy sequence. □

Theorem 19. *Let α be a proper fraction and $0 < \inf p_r \leq p_r \leq \sup p_r < \infty$. Then $N(\Delta_\nu^\alpha, p)_\theta \subset \Delta_\nu^\alpha(S_\theta)$.*

Proof. Suppose that $x = (x_k) \in N(\Delta_\nu^\alpha, p)_\theta$ and $\sum_*$ denote the sum over $k \in (k_{r-1}, k_r]$ such that $|\Delta_\nu^\alpha(x_k) - L| \geq \varepsilon$. Therefore we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\left|\Delta_\nu^\alpha(x_k) - L\right|^{p_r} &\geq \frac{1}{h_r}\sum_*\left|\Delta^\alpha x_k - L\right|^{p_r} \\ &\geq \frac{1}{h_r}\sum_*\varepsilon^{\inf p_r} \\ &\geq \frac{1}{h_r}\left|\left\{k \in I_r : \left|\Delta^\alpha(x_k) - L\right| \geq \varepsilon\right\}\right|\varepsilon^{\inf p_r}.\end{aligned} \tag{25}$$

Taking the limit as $r \to \infty$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in I_r : \left|\Delta_\nu^\alpha(x_k) - L\right| \geq \varepsilon\right\}\right| \leq \frac{1}{\varepsilon^{\inf p_r}}\left(\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left|\Delta_\nu^\alpha x_k - L\right|^{p_r}\right) = 0 \tag{26}$$

implies that $x \in \Delta_\nu^\alpha(S_\theta)$. Hence $N(\Delta_\nu^\alpha, p)_\theta \subset \Delta_\nu^\alpha(S_\theta)$. □

5. Concluding Remarks

In this paper, certain results on some lacunary statistical difference sequence spaces of order m ($m \in \mathbb{N}$) have been extended to the difference sequence spaces of fractional order α. The results presented in this paper not only generalize the earlier works done by several authors [2, 3, 15, 20, 21] but also give a new perspective regarding the development of difference sequences. As a future work we will study certain matrix transformations of these spaces.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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